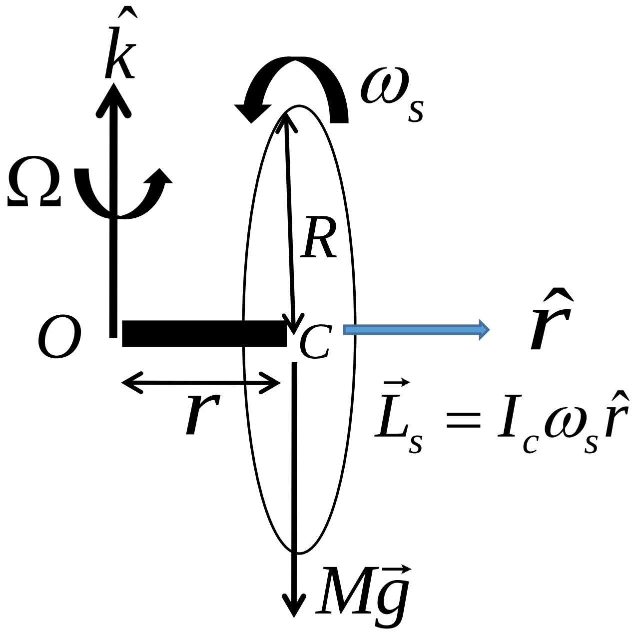
Rigid Body Dynamics (Contd)

(Rotation + Revolution)

Precession of bicycle wheel



$$\vec{\tau}_O = Mgr\hat{\theta}$$

$$\vec{L}_s = I_C \omega_s \hat{r}, \;\; \vec{L}_O = I_O \Omega \hat{k}$$

$$\vec{L} = \vec{L}_O + \vec{L}_s$$

$$\vec{\tau}_O = \frac{d\vec{L}}{dt} = \frac{d\vec{L}_s}{dt} = \left(\frac{d\vec{L}_s}{dt} \right)_{\text{body}} + \;\; \Omega \hat{k} \times \vec{L}_s$$

$$= \Omega \hat{k} \times \vec{L}_s = I_C \omega_s \Omega \hat{\theta}$$

$$I_C \omega_s \Omega = Mgr, \qquad I_C = MR^2$$

$$\Omega = \frac{gr}{R^2 \omega_s}, \quad \text{angular frequency of precession.}$$

$\vec{L}_s$ will follow $\vec{\tau}_O$.

Gyroscope Motion

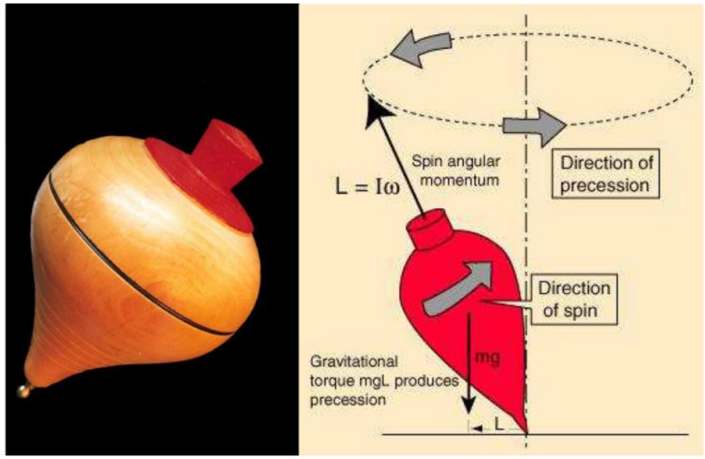
Equation of motion

In the body frame: $\vec{L} = \overleftrightarrow{I}\vec{\omega}$

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{fix}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{L}, \qquad \text{Therefore,} \qquad \vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{L}$$

If the body frame coincides with the principal axes of the system, the moment of inertia tensor will be given by a diagonal matrix.

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \text{ Hence, } \vec{L} = \overleftrightarrow{I}\vec{\omega} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix},$$



All points of the top are restricted to a spherical surface.

$$\text{Then, } \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} = \begin{pmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{pmatrix} \text{ and } \vec{\omega}\times\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix} = \begin{pmatrix} (I_3 - I_2)\omega_3\omega_2 \\ (I_1 - I_3)\omega_1\omega_3 \\ (I_2 - I_1)\omega_2\omega_1 \end{pmatrix}$$

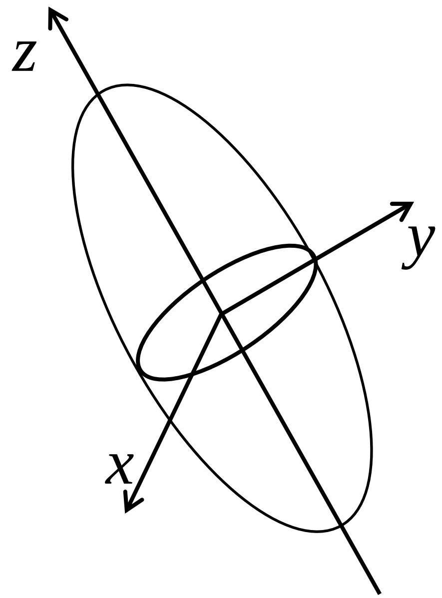
Hence,

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2$$
$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3$$
$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1$$

Euler's equations are in body frame.

Wolfgang Pauli and Niels Bohr stare in wonder at a spinning top.

The Free Symmetric Top:



The symmetric top is an object with $I_1 = I_2 \neq I_3$.

$$\left(L_1, L_2, L_3\right) = \left(I_1\omega_1, I_1\omega_2, I_3\omega_3\right) : \vec{L} \text{ and } \vec{\omega} \text{ are not parallel.}$$

Euler's equations become

$$\begin{aligned} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) &= 0 \\ I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) &= 0 \\ I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_1 - I_3) \\ I_2\dot{\omega}_2 &= -\omega_1\omega_3(I_1 - I_3) \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

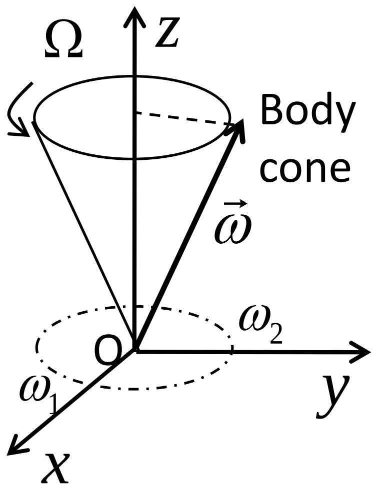
$$\dot{\omega}_3 = 0, \quad \omega_3 = \text{constant.}$$

$$\dot{\omega}_1 = -\Omega\omega_2,\ \dot{\omega}_2 = \Omega\omega_1,\ \ \Omega = \frac{\left(I_3 - I_1\right)\omega_3}{I_1}, \quad \left(\dot{\omega}_1 + i\dot{\omega}_2\right) - i\Omega\left(\omega_1 + i\omega_2\right) = 0,$$

$$\xi = \omega_1 + i\omega_2, \quad \dot{\xi} - i\Omega\xi = 0, \ \ \xi = \omega_0 e^{i\Omega t}, \ \ \omega_1 + i\omega_2 = \omega_0 \cos\Omega t + i\omega_0 \sin\Omega t$$

$$\left(\omega_1, \omega_2\right) = \left(\omega_0 \cos\Omega t, \omega_0 \sin\Omega t\right),\ \omega_0 \text{ is a constant.}$$

The magnitude of ω is also constant: $\left|\vec{\omega}\right| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\omega_0^2 + \omega_3^2} = \text{constant}$



The angular velocity vector ω revolves or precesses about the body z-axis. In the body frame, ω traces out a cone around the body symmetry axis, called **body cone**.

Since there is no force, there is no torque at the CM (O) . Hence angular momentum ***L*** is also constant in time and its direction is fixed in space.

For the force free motion and CM is fixed, the rotational KE is also constant

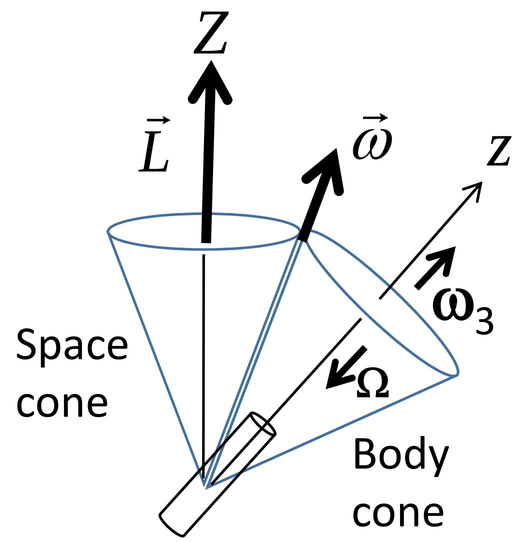
$$K_{\text{rot}} = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \text{constant.}$$

Since ***L*** is constant, ω must move such that its projection on the stationary angular momentum vector is constant. In that case, ***L***, ω, and body z-axis must lie in a plane. Check that $\vec{L}\cdot\left(\vec{\omega}\times\hat{k}\right)=0.$

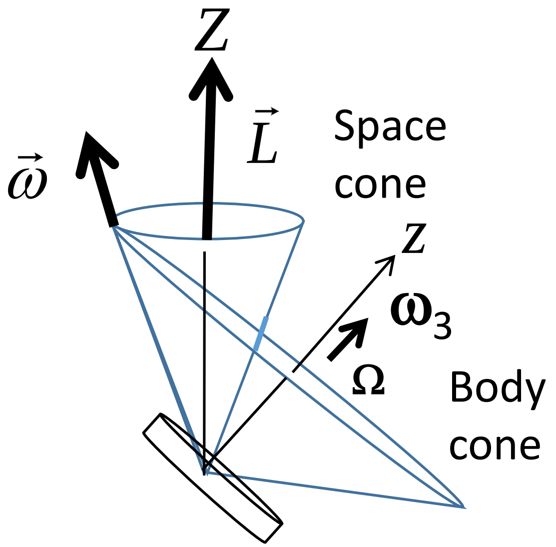
$$\vec{\omega}\times\hat{k} = \omega_2\hat{i} - \omega_1\hat{j}, \text{ then } \vec{L}\cdot\left(\vec{\omega}\times\hat{k}\right) = I_1\omega_1\omega_2 - I_2\omega_1\omega_2 = 0 \quad \left(I_1 = I_2\right)$$

If ***L*** coincides with the Z-axis of the space fixed frame, then ω traces out a cone around the fixed Z-axis, called **space cone**.

Body cone and space cone:



Prolate: $I_1 > I_3$



Oblate: $I_3 > I_1$