PH101

Lecture 8

24th Aug 2017

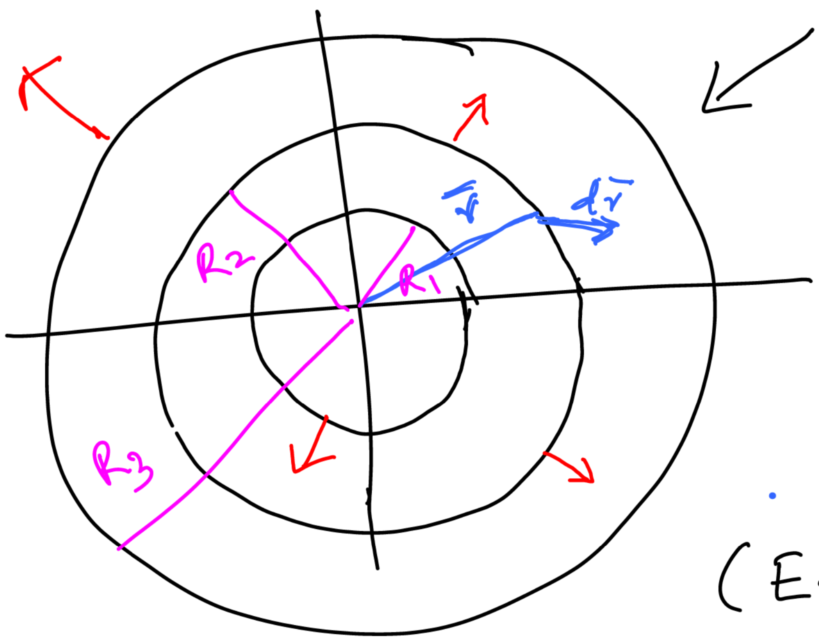
Stokes' Theorem

Physical Interpretation of $\bar{\nabla}$

The $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ & $\frac{\partial u}{\partial z}$ (Surely mean the 'rate' of change in u wrt space.)

Direction of $\bar{\nabla}\phi$?

Let, $\phi(x,y) = x^2 + y^2$



ISO-'Surfaces' or Contours of $\phi = R_1^2$, $\phi = R_2^2$, $\phi = R_3^2$

$$d\phi = \bar{\nabla}\phi \cdot d\bar{r}$$

(change in ϕ if we move along $d\bar{r}$)

(Equipotential surface/curve)

If the magnitude of $d\bar{r}$ is kept constant and vary only its direction at $\bar{r}$.

When do we get maximum in du?

That's in which direction do we need to walk to get maximum change in u.

If the two vectors are parallel, we get max. of their dot product!
($\bar{\nabla} u$ & $d\bar{r}$)

Conclusion:

$\bar{\nabla}\phi$ points in the direction of maximum change in ϕ.

$\bar{\nabla}\phi$ is perpendicular to the iso-surface!

(Always!)

For $\phi = x^2 + y^2$

$$\bar{\nabla}\phi = 2(x\hat{i} + y\hat{j}) = 2\bar{r} = \underline{\underline{2r\hat{r}}}$$

Let's note: If $d\bar{r}$ is taken along $\hat{\theta}$,

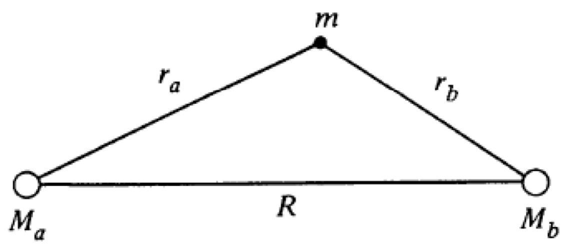
$$du = \bar{\nabla}\phi \cdot \hat{\theta}$$

$$du = 2(x\hat{i} + y\hat{j}) \cdot d\bar{r} = 2(x\hat{i} + y\hat{j}) \cdot \hat{\theta}$$

$$= 2(r\cos\theta^{-}\sin\theta + r\sin\theta\cos\theta) = 0$$

$(\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$ (Naturally as we are walking on a circle when $\phi = R^2 = \text{Const.}$)

Equipotential Surface of a binary star (See Kleppner)



$$U = -\frac{G M_a m}{r_a} - \frac{G M_b m}{r_b}$$

$r_a \ll r_b$ $\qquad$ $r_a \gg r_b$

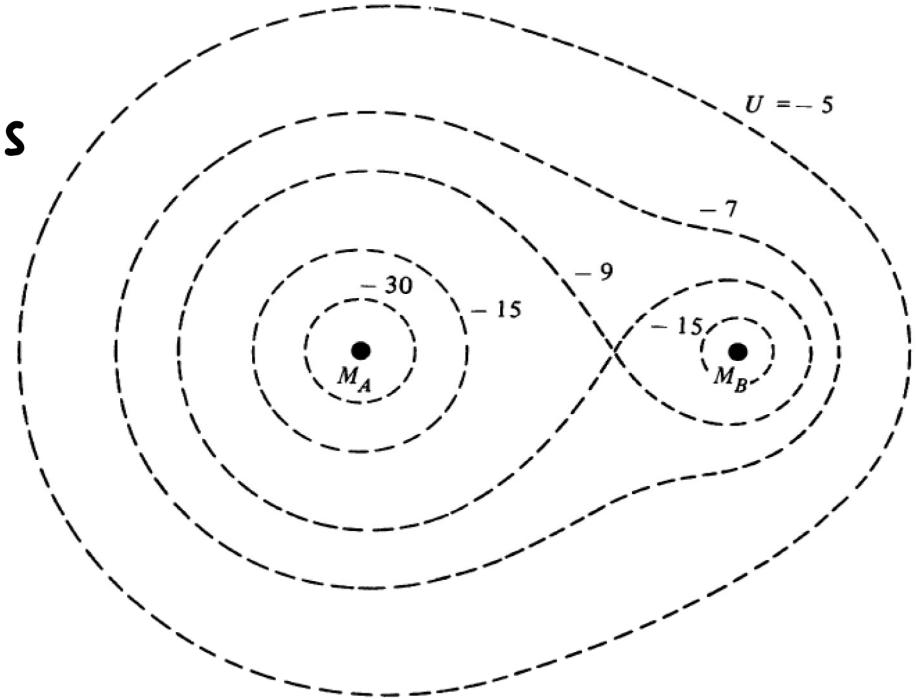
M_a $\qquad$ M_b

(No need to worry about directions!)

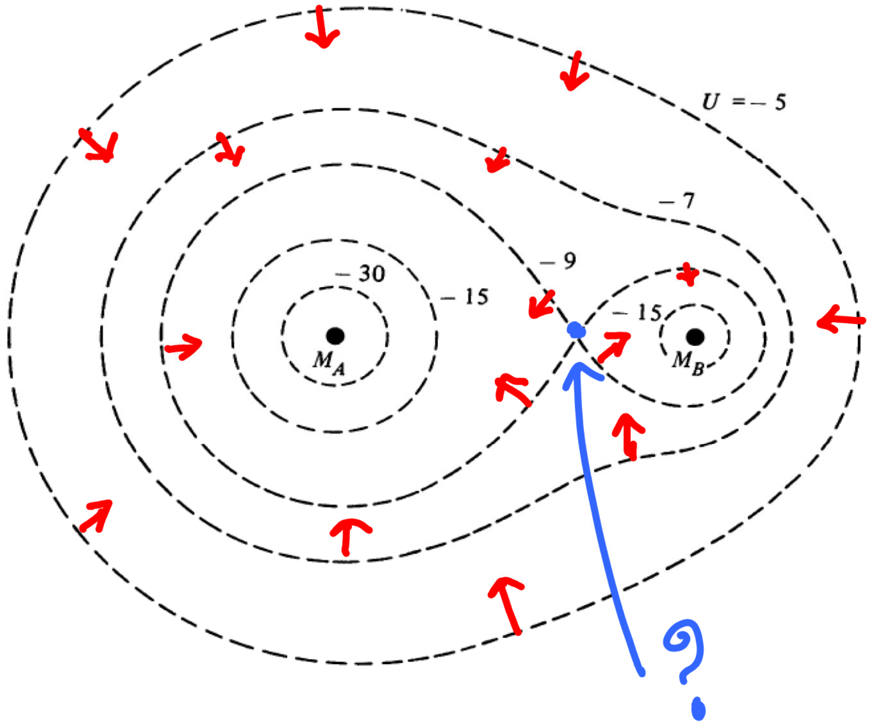
Equipotential contours

i.e. $U = -C$

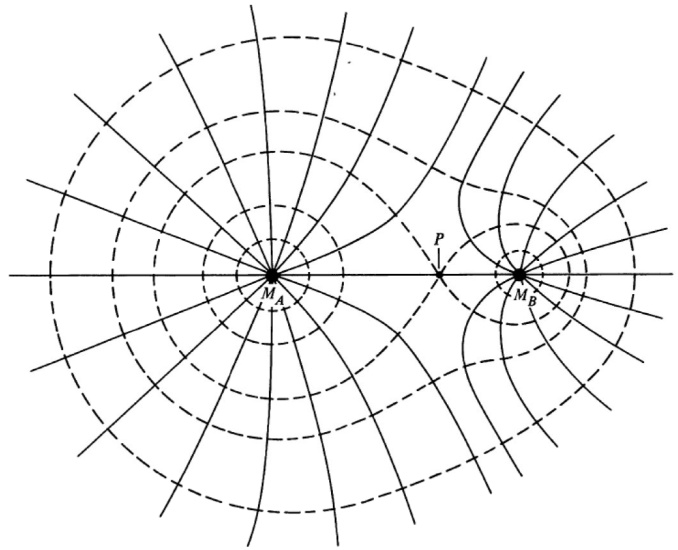
$\bar{F} = -\bar{\nabla}\phi$?



Direction of F



Vector field of $\bar{F}$



Stokes Theorem

We $\oint \bar{F}.d\bar{r} = 0$ if $\bar{F}$ is conservative.

& $\oint \bar{F}.d\bar{r} \neq 0$ if $\bar{F}$ is non conservative.

Stokes theorem:

$$\oint \bar{F}.d\bar{r} = \int_S (\bar{\nabla} \times \bar{F}) \cdot d\bar{A}$$

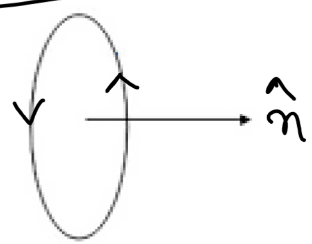
$$\bar{a} = a_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$

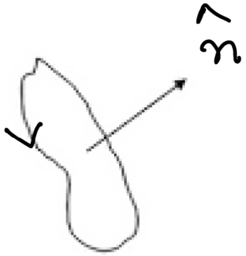
$$\bar{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$

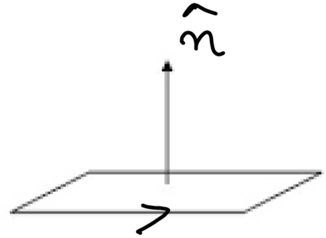
$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Area as a vetor



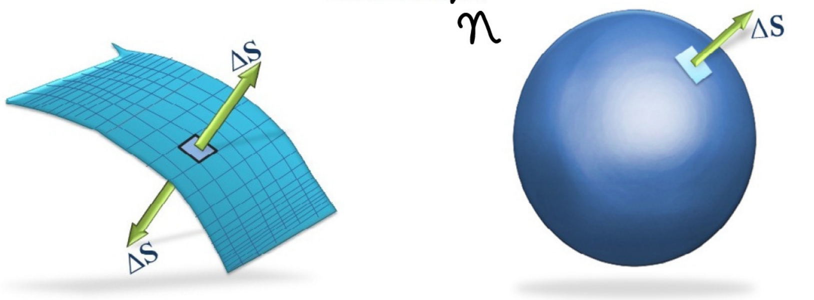




$\hat{n}$

Area Vectors

$\hat{n}$



$$\overline{dA} = dA\,\hat{n}$$

Open Surface

(Any one of the *two normals* can be taken as the *outward normal*)

Closed Surface

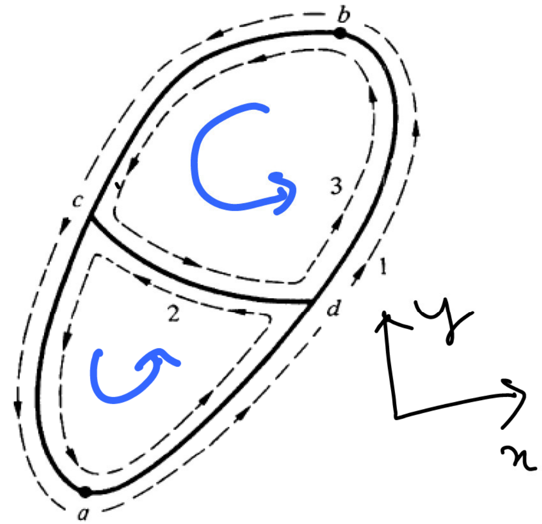
(Only *one* possible *outward normal*)

Proof (for 2D)

Let $\bar{F} = F_x(x,y)\,\hat{i} + F_y(x,y)\,\hat{j}$

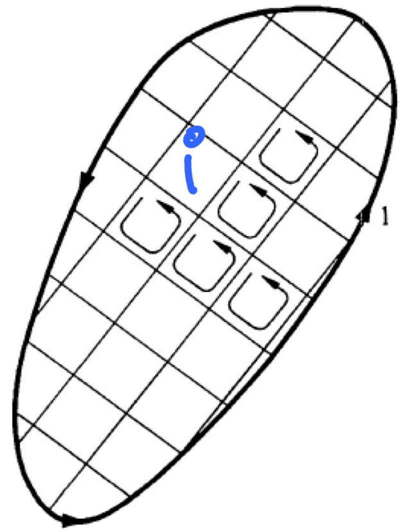
Loop on the xy-plane

is area of the loop, $\bar{A} = A\hat{k}$



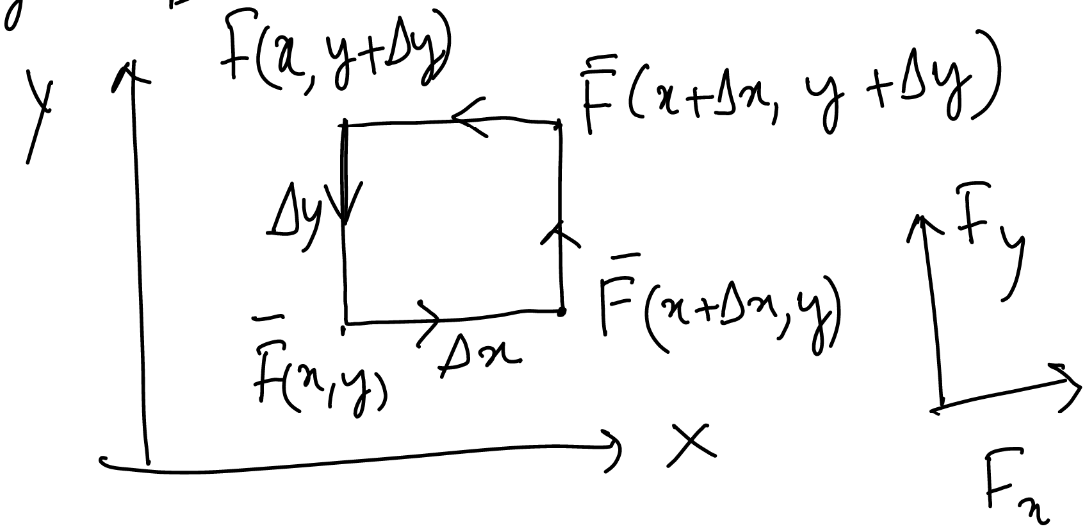
$$\oint_{①} \bar{F}\cdot d\bar{r} = \oint_{②} \bar{F}\cdot d\bar{r} + \oint_{③} \bar{F}\cdot d\bar{r}$$

Note: $\displaystyle\int_d^c \bar{F}\cdot d\bar{r} = -\int_c^d \bar{F}\cdot d\bar{r}$



$$\oint_{①} \bar{F}\cdot d\bar{r} = \sum_{i=1}^{\text{all loops}} \oint_{ⓘ} \bar{F}\cdot d\bar{r}$$

Let's take one tiny loop,



$$\oint_i \bar{F}\cdot d\bar{r} = F_x(x,y)\,\Delta x + \underline{F_y(x+\Delta x, y)\,\Delta y} - \underline{F_x(x, y+\Delta y)\,\Delta x} - F_y(x,y)\,\Delta y$$

$$F_y(x+\Delta x, y)\,\Delta y = \left(F_y(x,y) + \frac{\partial F_y(x,y)}{\partial x}\Delta x\right)\Delta y$$

$$F_x(x, y+\Delta y)\,\Delta x = \left(F_x(x,y) + \frac{\partial F_x(x,y)}{\partial y}\Delta y\right)\Delta x$$

$$\oint_{(i)} \bar{F} \cdot d\bar{r} = \left(\frac{\partial F_y(x,y)}{\partial x} - \frac{\partial F_x(x,y)}{\partial y} \right) \Delta x \Delta y$$

$$\text{So} \quad \oint_{(1)} \bar{F} \cdot d\bar{r} = \sum_i \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)_{(i)} \Delta x \Delta y$$

$|\; dA = dx\,dy$

$$\oint \bar{F} \cdot d\bar{r} = \int_{Surface} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dA$$

$$\text{Now} \quad \bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & 0 \end{vmatrix}$$

Since $\bar{F} = F_x(x,y)\,\hat{i} + F_y(x,y)\,\hat{j}$

$$\bar{\nabla}\times\bar{F} = \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\frac{\partial F_y}{\partial z} = 0$$

$$\frac{\partial F_x}{\partial z} = 0$$

So $\displaystyle\int_{Surf} (\bar{\nabla}\times\bar{F}) \cdot \overline{dA}$

$$\underline{\underline{\overline{dA} = \hat{k}\, dA}}$$

loop in x-y plane!

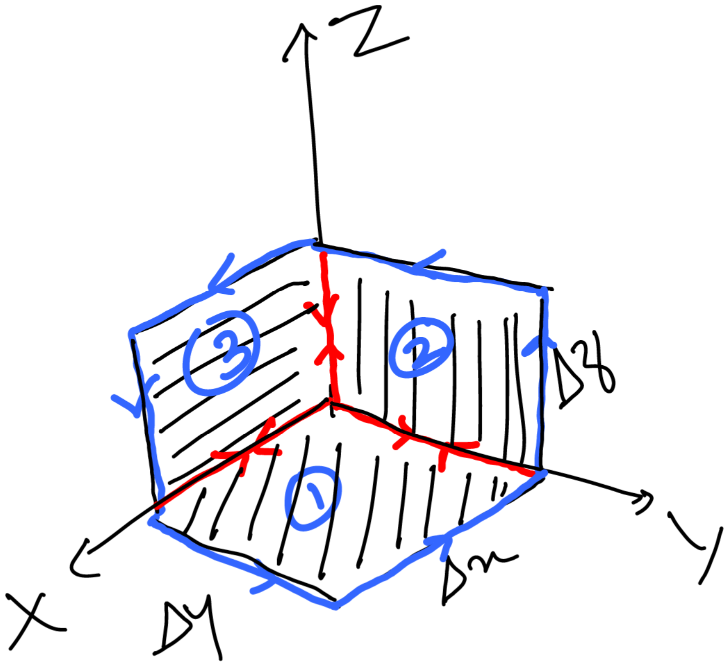
$$= \int_{Surf} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k} \cdot dA\,\hat{k}$$

$$= \underline{\underline{\int_{Surf} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dA = LHS}}$$

We have actually proved the

Green's Theorem (!) – the 2D case of

Stokes Theorem.



By constructing a '3-D' surface like this and finding $\oint \bar{F}\cdot d\bar{r}$ for each of them we get the general result

$$\oint \bar{F}\cdot d\bar{l} = \int_S (\bar{\nabla}\times\bar{F})\cdot d\bar{A}$$

① $\Rightarrow$ $$\bar{F}\cdot d\bar{r} = \left(F_y + \frac{\partial F_y}{\partial x}\Delta x\right)\Delta y - \left(F_x + \frac{\partial F_x}{\partial y}\Delta y\right)\Delta x$$

$$= F_y\Delta y - F_x\Delta x + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x\Delta y$$

② $\Rightarrow$ $$\bar{F}\cdot d\bar{r} = \left(F_z + \frac{\partial F_z}{\partial y}\Delta y\right)\Delta z - \left(F_y + \frac{\partial F_y}{\partial z}\Delta z\right)\Delta y$$

$$= F_z\Delta z - F_y\Delta y + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\Delta y\Delta z$$

③ $\Rightarrow$ $$\bar{F}\cdot d\bar{r} = \left(F_x + \frac{\partial F_x}{\partial z}\Delta z\right)\Delta x - \left(F_z + \frac{\partial F_z}{\partial x}\Delta x\right)\Delta z$$

$$= F_x\Delta x - F_z\Delta z + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\Delta x\Delta y$$

① + ② + ③ $\Rightarrow$ $$\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x\Delta y + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\Delta y\Delta z + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\Delta x\Delta y$$

Now $(\bar{\nabla}\times\bar{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

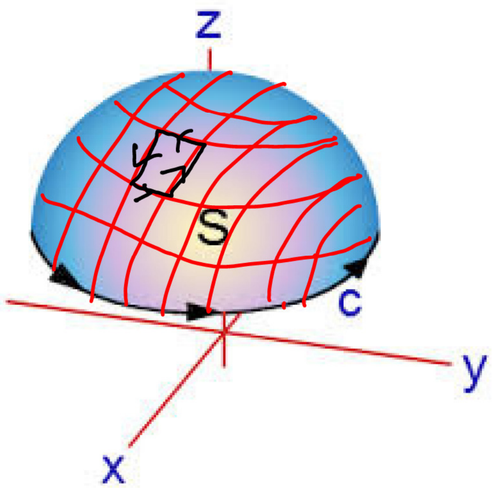
$$= \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

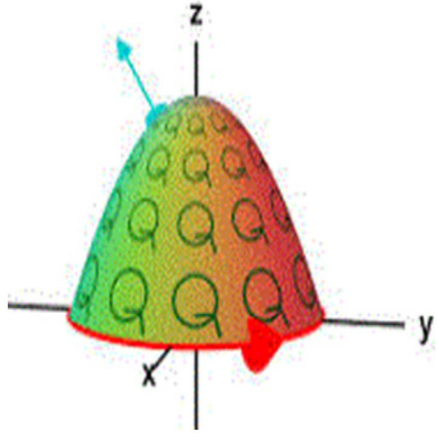
$$(\bar{\nabla}\times\bar{F})\cdot\bar{dA} = (\bar{\nabla}\times\bar{F})\cdot\big(\underset{1}{\Delta x\Delta y\,\hat{k}} + \underset{2}{\Delta y\Delta z\,\hat{i}} + \underset{3}{\Delta x\Delta z\,\hat{j}}\big)$$

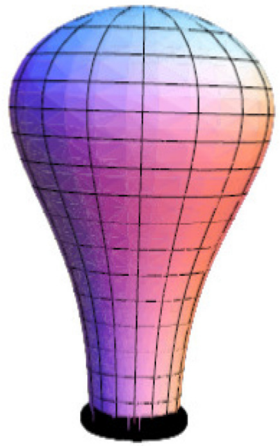
$= L.H.S$

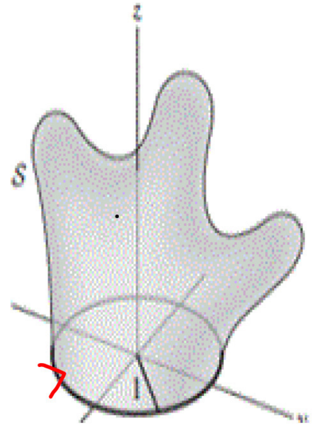
$$\oint \bar{F}\cdot\bar{dl} = \int_S (\bar{\nabla}\times\bar{F})\cdot\bar{dA}$$ **Stokes Theorem**

By the way $\bar{\nabla} \times \bar{F}$ – Curl of F or Curl(F)



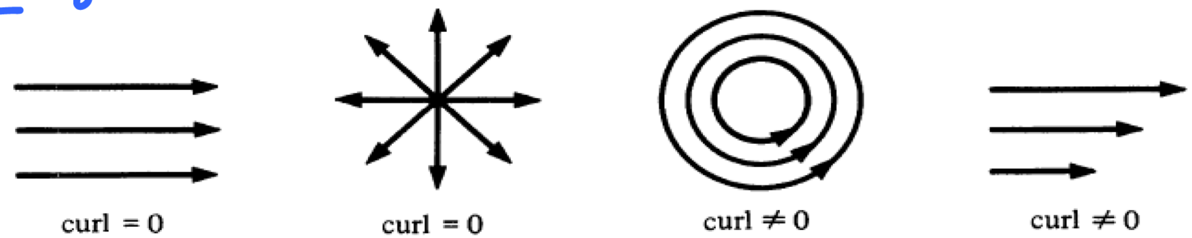






Line integral of $\bar{F}$ over a closed loop in space is equal to area integral of $\bar{\nabla} \times \bar{F}$ over any surface bounded by the loop.

Curl ?



$\bar{\nabla} \times \bar{F}$ involves $\frac{\partial F_x}{\partial y}$, $\frac{\partial F_y}{\partial x}$, i.e

variation of components in <u>other</u> directions.

So, $\bar{\nabla} \times \bar{F}$ is a measure of <u>angular</u> variations in a <u>vector field</u>. (See Kleppner)

Curl ($\bar{\nabla} \times \bar{F}$) is sometimes called <u>"rot"</u>!

How is this useful for us?

For conservative forces,

$$\oint \bar{F} \cdot d\bar{r} = 0$$ for any loop!

i.e. $$\int_{Surf} (\bar{\nabla} \times \bar{F}) \cdot d\bar{A} = 0$$ over any possible bounding surfaces!

Hence $\bar{\nabla} \times \bar{F} = 0$ everywhere in space

If $\bar{F}$ is conservative!