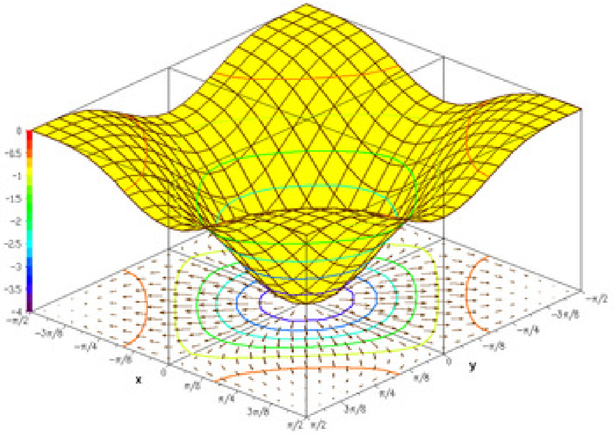
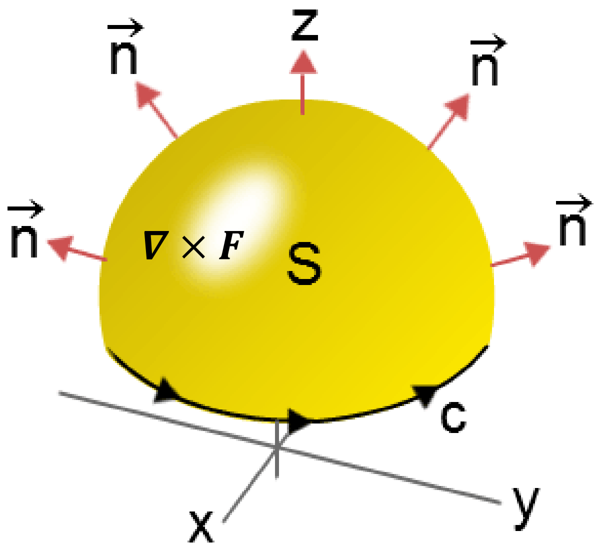


PH101
Lecture -9
30 August 2017

Stokes' Theorem Examples, Harmonic Approximation, …

Utility of Stokes' Theorem



$$\oint \boldsymbol{F} \cdot \boldsymbol{dr} = \int_{Surf} (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{A}$$

For conservative forces $\oint \boldsymbol{F} \cdot \boldsymbol{dr}$=0 for any closed loop.

Hence $\int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{A}$=0 for all surfaces.

In other words,

$\boldsymbol{\nabla} \times \boldsymbol{F}$=0 everywhere in space for conservative forces !

Curl of Gradient

$$\bar{\nabla} \times (\nabla \phi(x,y,z)) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) + \hat{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

For double differentiable countinuous functions
The order of differentiation does not matter!

So $\bar{\nabla} \times \bar{\nabla}\phi = 0$ (always!)

That is, if forces are derivable as <u>grad (function)</u> $(-\bar{\nabla} U)$ then it is <u>conservative</u>!

(as its <u>curl</u> is always zero!)

Examples: $(\bar{\nabla} \times \bar{F})$

1\. Let $\bar{F} = -mgz\,\hat{k}$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -mgz \end{vmatrix}$$

$$= \hat{i}\,\frac{\partial(-mgz)}{\partial y} - \hat{j}\,\frac{\partial(-mgz)}{\partial x} + \hat{k} \times 0$$

$$= \hat{i} \times 0 - \hat{j} \times 0 = \underline{\underline{0}}$$

(Conservative)

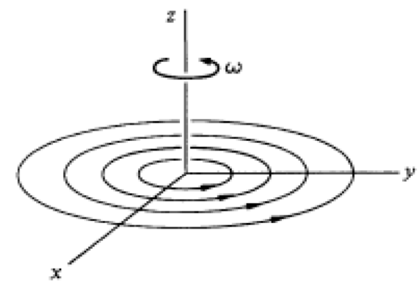
2. A Whirl pool whoes velocity of water given by,

$$\bar{v} = r\omega\,\hat{\theta} \quad \text{(polar)}$$

i.e. $\bar{v} = r\omega(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$\bar{v} = \omega(-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j})$

$= \omega(-y\hat{i} + x\hat{j})$



$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = \hat{k}(\omega - (-\omega))$$

$$= \underline{\underline{2\omega\hat{k}}} \quad (\neq 0)$$

<u>Expected</u> from the nature of $\bar{v}$ (see figure)!

Now verify Stokes theorem

$$\oint_{circle(R)} \bar{V} \cdot d\bar{r} = \int_{Circle(R)} (\bar{\nabla} \times \bar{V}) \cdot \bar{dA} \;?$$

$\bar{V} = r\omega \hat{\theta}$

$\bar{\nabla} \times \bar{F} = 2\omega \hat{k}$

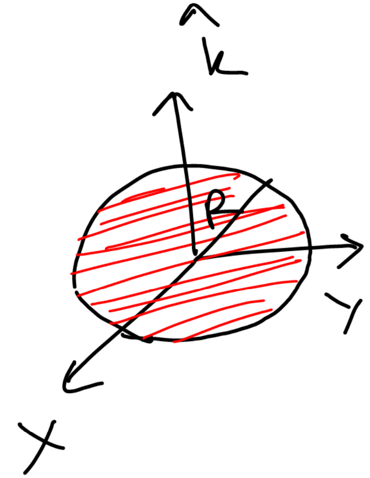
$\bar{dA} = dA\,\hat{k}$

$$d\bar{r} = R\,d\theta\,\hat{\theta}$$

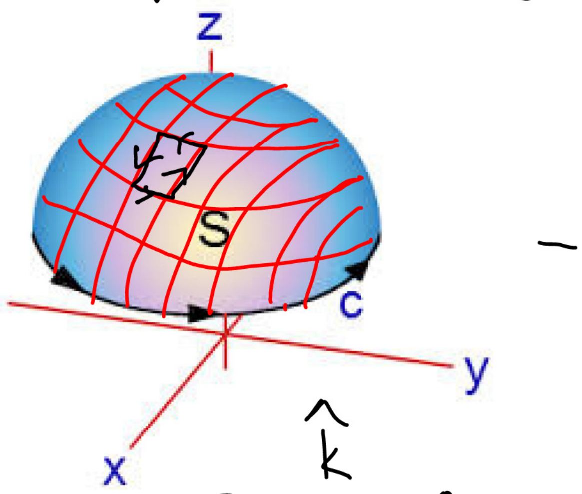
$$\oint \bar{V} \cdot d\bar{r} = \int_0^{2\pi} (R\omega)(R\,d\theta) = R^2 \omega \int_0^{2\pi} d\theta = \underline{\underline{2\pi R^2 \omega}}$$

$$\int_{circle} (2\omega \hat{k}) \cdot (r\,d\theta\,dr\,\hat{k}) = 2\omega \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{R} r\,dr$$

$$= 2\omega\,(2\pi)\,\frac{R^2}{2} = \underline{\underline{2\pi R^2 \omega}}$$ (Verified!)



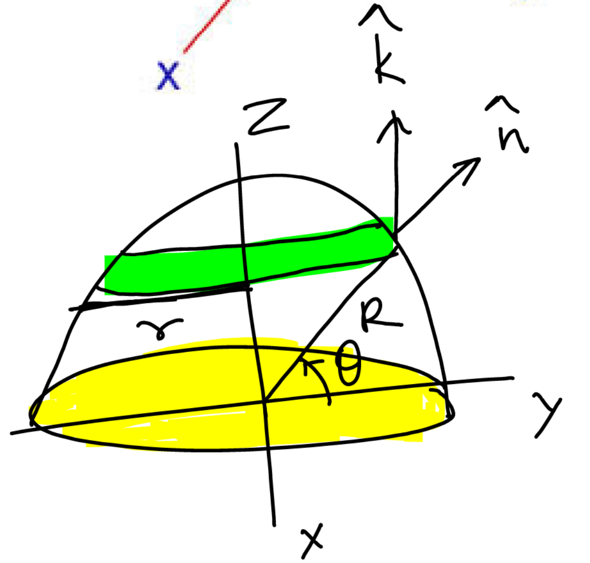
Now what if we take a hemisphere of radius R?



$$\oint_R \bar{v} \cdot d\bar{r} = 2\pi R^2 w$$

$$\int_{\text{hemi-sphere}} (\bar{\nabla} \times \bar{v}) \cdot d\bar{A} = 2w \int_{\text{hemisphere}} \hat{k} \cdot (2\pi r R \, d\theta) \hat{n}$$

$$= 4\pi R w \int_{\text{h.sphere}} r \, d\theta \; \hat{k} \cdot \hat{n}$$



$$\hat{n} = \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{R}$$

So $\hat{k} \cdot \hat{n} = z/R$

Now $r = R\cos\theta$, $z = R\sin\theta$ ↑

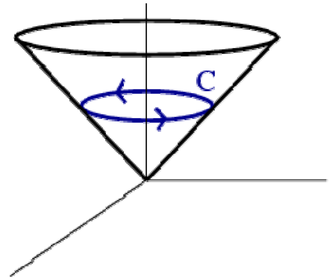
$$\int_{\text{h. sphere}} (\overline{\nabla} \times \overline{V}) \cdot d\overline{r} = 4\pi R \omega \int_{\theta=0}^{\pi/2} (R\cos\theta)\, d\theta \left(\frac{R\sin\theta}{R}\right)$$

$$= 4\pi R^2 \omega \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta = 2\pi R^2 \omega \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= 2\pi R^2 \omega \left. -\frac{\cos 2\theta}{2} \right|_0^{\pi/2} = -\pi R^2 \omega (-1-1)$$

$$= \underline{\underline{2\pi R^2 \omega}} \qquad (\text{Verified!})$$

Which Surface for this loop?



H.W Verify $\bar{F} = -\frac{GMm}{r^2}\hat{r}$ is conservative!

Equilibrium and Stability

If all forces acting on a body are conservative then the potential can be used to find the equilibrium points and the nature of the equilibrium easily.

$\boldsymbol{F} = -\boldsymbol{\nabla} U = 0$ will be a point of equilibrium since the force is zero.

Now consider the shape of the potential near an equilibrium.

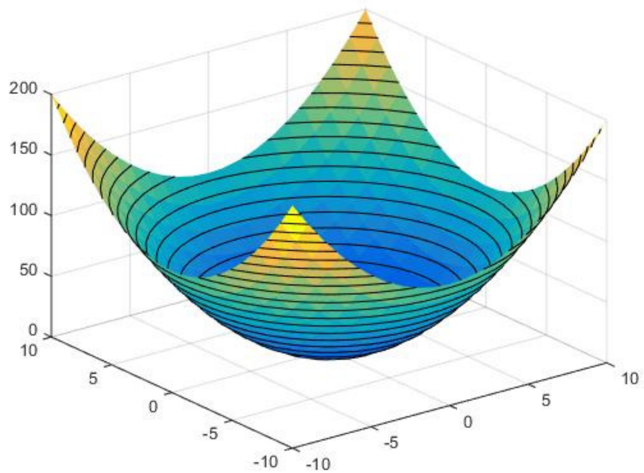
If the potential is minimum, then the equilibrium is stable, i.e if the body is pushed away from the equilibrium, it will try to go back to it.

In 1D, there can be three kinds of equilibrium, stable, unstable and neutral.

Stable Equilibrium

Consider a harmonic well potential. $U = (x^2 + y^2)$.
Consider equilibrium at (0,0).

$$\boldsymbol{F} = -\boldsymbol{\nabla} U = -2(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})$$

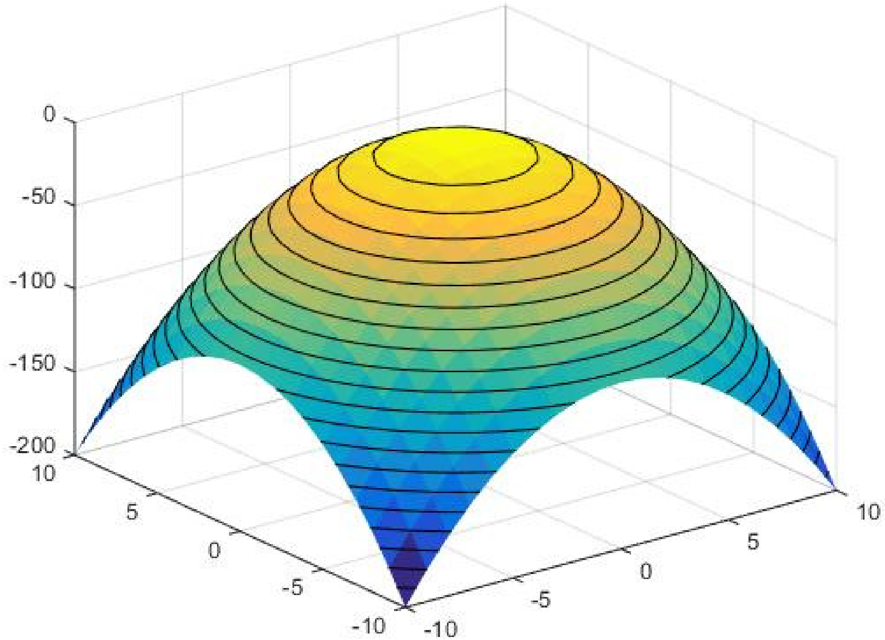


Unstable Equilibrium

Consider a harmonic well potential. $U = -(x^2 + y^2)$.

Consider equilibrium at (0,0).

$$\boldsymbol{F} = -\boldsymbol{\nabla} U = 2(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})$$

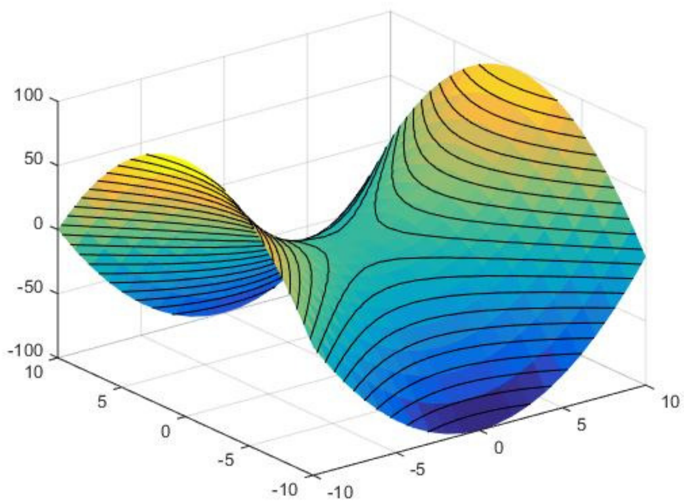


Saddle-Point

Consider a harmonic well potential. $U = (x^2 - y^2)$.

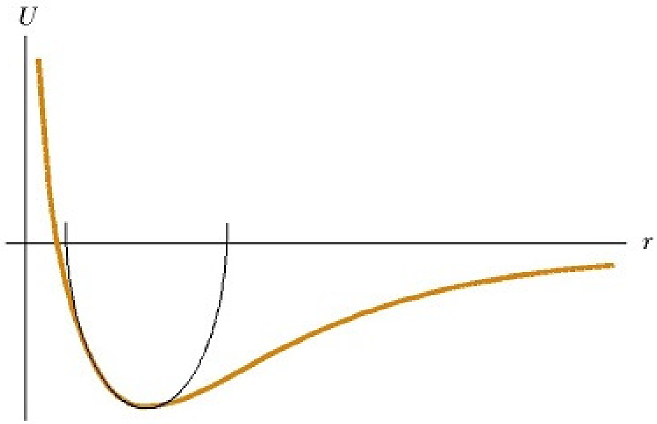
Consider equilibrium at (0,0).

$$\boldsymbol{F} = -\boldsymbol{\nabla} U = 2(-x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})$$

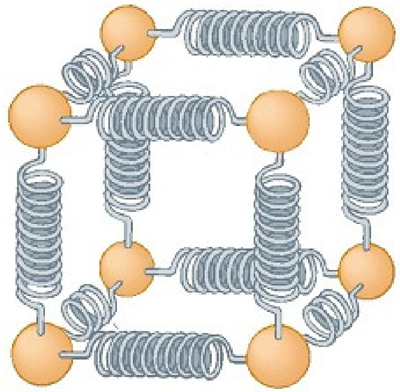


Harmonic Approximation

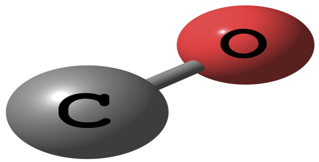
Harmonic potential is very important in physics such as in the analysis of molecular vibrations.



Harmonic approximation of the potential



Network of atoms



Carbon monoxide

Spectroscopy analysis can provide the frequencies

Taylor expansion of the potential

$$U(x) = U(x_0) + U'(x_0)(x - x_0) + \frac{1}{2!}U''(x_0)(x - x_0)^2 + \cdots$$

Here $U'(\mathrm{x}) = \frac{dU}{dx}$ and $\mathrm{U}''(\mathrm{x}) = \frac{d^2U}{dx^2}$

Here we are taking the expansion around the equilibrium distance x_0.

Hence $U'(x_0) = 0$ since the force is zero (potential has an extremum).

Let us assume that $U(x_0)$=0, the potential at the equilibrium (reference) is zero.

$$U(x) = \frac{1}{2!}U''(x_0)\,(x - x_0)^2$$

Harmonic Approximation of U(x)

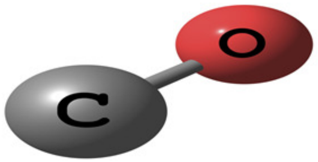
$$U(x) = \frac{1}{2!} U''(x_0)\,(x - x_0)^2$$

Spring constant, k $= U''(x_0)$.

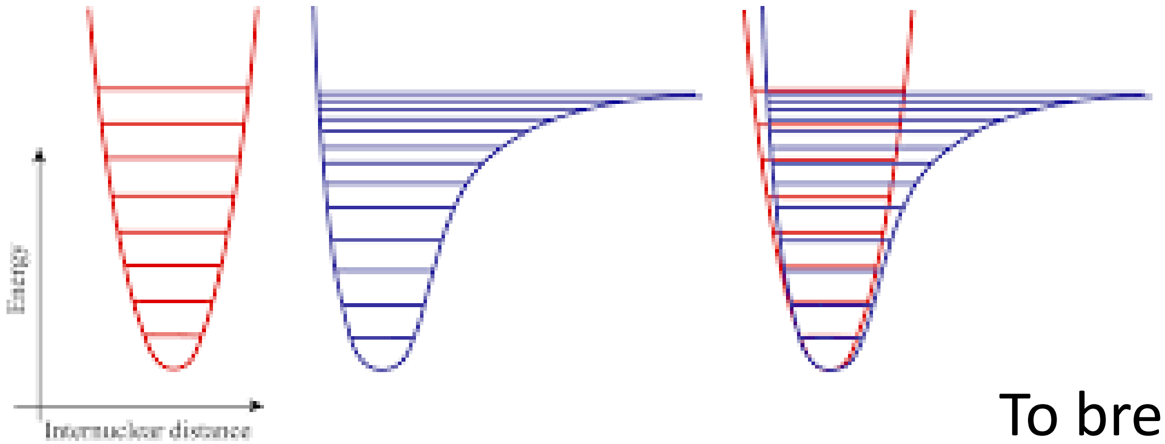
The frequency of vibration about the minimum is $\omega = \sqrt{k/\mu}$ where μ is the reduced mass of the oscillator.

Eg. diatomic molecules N_2, O_2, Cl_2

Harmonic Approximation to Morse Potential



$$U(x) = D(1 - e^{-\alpha(x - x_0)})^2$$



$$U(x_0) = 0$$
$$U(\propto) = D$$

To break the molecule one has to supply energy D. This is a convenient model for diatomic molecules.

$$U(x) = D(1 - e^{-\alpha(x-x_0)})^2$$

First find the equilibrium

$$U'(x) = 2D\alpha\left(1 - e^{-\alpha(x-x_0)}\right)\ e^{-\alpha(x-x_0)} = 0$$

Solving, at equilibrium $x = x_0$

Now $U''(x) = 2D\alpha\left(-\alpha e^{-\alpha(x-x_0)} + 2\alpha e^{-2\alpha(x-x_0)}\right)$

At equilibrium $U''(x_0) = 2D\alpha^2 \approx k$

$$\omega = \sqrt{\frac{k}{\mu}} = \alpha\sqrt{2D/\mu}$$