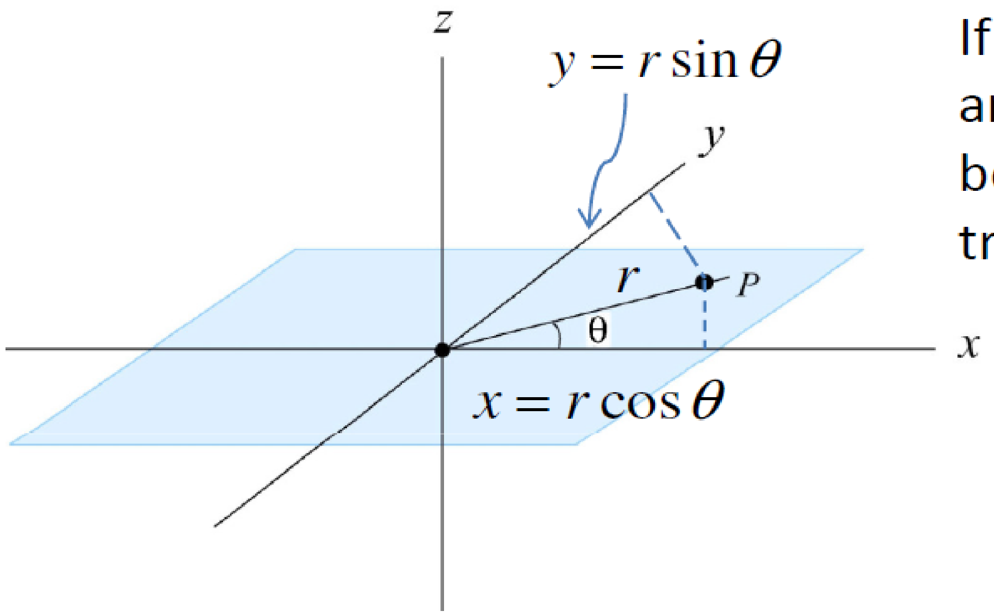
# Polar Coordinates

# Polar and Cartesian coordinates:




If polar coordinates $(r, \theta)$ of a point in the plane are given, the Cartesian coordinates $(x, y)$ can be determined from the coordinate transformations

$$x = r\cos\theta \qquad r = +(x^2 + y^2)^{1/2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}(y/x)$$

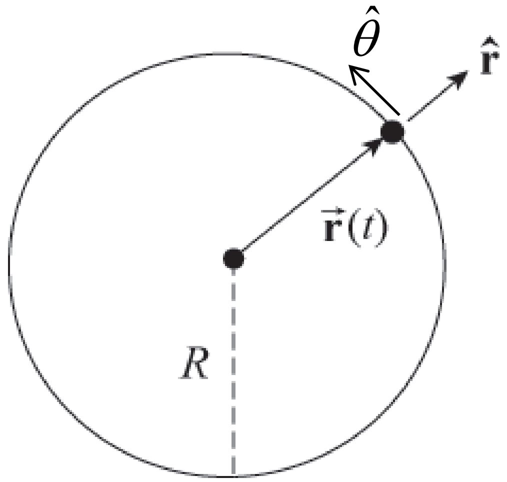
Note: $r \geq 0$ so take the positive square root only.

Since $\tan\theta = \tan(\theta + \pi)$

For $0 \leq \theta \leq \pi/2$ $x \geq 0$ and $y \geq 0$

For $(-x, -y)$ take $\theta + \pi$

# Unit Vectors in Polar coordinates




The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

In Cartesian coordinate: $\vec{r} = x\hat{i} + y\hat{j}$

By coordinate transformations:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Therefore: $\vec{r} = r\cos\theta\hat{i} + r\sin\theta\,\hat{j}$

The unit vectors are defined as :

$$\hat{r} = \frac{\partial\vec{r}/\partial r}{\left|\partial\vec{r}/\partial r\right|} = \cos\theta\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{r}\times\hat{\theta} = \hat{k}$$

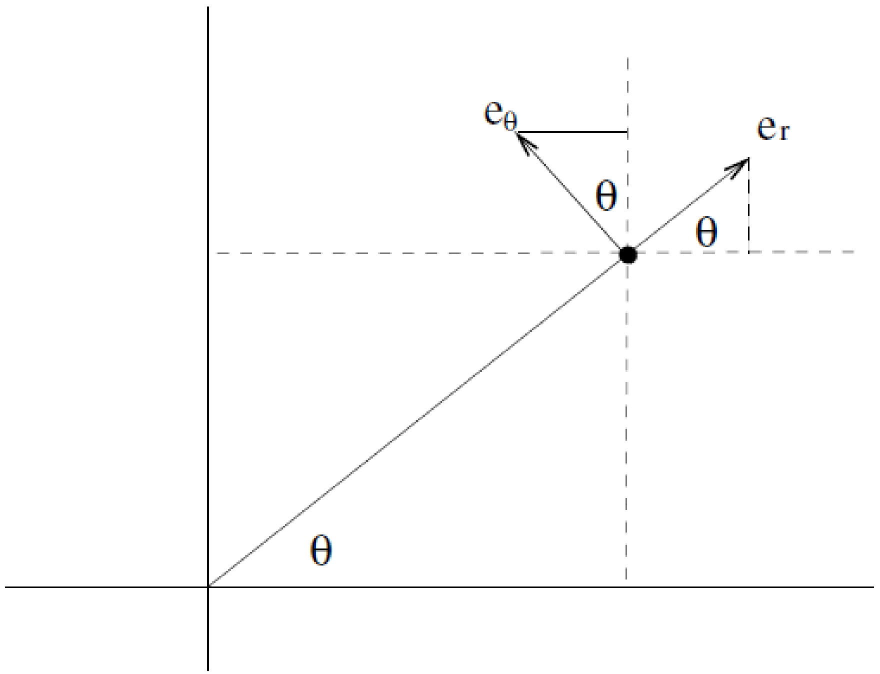
$$\hat{i} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$
$$\hat{j} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{\left|\partial\vec{r}/\partial\theta\right|} = -\sin\theta\hat{i} + \cos\theta\,\hat{j}$$

# Unit Vectors in Polar coordinates

Define at each point, a set of two unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$ as shown in the figure.

$$\hat{\mathbf{r}} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$
$$\hat{\theta} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$




**Unit vectors only depend on θ**

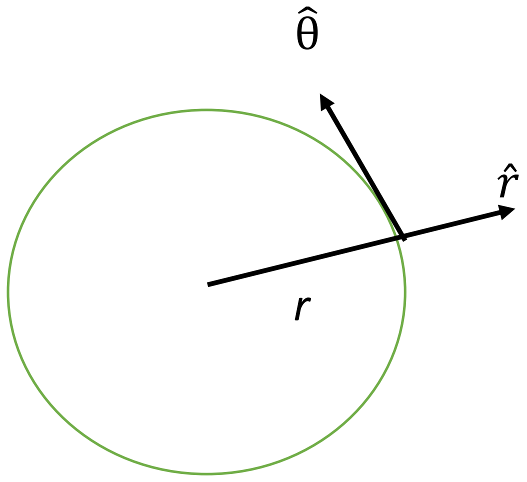
$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\theta}$$
$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{\mathbf{r}}$$

unit vectors are functions of the polar coordinates

# Motion in Plane Polar Coordinates

# Velocity and acceleration in polar coordinates

# Velocity in polar coordinate:




The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

And the unit vectors are: $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ & $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$

Since the unit vectors are not constant and changes with time, they should have finite time derivatives:

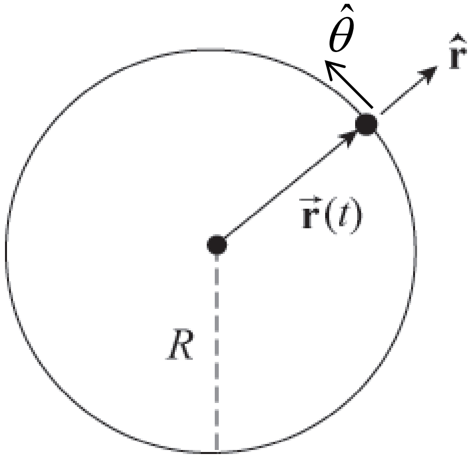
$$\dot{\hat{r}} = \dot{\theta}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) = \dot{\theta}\hat{\theta} \quad \text{and} \quad \dot{\hat{\theta}} = \dot{\theta}\left(-\cos\theta\hat{i} - \sin\theta\hat{j}\right) = -\dot{\theta}\hat{r}$$

Therefore the velocity is given by: $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

**Radial velocity + tangential velocity**

**In Cartesian coordinates** $= \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j}$

# Example-1: Uniform Circular Motion




$$\vec{\mathbf{r}}(t) = R\hat{\mathbf{r}}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Since $\dot{r} = \frac{dR}{dt} = 0$ and $\omega = \frac{d\theta}{dt} = \dot{\theta}$

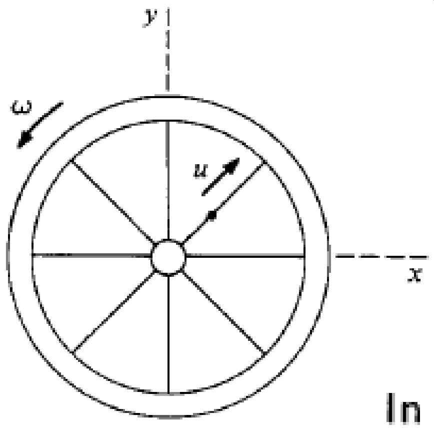
$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = R\omega\hat{\boldsymbol{\theta}}(t)$$

Since $\vec{v}$ is along $\hat{\theta}$ it must be perpendicular to the radius vector $\vec{r}$ and it can be shown easily

$R^2 = \vec{\mathbf{r}} \cdot \vec{\mathbf{r}}$ $\qquad$ $\frac{d}{dt}R^2 = \frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = 0$, $\qquad$ $\vec{\mathbf{r}} \perp \vec{\mathbf{v}}$

# Why polar coordinates?

## Example-2: Velocity of a Bead on a Spoke




A bead moves along the spoke of a wheel at constant speed $u$ meters per second. The wheel rotates with uniform angular velocity $\dot{\theta} = \omega$ radians per second about an axis fixed in space. At $t = 0$ the spoke is along the $x$ axis, and the bead is at the origin. Find the velocity at time $t$

In polar coordinates : $r = ut, \dot{r} = u, \dot{\theta} = \omega.$ Hence

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

To specify the velocity completely, we need to know the direction of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. This is obtained from $\mathbf{r} = (r,\theta) = (ut,\omega t)$.

In cartesian coordinates. : $v_x = v_r \cos\theta - v_\theta \sin\theta \qquad v_y = v_r \sin\theta + v_\theta \cos\theta.$

Since $v_r = u$, $v_\theta = r\omega = ut\omega$, $\theta = \omega t$,

$$\mathbf{v} = (u \cos \omega t - ut\omega \sin \omega t)\hat{\mathbf{i}} + (u \sin \omega t + ut\omega \cos \omega t)\hat{\mathbf{j}}$$

Note how much simpler the result is in plane polar coordinates.

**Symmetry is important.**

# Acceleration in Polar coordinate:

$$\mathbf{a} = \frac{d}{dt}\mathbf{v}$$
$$= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})$$
$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}.$$

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta},\ \dot{\hat{\theta}} = -\dot{\theta}\hat{r}$$

$$\mathbf{a} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}}$$
$$= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

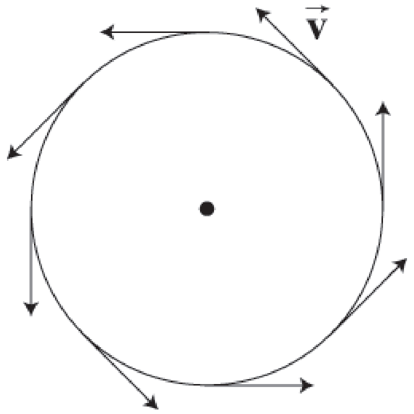
The term $\ddot{r}\hat{\mathbf{r}}$ is a linear acceleration in the radial direction due to change in radial speed. Similarly, $r\ddot{\theta}\hat{\boldsymbol{\theta}}$ is a linear acceleration in the tangential direction due to change in the magnitude of the angular velocity.

The term $-r\dot{\theta}^2\hat{\mathbf{r}}$ is the centripetal acceleration

Finally, the Coriolis acceleration $2\dot{r}\dot{\theta}\hat{\theta}$

Usually, Coriolis force appears as a fictitious force in a rotating coordinate system. However, the Coriolis acceleration we are discussing here is a real acceleration and which is present when *r* and $\theta$ both change with time.

# Example-1: Circular motion




$$\vec{a} = a_r\hat{r} + a_\theta\hat{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

For a circular motion, $r = R,$ the radius of the circle.

Hence, $\dot{r} = \ddot{r} = 0$

So, $a_\theta = R\ddot{\theta}$ and $a_r = -R\dot{\theta}^2$

For uniform circular motion, $\dot{\theta} = \omega =$ constant. Hence, $a_\theta = R\dfrac{d\omega}{dt} = 0$

For non-uniform circular motion, $\omega$ is function of time. Hence, $a_\theta = R\dfrac{d\omega}{dt} = R\alpha,$

where $\alpha = \dfrac{d\omega}{dt}$ is the angular acceleration.

However, the radial acceleration is always $a_r = -R\dot{\theta}^2 = -R\omega^2$

Therefore, an object traveling in a circular orbit with a constant speed is always accelerating towards the center. Though the magnitude of the velocity is a constant, the direction of it is constantly varying. Because the velocity changes direction, the object has a nonzero acceleration.

# Example-2: **Acceleration of a Bead on a Spoke**

A bead moves outward with constant speed $u$ along the spoke of a wheel. It starts from the center at $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where $\omega$ is a constant. Find the velocity and acceleration.

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

We are given that $\dot{r} = u$ and $\dot{\theta} = \omega$. The radial position is given by $r = ut$, and we have

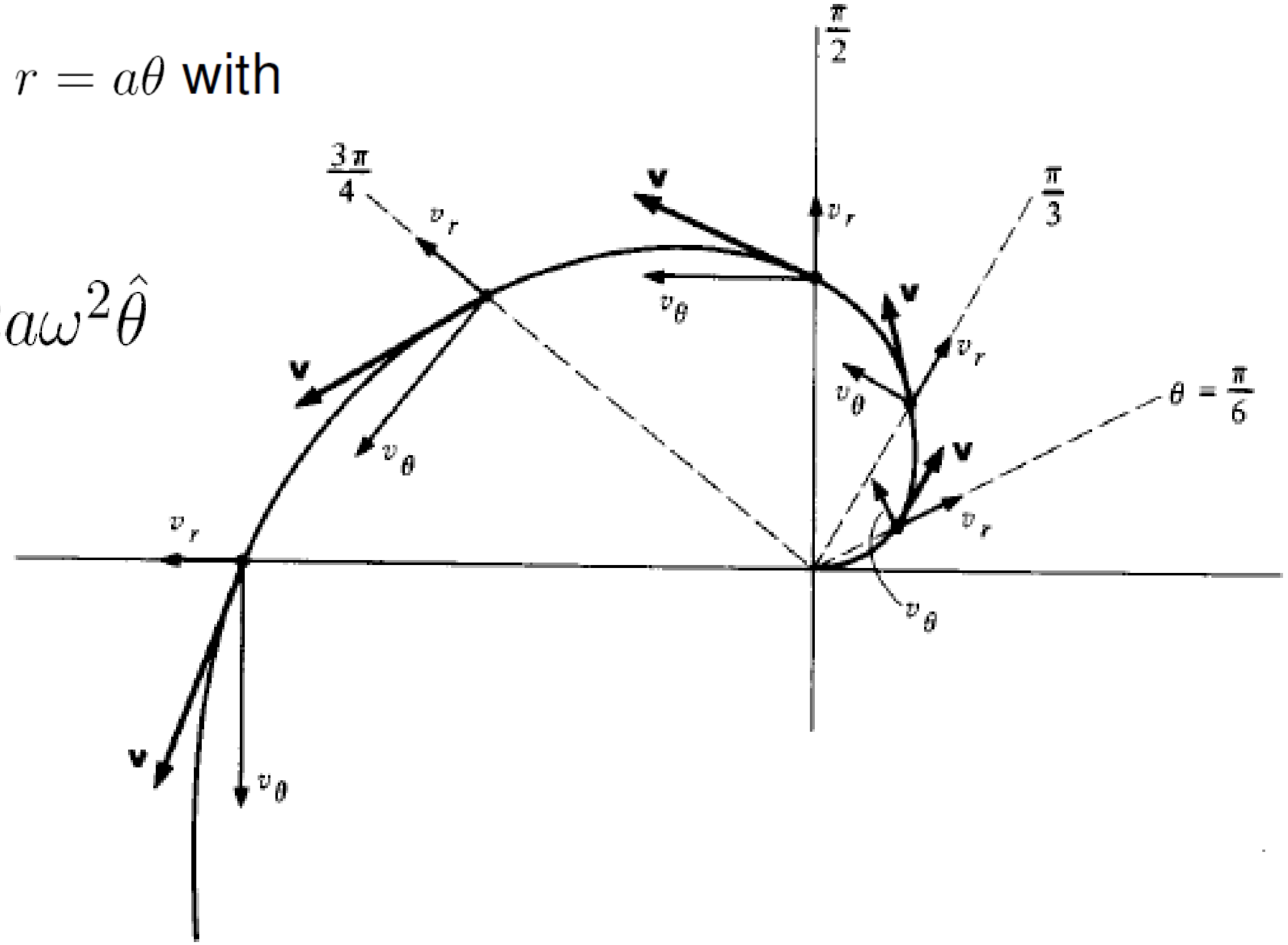
$$\mathbf{v} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

The acceleration is

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= -ut\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\boldsymbol{\theta}}.\end{aligned}$$

Consider a particle moving on a spiral given by $r = a\theta$ with a uniform angular speed $\omega$. Then $\dot{r} = a\dot{\theta} = a\omega$.

$$\mathbf{v} = a\omega\hat{\mathbf{r}} + a\omega^2 t\hat{\theta} \text{ and } \mathbf{a} = -a\omega^3 t\hat{\mathbf{r}} + 2a\omega^2\hat{\theta}$$




The velocity is shown in the sketch for several different positions of the wheel. Note that the radial velocity is constant. The tangential acceleration is also constant—can you visualize this?

**Though the magnitude of radial velocity is constant there is a radial acceleration.**

# Motion: Kinematics in 1D

The motion of the particle is described specifying the position as a function of time, say, $x\left(t\right)$.

The instantaneous velocity is defined as

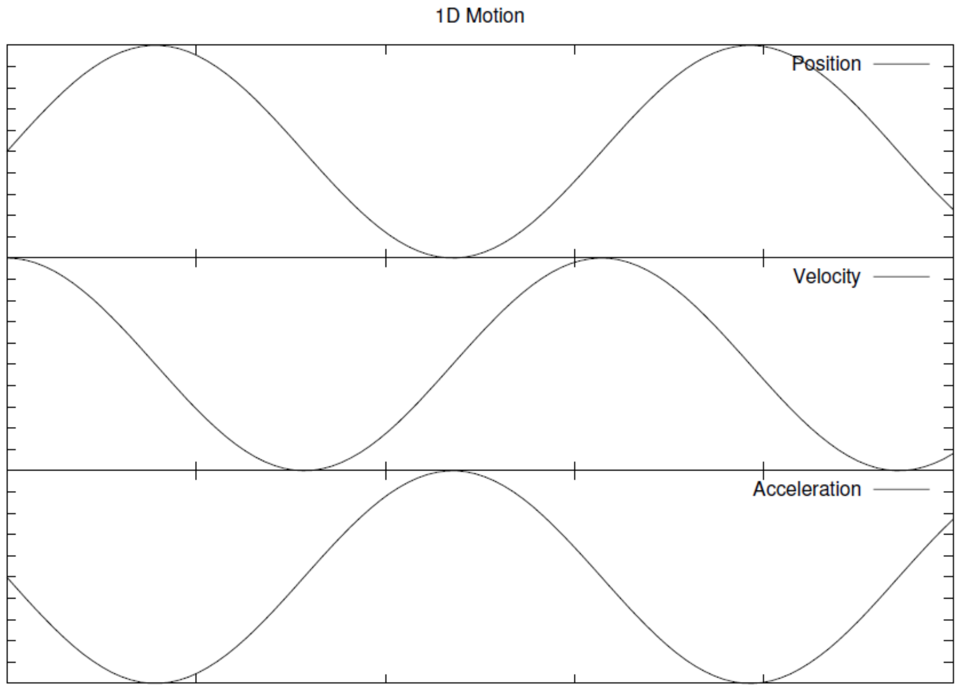
(1) $$v\left(t\right)=\frac{dx}{dt}$$

and instantaneous acceleration, as

(2) $$a\left(t\right)=\frac{dv}{dt}=\frac{d^2x}{dt^2}$$

**Example**

If $x\left(t\right)=\sin(t)$, then $v\left(t\right)=\cos(t)$ and $a\left(t\right)=-sin(t)$.



Usually the $x(t)$ is not known in advance!

But the acceleration $a(t)$ is known and at some given time, say $t_0$, position $x(t_0)$ and velocity $v(t_0)$ are known.

The formal solution to this problem is

$$\begin{aligned} v(t) &= v(t_0) + \int_{t_0}^{t} a(t')\,dt' \\ x(t) &= x(t_0) + \int_{t_0}^{t} v(t')\,dt' \end{aligned}$$

Let the acceleration of a particle be $a_0$, a constant at all times. If, at $t = 0$ velocity of the particle is $v_0$, then

$$\begin{aligned} v(t) &= v_0 + \int_0^t a_0 dt \\ &= v_0 + a_0 t \end{aligned}$$

And if the position at $t = 0$ is $x_0$,

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2$$

More complex situations may arise, where an acceleration is specified as a function of position, velocity and time. $a\left(x,\dot{x},t\right)$ . In this case, we need to solve a differential equation

$$\frac{d^2x}{dt^2} = a\left(x,\dot{x},t\right)$$

which may or may not be simple.

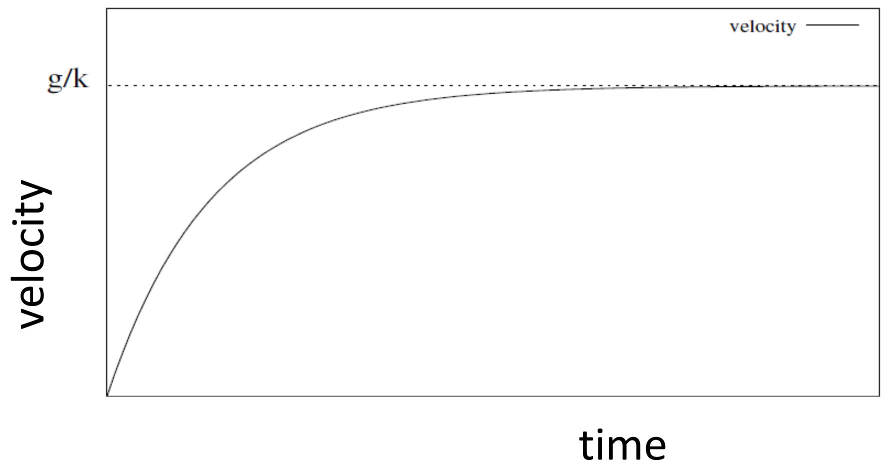
**Example**

Suppose a ball is falling under gravity in air, resistance of which is proportional to the velocity of the ball.

$$a\left(\dot{y}\right) = -g - k\dot{y}$$

If the ball was just dropped, velocity of the ball after time then

$$v(t) = -\frac{g}{k}\left(1 - e^{-kt}\right)$$



# Kinematics in 2D

The instantaneous velocity vector is defined as

$$\begin{aligned}\mathbf{v}(t) &= \frac{d}{dt}\mathbf{r} \\ &= \lim_{dt\to 0}\frac{\mathbf{r}(t+dt)-\mathbf{r}(t)}{dt} \\ &= \lim_{dt\to 0}\frac{x(t+dt)-x(t)}{dt}\mathbf{i}+\lim_{dt\to 0}\frac{y(t+dt)-y(t)}{dt}\mathbf{j} \\ &= \frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}\end{aligned}$$

The instantaneous acceleration is given by:

$$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i}+\frac{d^2y}{dt^2}\mathbf{j}$$

# Kinematics in 2D

In this case, we have to solve two differential equations

$$\frac{d^2x}{dt^2} = a_x$$
$$\frac{d^2y}{dt^2} = a_y$$

**Example**

A ball is projected at an angle $\theta$ with a speed $u$. The net acceleration is in downward direction. Then $a_x = 0$ and $a_y = -g$. The equations are

$$\frac{d^2x}{dt^2} = 0$$
$$\frac{d^2y}{dt^2} = -g$$

# Charge particle in a magnetic field

A particle has a velocity v in XY plane. Magnetic field is in z direction The acceleration is given by $\frac{q}{m}\mathbf{v} \times \mathbf{B}$

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{qB}{m}v_y \\ \frac{d^2y}{dt^2} &= -\frac{qB}{m}v_x \end{aligned}$$

Solution is rather simple, that is circular motion in xy plane.

# Equation of motion of a chain

A uniform chain of length ‘a’ is placed on a horizontal frictionless table, so that a length ‘b’ of the chain dangles over the side. How long will it take for the chain to slide off the table?