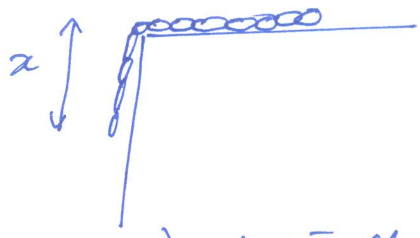
Ex. A uniform chain of length 'a' is placed on a horizontal frictionless table, so that a length 'b' of the chain dangles over the side. How long will it take for the chain to slide off the table?



Let at time t, the chain dangles by an amount x.

Assume the density (mass/unit length) of the chain is σ. By Newton's law

$$(\sigma g)x = \sigma a \frac{dv}{dt}.$$

$$(\sigma g)x = \sigma a \cdot \frac{dv}{dx} \cdot \frac{dx}{dt} = \sigma a v \frac{dv}{dx}$$

Thus, $$v\,dv = \frac{g}{a} x\,dx.$$

$$\frac{v^2}{2} = \frac{g}{2a}x^2 + c$$

$v = 0$ for $x = b$.

$$0 = \frac{g b^2}{2a} + c \Rightarrow c = -\frac{g b^2}{2a}.$$

$$v = \sqrt{\frac{g}{a}}\sqrt{x^2 - b^2} = \frac{dx}{dt}.$$

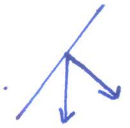
Again, $$\frac{dx}{\sqrt{x^2-b^2}} = \sqrt{\frac{g}{a}}\,dt.$$

Integrating, $$\ln\left(\frac{x + \sqrt{x^2-b^2}}{b}\right) = \sqrt{\frac{g}{a}}\,t.$$

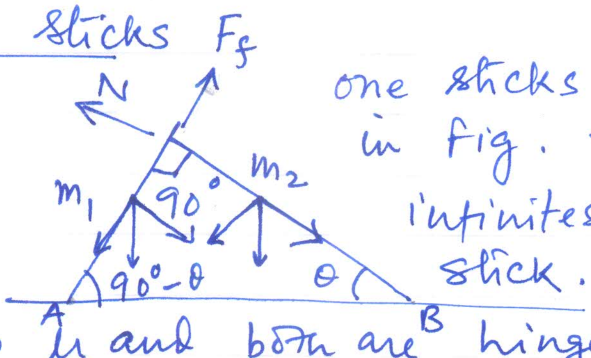
For the chain to completely slide off $x = a$.

$$T = \sqrt{\frac{a}{g}} \ln\left(\frac{a + \sqrt{a^2-b^2}}{b}\right).$$ (Ans).

Examples



Leaning Sticks



One sticks leans on another as shown in fig. The left stick extends infinitesimally beyond the right stick. Coefficient of friction betn. sticks is μ and both are hinged at the ground. For what θ, the arrangement is stable?

Solⁿ. Let m_1 and m_2 be the masses of the sticks.

$\cancel{m_1 g \sin\theta = m_2 g \sin\theta}$ $\qquad$ $m_1 g \cos\theta = m_2 g \sin\theta = F_f$.

$\Rightarrow \cancel{\frac{m_1}{m_2} =}$ $\qquad \Rightarrow \frac{m_1}{m_2} = \tan\theta \qquad (1)$

$$F_f \leq \mu N. \qquad (2)$$

Balancing torque on left stick about A,

$$l_1 \cdot N = m_1 g \sin\theta \left(\frac{l_1}{2}\right) \Rightarrow N = \frac{m_1 g \sin\theta}{2}$$

Similarly for right stick about B,

$$f_f = \frac{m_2 g \cos\theta}{2}.$$

Thus, $\quad m_2 \cos\theta \leq \mu m_1 \sin\theta$

$$\Rightarrow \tan^2\theta \geq \frac{1}{\mu}.$$

For $\mu \to 0$, $\theta \sim \pi/2$, the sticks have to be very close to each other.

For $\mu \to \infty$, $\theta \sim 0$.

Example

Charged particle in a magnetic field – An advanced example of motion in 2 & 3 dimensions.

The force expression is given by the Lorentz force

$$\vec{F} = m\frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B}).$$

The particle executes a circular motion when $\vec{v}$ is perpendicular to $\vec{B}$ whose radius is given by $r = \frac{mv}{qB}$ and the angular velocity $\omega = \frac{qB}{m}$.

We shall aim for a complete solution of the problem. Let $\vec{B} = (0, 0, \hat{B})$.

Thus $\vec{v} \times \vec{B} = (v_y B, -v_x B, 0)$.

Writing the equations of motion componentwise,

$$m\frac{dv_x}{dt} = qBv_y \Rightarrow \frac{dv_x}{dt} = \frac{qB}{m} v_y$$

$$m\frac{dv_y}{dt} = -qBv_x \Rightarrow \frac{dv_y}{dt} = -\frac{qB}{m} v_x$$

$$m\frac{dv_z}{dt} = 0.$$

Introduce, $\omega_c = \frac{qB}{m}$ (the subscript 'c' stands for cyclotron)

$$\frac{dv_x}{dt} = \omega_c v_y \qquad (1)$$

$$\frac{dv_y}{dt} = -\omega_c v_x \qquad (2)$$

This yields coupled sets of equation

contd.

However differentiate (1) wrt time and use (2)

$$\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = 0. \qquad (3)$$

– This is known to be an equation of SHM. The solⁿ are

$$v_x(t) = v_0 \cos(\omega_c t) \qquad (4)$$

v_0 : initial velocity.

Since $v_y(t) = \frac{1}{\omega_c} \frac{dv_x}{dt}$

$$v_y(t) = -v_0 \sin(\omega_c t). \qquad (5).$$

Integrating (4) & (5), one gets,

$$x = x_0 + \frac{v_0}{\omega_c} \sin(\omega_c t) \qquad (6)$$

$$y = y_0 + \frac{v_0}{\omega_c} \cos(\omega_c t) \qquad (7)$$

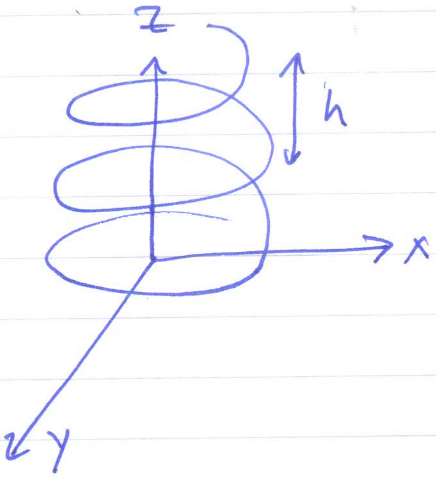
Eq (6) & (7) together represents the equ. of a circle with center at (x_0, y_0) and radius, $r_c = \frac{v_0}{\omega_c} = \frac{m v_0}{qB}$

The third equation of motion

$m \frac{dv_z}{dt} = 0$ is useful when the particle has an initial z-component of velocity v_{0z} (parallel to the dirⁿ of the magnetic field), then

$z = v_{0z} t$ which is a uniform motion in z-direction. Thus we have a circular motion in the planar dirⁿ (x-y) and the free motion in z-dirⁿ.

Contd.



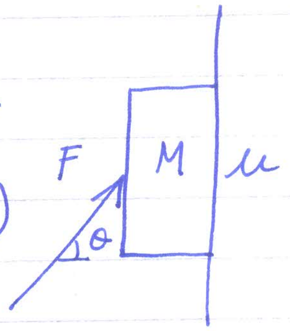
The pitch of the helix is defined as the z-distance gained through one revolution, i.e in time $T_c = 2\pi/\omega_c$

$$h = v_z T_c = v_z \frac{2\pi}{\omega_c} = \frac{2\pi m v_z}{qB}.$$

Example: Thomson's experiment of measuring e/m.

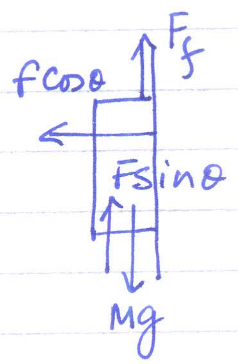
Pushing a book up in the vertical wall

8. A book of mass M is positioned against a vertical wall by a force F applied at angle θ $(-\pi/2 < \theta < \frac{\pi}{2})$



a) What is F_{min} for a given θ?
b) for what θ, F_{min} is smallest? What is the magnitude of this F_{min}?
c) What is the minimum θ, below which any F will not keep the book up?

Ans.



At rest,

$$F\sin\theta + F_f - Mg = 0. \quad (1)$$

The maximum $F_f = \mu F\cos\theta \; (= \mu N)$

So the second condition is $F_f \leq \mu F\cos\theta$ (2)

Putting (2) in (1)

$$F(\sin\theta + \mu\cos\theta) \geq Mg$$

So, $F \geq \dfrac{Mg}{\sin\theta + \mu\cos\theta}$ for the book to remain vertical.

(a) So $F_{min} = \dfrac{Mg}{\sin\theta + \mu\cos\theta}$ assuming $(\sin\theta + \mu\cos\theta) > 0$ Otherwise there is no solution.

(b) To minimize F_{min}, maximize the denominator. Take derivative of $\sin\theta + \mu\cos\theta \Rightarrow$ put it to zero.

$$\cos\theta - \mu\sin\theta = 0 \Rightarrow \tan\theta = 1/\mu$$

Contd.

This gives $f_{min} = \frac{mg}{\sqrt{1+\mu^2}}$

This is the smallest possible f that keeps the hook up where the angle must be $\theta = \tan^{-1}(1/\mu)$. Thus if μ is very small, then f has to be applied near vertical$(\frac{\pi}{2})$ with magnitude mg. and for Large μ, f can be applied horizontal $(\theta \simeq 0)$ with magnitude mg/μ.

(c) Basically it asks for an f for which the problem has no solution. This happens when the coefficient of f, i.e.

$$\sin\theta + \mu\cos\theta = 0 \quad \text{or negative.}$$

$$\text{i.e.} \quad \tan\theta = -\mu$$

Since tan is a monotonic fn, if θ is more negative than this, then no matter how much f is applied, one can not keep it vertical.