PH101
Lecture -7
23rd August 2017

Work Energy Theorem
Conservation of Mechanical Energy

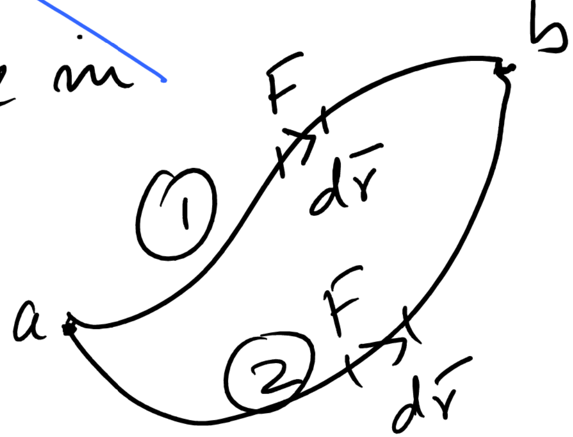
Work done by a force F

$$W_{ba} = \oint_a^b \bar{F} \cdot d\bar{r}$$

C- implies a curve /path

$\oint$ – called line integral !

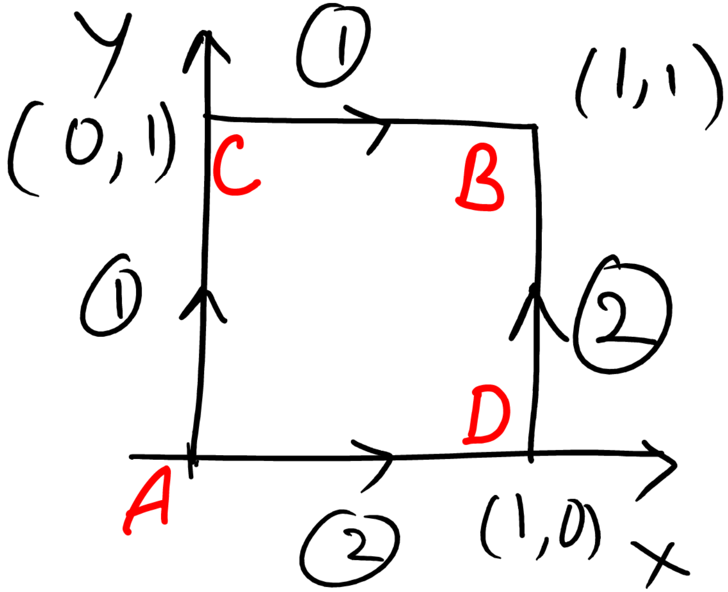
In general W_{ba} depend on the path traced by the particle in moving from a ~~to~~ b.



$$W_{ba}^{①} \neq W_{ba}^{②}$$

Example

Let $\bar{F} = (xy\,\hat{i} + y^2\,\hat{j})$



$$W_{BA} = \oint_A^B \bar{F}\cdot d\bar{r}$$

$$= \oint_A^B (xy\,\hat{i} + y^2\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

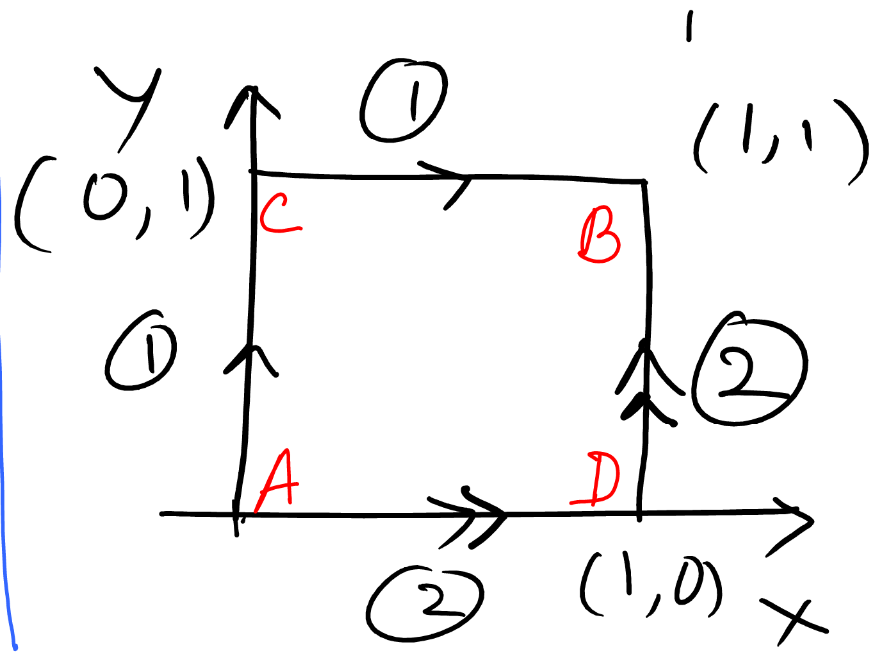
$$= \oint_A^B xy\,dx + \oint_A^B y^2\,dy$$

Path ①,

$$W_{BA}^{①} = W_{CA} + W_{CB}$$

Along AC, $x=0$

Along CB, $y=1$ & $dy=0$



$$W_{CA} = \int_0^1 y^2 dy = \frac{1}{3}$$

$$W_{BC} = \int_0^1 x\,dx = 1/2$$

$$W_{BA}^{①} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$$W_{BA}^{②} = W_{DA} + W_{BD}$$

Along AD, $y=0$ so $W_{DA}=0$

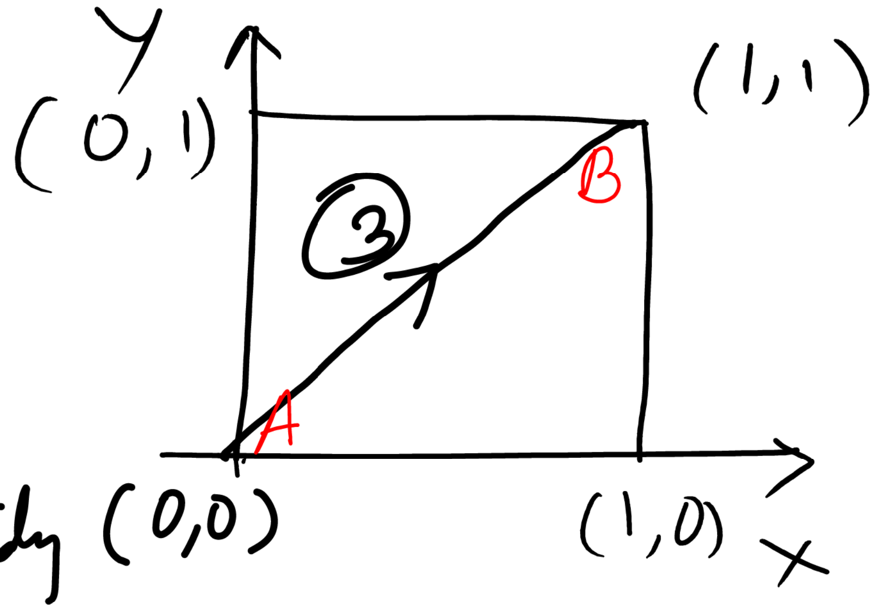
Along DB, $x=1$ & $dx=0$

$$W_{BD} = \int_0^1 y^2\,dy = \underline{\underline{1/3}}$$

So $W_{BA}^{②} = 1/3 \neq W_{BA}^{①} = \frac{5}{6}$

③ $y = x$

$dx = dy$



$$W^{③}_{BA} = \int_0^1 x^2\,dx + \int_0^1 y^2\,dy$$

$$= 1/3 + 1/3 = \underline{\frac{2}{3}} \quad \left(\neq W^{①}_{BA} \neq W^{②}_{BA}\right)$$

So the <u>Path Matters</u>!

(Not only the destination!)

Work - Energy theorem

Let $\bar{F}$ be the total (external) force

$$W_{ba} = \oint_a^b \bar{F}\cdot d\bar{r} = \oint_a^b m\left(\frac{d\bar{v}}{dt}\cdot\bar{v}\right)dt$$

Now,

$$\frac{d(v^2)}{dt} = \frac{d(\bar{v}\cdot\bar{v})}{dt} = \bar{v}\cdot\frac{d\bar{v}}{dt} + \frac{d\bar{v}}{dt}\cdot\bar{v}$$

$$= 2\bar{v}\cdot\frac{d\bar{v}}{dt} = 2\,\frac{d\bar{v}}{dt}\cdot\bar{v}$$

$$W_{ba} = \oint_a^b m\,\frac{1}{2}\frac{d(v^2)}{dt}\,dt = \frac{m}{2}v^2\Big|_a^b$$

ie $$W_{ba} = \frac{m}{2} v_b^2 - \frac{m}{2} v_a^2 = K_b - K_a$$

K - kinetic energy

ie Work done (by the ~~total~~ force) in moving a particle from a to b equal's change in K.E at b & a.

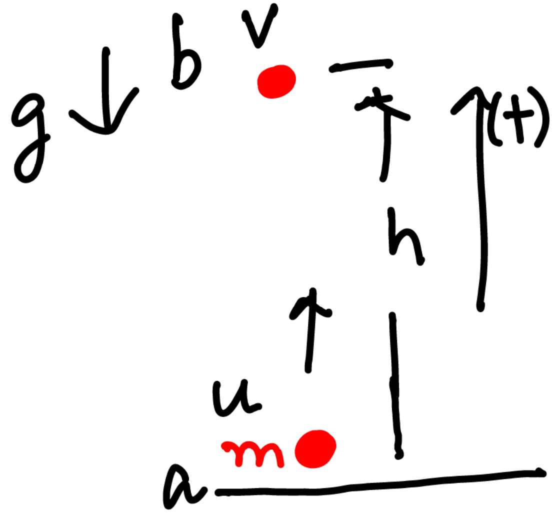
This is valid for all types & kinds of forces!

Example

$$W_{ba} = \oint_a^b \bar{F} \cdot d\bar{r}$$

$$= \int_a^b -mg \, dr = -mg(b-a)$$

$$= -mgh$$

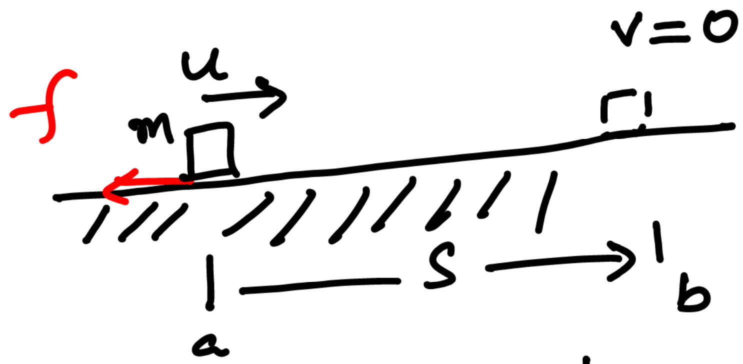


$$v^2 = u^2 - 2gh$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = -mgh$$

$$\underline{\underline{K_b - K_a = W_{ba}}}$$

Example 2



Find the coefficient of friction, μ

$$W_{ba} = -fS = \underbrace{\frac{1}{2}mv^2}_{(=0)} - \frac{1}{2}mu^2$$

ie, $f = \mu N = \mu mg = \frac{1}{2}\frac{mu^2}{S}$

or $$\mu = \frac{u^2}{2gS}$$

Work energy theorem applies to system of particles as well.

$$W_{ba} = \oint_a^b \bar{F}^{ext} \cdot d\bar{r} = \frac{1}{2} M \dot{x}^2(b) - \frac{1}{2} M \dot{x}^2(a)$$

x - center of mass & $\frac{1}{2} M \dot{x}^2$ - translational K.E

$\bar{F}^{ext}$ - total external force on the system.

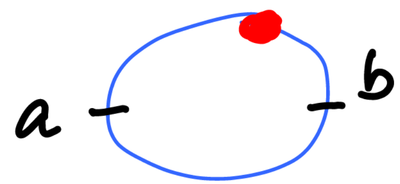
Use?

Path must be known appriori!

Constrained motion are fine!

1-D cases & Constrained motion are fine!

[Such as a bead on a circular ring



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Central forces

If $\bar{F} = f(r)\hat{r}$ (eg. gravitational, Coulomb)

$$W_{ba} = \oint_a^b \bar{F}\cdot d\bar{r} = \oint_a^b f(r)\,\hat{r}\cdot(\hat{r}\,dr + r\,d\theta\,\hat{\theta})$$

$$= \oint_a^b f(r)\,dr$$

(1-D integral (effectively) has no choice of path. So depend only on "a" & "b")

ie $W_{ba} = \int_a^b f(r)\,dr$ (Path Independent)

Forces whose work done are path independent are called "Conservative" forces!

ie for a conservative force, $\bar{F}$

$$W_{ba} = \oint_a^b \bar{F}.d\bar{r} = \int_a^b \bar{F}.d\bar{r}$$

Also,

$$\int_a^a \bar{F}.d\bar{r} = \oint \bar{F}.d\bar{r} = 0$$

$\oint$ – signifies integration over any closed loop/path.

Examples of Conservative forces

$\bar{F} = mg\hat{k}$ (gravity)

$$\bar{F} = -\frac{GMm}{r^2}\hat{r} \quad \text{(Gravitational)}$$

$$\bar{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r} \quad \text{(Coulomb)}$$

$F = -k(x-l)\hat{i}$ (spring force)

Counter examples: Friction

viscous drag

For Conservative forces

We can define potential energy

as, $W_{ba} = \oint_a^b \bar{F} \cdot d\bar{r} = \int_a^b \bar{F} \cdot d\bar{r}$

(No path dependence)

$$W_{ba} = -U_b + U_a$$

or, $U_b - U_a = -\int_a^b \bar{F} \cdot d\bar{r}$

U_b - the Potential energy at $\underline{\underline{b}}$ w.r.t. $\underline{\underline{a}}$

Now

$$U_b - U_a = -\int_a^b \bar{F} \cdot d\bar{r} = -W_{ba} = -(K_b - K_a)$$

$$\boxed{U_b + K_b = U_a + K_a}$$

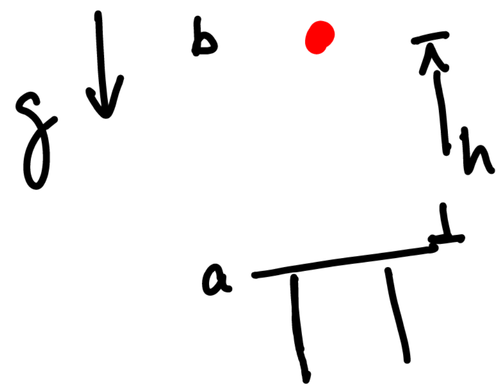
Conservation of total mechanical energy under the action of conservative force.

Now, $U_b - U_a = -\int_a^b \bar{F} \cdot d\bar{r}$ (we can in principle talk only above change in U !)

But with a convenient choice of reference we talk of P.E at a location, $\bar{r}$.

Examples

I) $$U(b) = mgh$$
taking $U(a) = 0$ (ref)



II) $$\bar{F} = -\frac{GMm}{r^2}\hat{r}$$

$$U(\bar{r}_b) - U(\bar{r}_a) = +\int_a^b \frac{GMm}{r^2}\,dr$$

$$= -GMm\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

if $r_a = \infty$ & $r_b = r$

$$U(r) - U(\infty) = -\frac{GMm}{r}$$

taking $U(\infty) = 0$ we define $\underline{\underline{U(r) = -\frac{GMm}{r}}}$

Back to P.E

$$U_b - U_a = -\int_a^b \bar{F} \cdot d\bar{r}$$

or

$$dU = -\bar{F} \cdot d\bar{r}$$

ie

$$F = -\frac{dU}{dx} \quad (\text{1-D case})$$

What about 3-D case.

$$\cancel{\bar{F} = -\frac{dU}{d\bar{r}}} \, ?$$

Let's note that $\quad U = U(x, y, z)$

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

i.e. Infinitesimal change in u correspond to Sum of changes in u due x $\left(\frac{\partial u}{\partial x}dx\right)$, y & z.

Now,

$$du = \underbrace{\left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right)}_{-\bar{F}} \cdot \underbrace{(dx\hat{i} + dy\hat{j} + dz\hat{k})}_{d\bar{r}}$$

i.e.

$$\bar{F} = -\left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right)$$

Notation: $\bar{\nabla}(\,) = \hat{i}\frac{\partial(\,)}{\partial x} + \hat{j}\frac{\partial(\,)}{\partial y} + k\frac{\partial(\,)}{\partial z}$

A vector operator called "del".

(nabla: inverted delta: "Δ")

(i $\phi(x, y, z)$

$$\bar{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$
)

$$\boxed{\bar{F} = -\bar{\nabla} U}$$

$\bar{F}$ = -ve Gradient of U

Examples

1) Let $U = \frac{k}{2}(x^2 + y^2 + z^2)$

$$\bar{F} = -\nabla U = -k(x\hat{i} + y\hat{j} + z\hat{k}) \quad (= -k\bar{r})$$

2) $U = mgz$

$$\bar{F} = -mg\hat{k}$$

3)

$$U = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{(x^2+y^2+z^2)}}$$

$$\frac{\partial U}{\partial x} = -GMm \times -\frac{1}{2(x^2+y^2+z^2)^{3/2}} \times 2x$$

$$= \frac{GMm\,x}{(r^2)^{3/2}} = \frac{GMm\,x}{r^3}$$

llly

$$\frac{\partial U}{\partial y} = \frac{GMm\,y}{r^3} \quad \& \quad \frac{\partial U}{\partial z} = \frac{GMm\,z}{r^3}$$

$$\therefore \quad \vec{F} = -\left(\hat{i}\,\frac{GMm\,x}{r^3} + \hat{j}\,\frac{GMm}{r^3}\,y + \hat{k}\,\frac{GMm\,z}{r^3}\right)$$

$$= -\frac{GMm}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

i.e. $\bar{F} = -\frac{GMm}{r^2}\left(\frac{\bar{r}}{r}\right) = -\frac{GMm}{r^2}\hat{r}$

slightly easier:

$$\frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x} = -\frac{GMm}{r^2}\frac{\partial r}{\partial x}$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r}$$

i.e. $$\frac{\partial U}{\partial x} = -\frac{GMm}{r^2}\frac{x}{r}$$

Let's Note.

$$\bar{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

i.e. $F_x = -\frac{\partial U}{\partial x}$ & $F_y = -\frac{\partial U}{\partial y}$ & $F_z = -\frac{\partial U}{\partial z}$