Rigid Body Dynamics

Angular Velocity as Vector

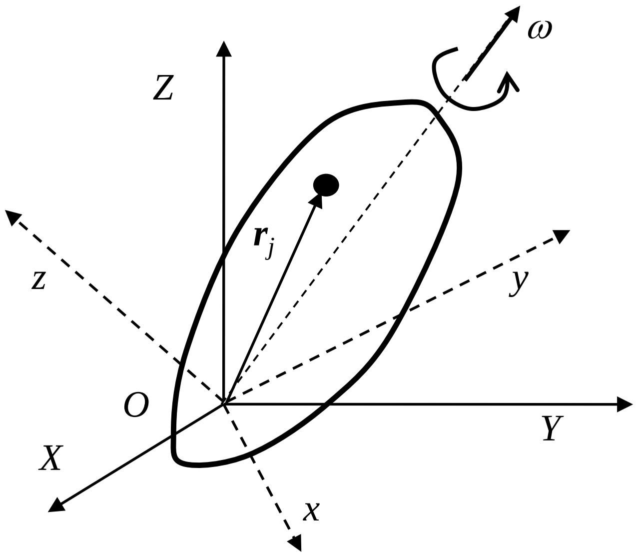
- angular velocity vector is defined for fixed axis motion.
- Generalize to instantaneous angular velocity vector.
- Define: $\vec{\omega}$ such that the instantaneous velocity $\vec{v}_i$ of each particle can be written as

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

If we know $\vec{\omega}(t)$, the motion can be found.

Angular momentum of a rotating rigid body

Consider a rigid body composed of N particles with masses m_1, m_2, . . . , m_j, . . . , m_N. One of the points O of the body is fixed and the body is rotating about an arbitrary axis passing through the fixed point with an angular velocity ω.



XYZ: Fixed frame, *xyz*: Body frame

Take *XYZ* and *xyz* are the fixed and body frames respectively with common origin at *O*. Then, the coordinates of the jth point mass are same in the body and fixed frames. Let us tale it as r_j.

The velocity v_j of the jth particle in the fixed frame is

$$\vec{v}_j = \left(\frac{d\vec{r}_j}{dt}\right)_{\text{fix}} = \left(\frac{d\vec{r}_j}{dt}\right)_{\text{rot}} + \left(\vec{\omega}\times\vec{r}_j\right) = \vec{\omega}\times\vec{r}_j$$

$$\left(\frac{d\vec{r}_j}{dt}\right)_{\text{rot}} = 0, \text{ in a rigid body.}$$

Instantaneous Angular Momentum

At any instant t, there is $\vec{\omega}$, such that

$$\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{r}_i \times P_i \\
&= \sum_i m_i \mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i) \\
&= \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i (\mathbf{r}_i \cdot \vec{\omega}) \mathbf{r}_i
\end{aligned}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

If

$$\begin{aligned}
\vec{\omega} &= \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \\
\mathbf{r}_i &= x_i \mathbf{i} + y_i \mathbf{j} + z_i \mathbf{k}
\end{aligned}$$

Instantaneous Angular Momentum

$$\mathbf{L} = \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i (\mathbf{r}_i \cdot \vec{\omega}) \mathbf{r}_i$$

The x component of Angular Momentum

$$\begin{aligned}
\mathbf{L}_x &= \sum_i m_i r_i^2 \omega_x - \sum_i m_i (\omega_x x_i + \omega_y y_i + \omega_z z_i) x_i \\
&= \left(\sum m_i (r_i^2 - x_i^2) \right) \omega_x + \left(- \sum m_i y_i x_i \right) \omega_y \\
&\quad + \left(- \sum m_i x_i x_i \right) \omega_z \\
&= I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z
\end{aligned}$$

Calculation details

Let us compute one component of ***L***, say L_x. Temporarily dropping the subscript j,

$$\boldsymbol{\omega} \times \mathbf{r} = (z\omega_y - y\omega_z)\hat{\mathbf{i}} + (x\omega_z - z\omega_x)\hat{\mathbf{j}} + (y\omega_x - x\omega_y)\hat{\mathbf{k}}.$$

$$\begin{aligned}[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_x &= y(\boldsymbol{\omega} \times \mathbf{r})_z - z(\boldsymbol{\omega} \times \mathbf{r})_y. \\ &= y(y\omega_x - x\omega_y) - z(x\omega_z - z\omega_x) \\ &= (y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z.\end{aligned}$$

Hence, $L_x = \Sigma m_j(y_j^2 + z_j^2)\omega_x - \Sigma m_j x_j y_j \omega_y - \Sigma m_j x_j z_j \omega_z.$

Let us introduce the following symbols:

$$I_{xx} = \Sigma m_j(y_j^2 + z_j^2) \qquad I_{xy} = -\Sigma m_j x_j y_j \qquad I_{xz} = -\Sigma m_j x_j z_j.$$

I_{xx} is the moment of inertia of the body about the x-axis of the body frame and I_{xy} and I_{xz} are called products of inertia. The products of inertia are symmetrical; for example,

$$I_{xy} = -\Sigma m_j x_j y_j = -\Sigma m_j y_j x_j = I_{yx}.$$

Instantaneous Angular Momentum

- $I_{xx} = (\sum m_i(r_i^2 - x_i^2))$ and I_{yy} and I_{zz} are called Moments of Inertia.
- $I_{xy} = (-\sum m_i y_i x_i)$, I_{xz} and I_{yz} are called Products of Inertia.
- Instantaneous Angular Momentum

(1)
$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

or

(2)
$$\mathbf{L} = I\vec{\omega}$$

Details

All the three components of angular momentum are given by:

$$
\begin{aligned}
L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\
L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\
L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z.
\end{aligned}
$$

For fixed axis rotation about the z-direction, $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$, L_z reduces to

$$L_z = I_{zz}\omega = \Sigma m_j(x_j^2 + y_j^2)\omega.$$

However, angular velocity in the z-direction can produce angular momentum about any of the three coordinate axes. For example, if $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$ then $L_x = I_{xz}\omega$ and $L_y = I_{yz}\omega$.

In fact, the angular momentum about one axis depends on the angular velocity about all three axes.

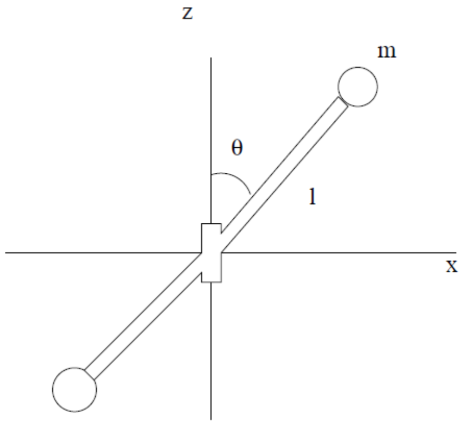
Both ***L*** and $\boldsymbol{\omega}$ are ordinary vectors, and ***L*** is proportional to $\boldsymbol{\omega}$ in the sense that doubling the components of $\boldsymbol{\omega}$ doubles the components of ***L***.

However, ***L*** does not necessarily point in the same direction as $\boldsymbol{\omega}$.

Example

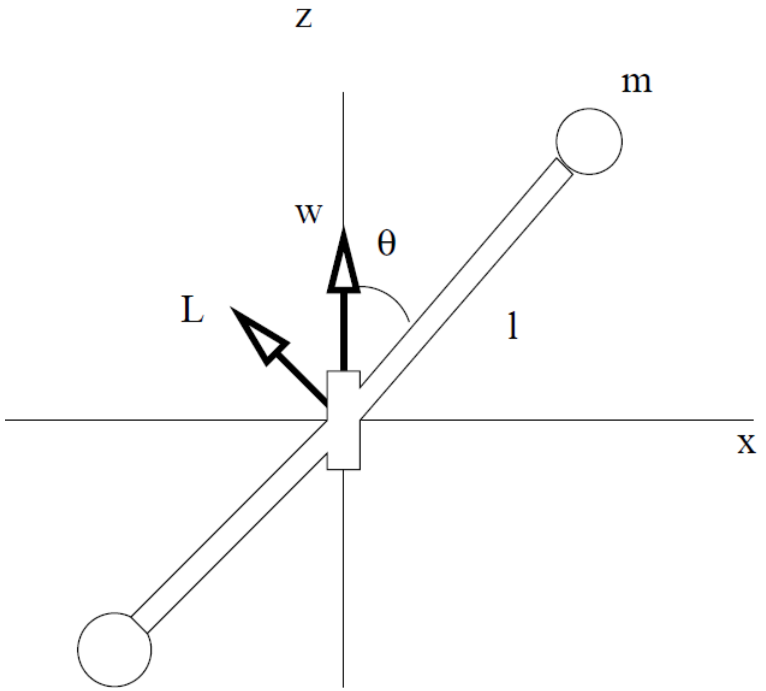
Moment of Inertia

$$I = \begin{pmatrix} 2ml^2\cos^2\theta & 0 & -2ml^2\cos\theta\sin\theta \\ 0 & 2ml^2 & 0 \\ -2ml^2\cos\theta\sin\theta & 0 & 2ml^2\sin^2\theta \end{pmatrix}$$



By the time body turns and lies in YZ plane, Moment of Inertia becomes

$$I = \begin{pmatrix} 2ml^2 & 0 & 0 \\ 0 & 2ml^2\cos^2\theta & -2ml^2\cos\theta\sin\theta \\ 0 & -2ml^2\cos\theta\sin\theta & 2ml^2\sin^2\theta \end{pmatrix}$$



If the body is spinning about z axis $\vec{\omega} = \omega_z \mathbf{k}$ then,

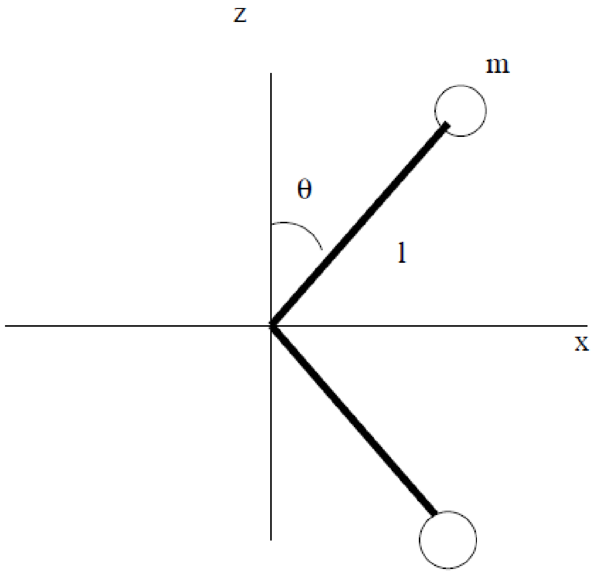
$$L = \begin{pmatrix} 2ml^2\cos^2\theta & 0 & -2ml^2\cos\theta\sin\theta \\ 0 & 2ml^2 & 0 \\ -2ml^2\cos\theta\sin\theta & 0 & 2ml^2\sin^2\theta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$$

$$= \begin{pmatrix} -2ml^2\cos\theta\sin\theta\omega_z \\ 0 \\ 2ml^2\sin^2\theta\omega_z \end{pmatrix}$$

Principle Axes

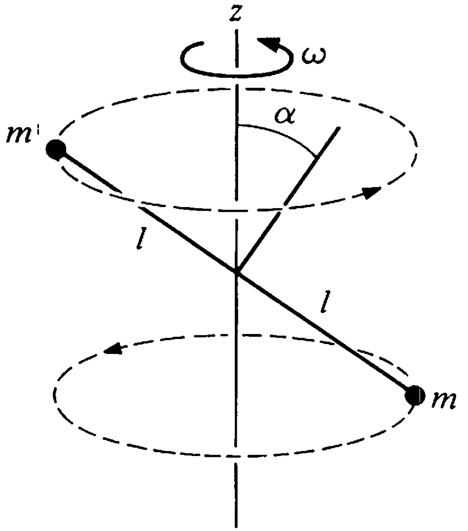
Moment of Inertia

$$I = \begin{pmatrix} 2ml^2\cos^2\theta & 0 & 0 \\ 0 & 2ml^2 & 0 \\ 0 & 0 & 2ml^2\sin^2\theta \end{pmatrix}$$



The three axes are called the Principle Axes of the body.

An example: Rotating Skew Rod

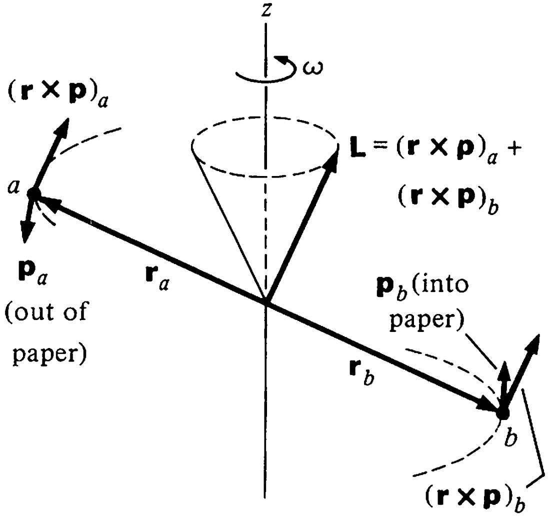


Consider a simple rigid body consisting of two particles of mass m separated by a massless rod of length $2l$. The midpoint of the rod is attached to a vertical axis which rotates at angular speed ω. The rod is skewed at angle α, as shown in the sketch. The problem is to find the angular momentum of the system.

Either

$$\mathbf{L} = \Sigma(\mathbf{r}_j \times \mathbf{p}_j). \quad |\mathbf{p}| = m\omega l \cos\alpha,$$

$$|\mathbf{r}| = l. \qquad |\mathbf{L}| = 2m\omega l^2 \cos\alpha.$$

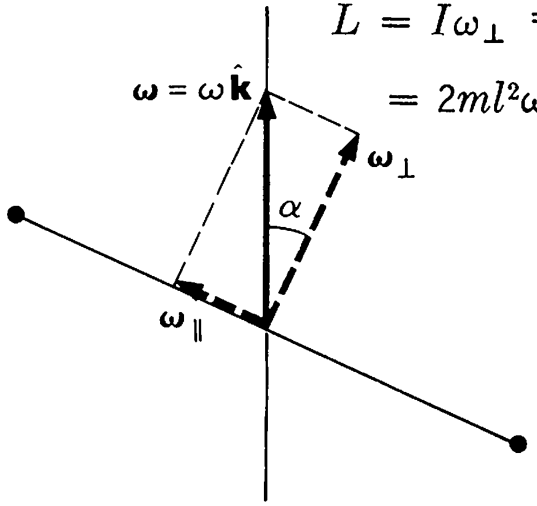


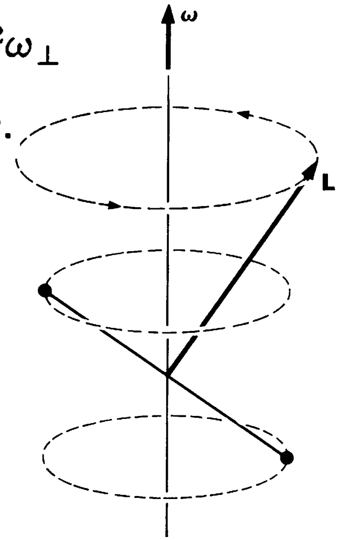
Or $\quad \boldsymbol{\omega} = \omega\hat{\mathbf{k}}$

$$\omega_\perp = \omega\cos\alpha, \text{ and } \omega_\parallel = \omega\sin\alpha.$$

$$L = I\omega_\perp = 2ml^2\omega_\perp$$

$$= 2ml^2\omega\cos\alpha.$$





$\boldsymbol{\omega}$ and $\boldsymbol{L}$ are not parallel.

Kinetic energy

$$K_{\text{rot}} = \tfrac{1}{2}\Sigma m_j \mathbf{v}_j'^2 = \tfrac{1}{2}\Sigma m_j(\boldsymbol{\omega} \times \mathbf{r}_j') \cdot (\boldsymbol{\omega} \times \mathbf{r}_j').$$

$$= \tfrac{1}{2}\Sigma m_j \boldsymbol{\omega} \cdot [\mathbf{r}_j' \times (\boldsymbol{\omega} \times \mathbf{r}_j')]$$

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

$$= \tfrac{1}{2}\boldsymbol{\omega} \cdot \Sigma m_j \mathbf{r}_j' \times (\boldsymbol{\omega} \times \mathbf{r}_j').$$

The sum in the last term is the angular momentum ***L***. Therefore,

$$K_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}.$$

Summary so far….

The nine components of MI can be tabulated In a 3 X 3 array:

$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$\overleftrightarrow{I}$ is called the moment of inertia tensor.

I_{xx}, I_{yy}, I_{zz} are the MI about the x,y and z-axis of the body frame respectively. Off-diagonal terms are the product of inertia.

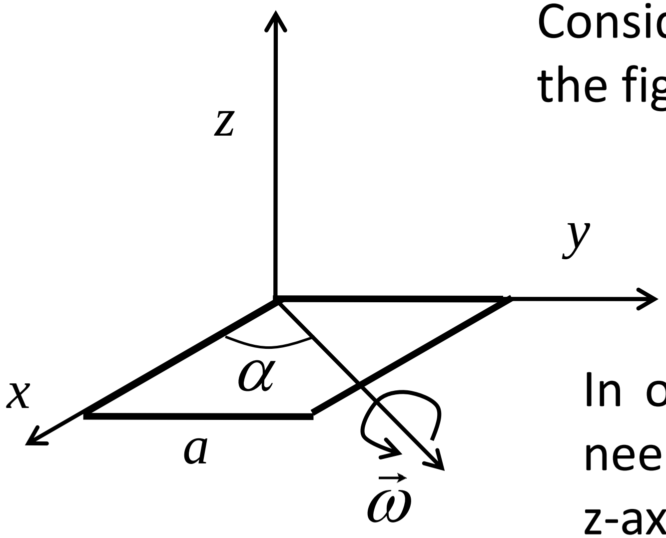
Since $I_{yx} = I_{xy}$, $I_{zx} = I_{xz}$, and $I_{yz} = I_{zy}$, out of the nine components, only six at most are different and the matrix is symmetric.

The angular momentum and the kinetic energy can be expressed as

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \quad \text{and} \quad T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

Rotation of a square plate

Consider rotation of a square plate of side *a* and mass *M* about an axis in the plane of the plate and making an angle α with the x-axis. What is the angular moment ***L*** about the origin?



Consider a body fixed coordinate system (x,y,z) as shown in the figure.

$$\vec{L} = \vec{I} \cdot \vec{\omega}, \quad \text{with} \quad \vec{\omega} = \begin{pmatrix} \omega \cos \alpha \\ \omega \sin \alpha \\ 0 \end{pmatrix}$$

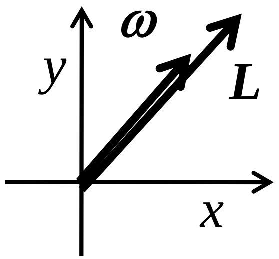
In order to estimate the moment of inertia tensor, one needs to calculate the moments of inertia about x, y, and z-axis and the products of inertia.

$$I_{xx} = \int_{x=0}^{a} \int_{y=0}^{a} \sigma y^2 dxdy = \frac{1}{3}\sigma a^4 = \frac{1}{3} Ma^2 = I_{yy},\ I_{zz} = I_{xx} + I_{yy} = \frac{2}{3} Ma^2$$

$$I_{xy} = -\int_{x=0}^{a} \int_{y=0}^{a} \sigma xydxdy = -\frac{1}{4}\sigma a^4 = -\frac{1}{4} Ma^2 = I_{yx},\ I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$$

$$\vec{L}=\begin{pmatrix} Ma^2/3 & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3 & 0 \\ 0 & 0 & 2Ma^2/3 \end{pmatrix}\begin{pmatrix} \omega\cos\alpha \\ \omega\sin\alpha \\ 0 \end{pmatrix}=\begin{pmatrix} Ma^2\omega\left(\frac{1}{3}\cos\alpha-\frac{1}{4}\sin\alpha\right) \\ Ma^2\omega\left(-\frac{1}{4}\cos\alpha+\frac{1}{3}\sin\alpha\right) \\ 0 \end{pmatrix}$$

For $\alpha=45^\circ$,



$$\vec{L}=\left(\frac{1}{12\sqrt{2}}Ma^2\omega,\frac{1}{12\sqrt{2}}Ma^2\omega,0\right) \quad \text{and} \quad \vec{\omega}=\begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$

For θ=45°, they are parallel to each other.

- For a rotation around a principal axis, a vanishing torque means that if the object is pivoted at the origin and if origin is the only place where any force is applied (implying that there is zero torque about it), then the object can undergo a rotation with constant angular velocity, ω. If you try it with a non-principal axis, it won't work.

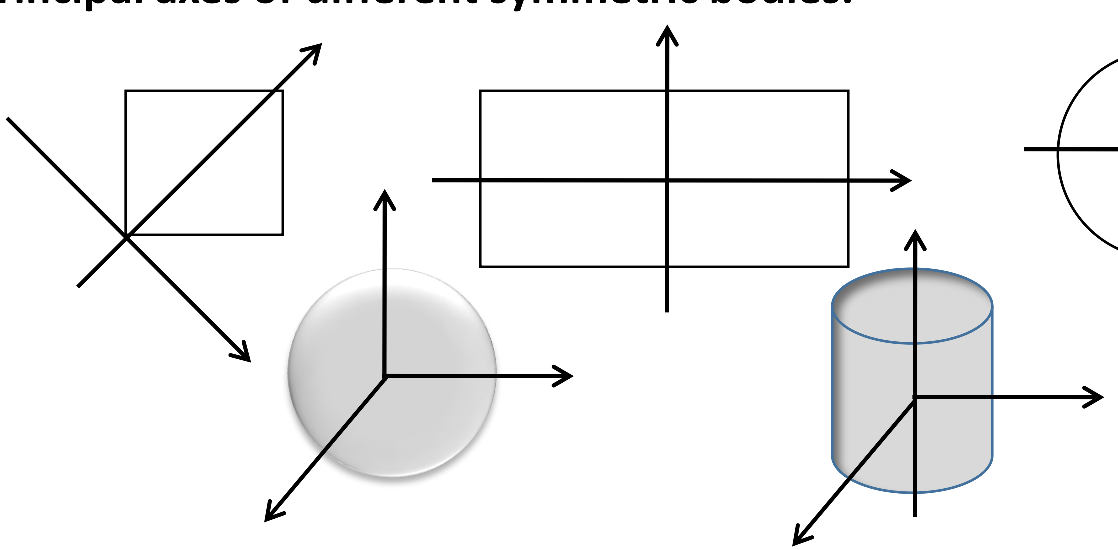
Principal axes and Moment of Inertia

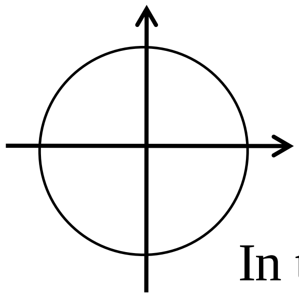
If the symmetry axes of a uniform symmetric body coincide with the coordinate axes, the products of inertia are zero. In this case the tensor of inertia takes a simple diagonal form:

$$\tilde{\mathbf{I}} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}.$$

For a body of any shape and mass distribution, it is always possible to find a set of three orthogonal axes such that the products of inertia vanish. Such axes are called principal axes. The tensor of inertia with respect to principal axes has a diagonal form.

Principal axes of different symmetric bodies:





In the principal axes system:

$$L_x = I_{xx}\omega_x,$$

$$L_y = I_{yy}\omega_y,$$

$$L_z = I_{zz}\omega_z,$$

Principal axes: Diagonalization of Inertia tensor:

Consider a rigid body with body axes *x-y-z,* Inertia tensor ***I*** is (in general) not diagonal.
But it can be made diagonal by $\overleftrightarrow{I}_d = R\overleftrightarrow{I}R^\dagger$

Rotate *x-y-z by R* to a new body axes x'-y'-z'.

$$\vec{\omega}' = R\vec{\omega},$$
$$\vec{L}' = R\vec{L} = R\overleftrightarrow{I}\vec{\omega} = R\overleftrightarrow{I}R^\dagger R\vec{\omega}, \quad R^\dagger R = 1$$
$$= \overleftrightarrow{I}_d \omega'$$

$$\overleftrightarrow{I}_d = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

One can choose a set of body axes that make the inertia tensor diagonal. These body axes are the **Principal Axes.**

How do one can find them?

Consider unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ along principal axes : $\overleftrightarrow{I}\hat{e}_i = \lambda_i \hat{e}_i, \; i = 1, 2, 3$

Express **I** in any body coordinates, and solve eigenvalue equation

$$\left|\overleftrightarrow{I} - \lambda I\right| = 0, \quad \Rightarrow \lambda = I_1, I_2, I_3 \quad (I \text{ is the identity matrix}).$$

Eigenvectors corresponding to each λ point the principal axes

One can often find the principal axes by just looking at the object.

Finding eigenvectors

<u>**Example:**</u> Find the eigenvalues and associated eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}.$$

First we compute $\det(\mathbf{A} - \lambda\mathbf{I})$ via a cofactor expansion along the second column:

$$\begin{aligned} \begin{vmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{vmatrix} &= (-2-\lambda)(-1)^4 \begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix} \\ &= -(2+\lambda)[(7-\lambda)(-8-\lambda)+54] \\ &= -(\lambda+2)(\lambda^2+\lambda-2) \\ &= -(\lambda+2)^2(\lambda-1). \end{aligned}$$

Thus $\mathbf{A}$ has two distinct eigenvalues, $\lambda_1 = -2$ and $\lambda_3 = 1$. (Note that we might say $\lambda_2 = -2$, since, as a root, -2 has multiplicity two. This is why we labelled the eigenvalue 1 as λ_3.)

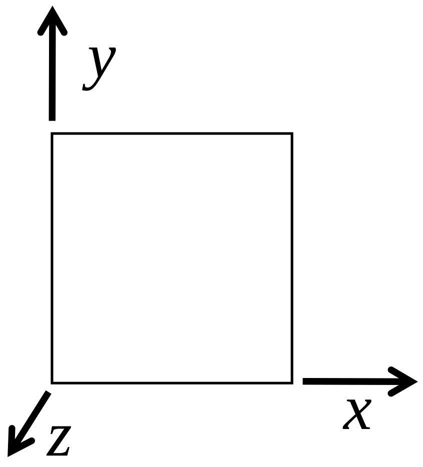
Now, to find the associated eigenvectors, we solve the equation $(\mathbf{A} - \lambda_j \mathbf{I})\mathbf{x} = \mathbf{0}$ for $j = 1, 2, 3$. Using the eigenvalue $\lambda_3 = 1$, we have

$$\begin{aligned}
(\mathbf{A} - \mathbf{I})\mathbf{x} &= \begin{bmatrix} 6x_1 - 3x_3 \\ -9x_1 - 3x_2 + 3x_3 \\ 18x_1 - 9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&\Rightarrow \quad x_3 = 2x_1 \qquad \text{and} \qquad x_2 = x_3 - 3x_1 \\
&\Rightarrow \quad x_3 = 2x_1 \qquad \text{and} \qquad x_2 = -x_1.
\end{aligned}$$

So the eigenvectors associated with $\lambda_3 = 1$ are all scalar multiples of

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Square lamina:



Mass *M* and side *a.*

$$\overleftrightarrow{I} = \begin{pmatrix} Ma^2/3 & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3 & 0 \\ 0 & 0 & 2Ma^2/3 \end{pmatrix}$$

$$\left| \overleftrightarrow{I} - \lambda I \right| = 0$$

$$\begin{vmatrix} Ma^2/3-\lambda & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3-\lambda & 0 \\ 0 & 0 & 2Ma^2/3-\lambda \end{vmatrix} = 0$$

$$\left(2Ma^2/3-\lambda\right)\left[\left(Ma^2/3-\lambda\right)^2 - \left(Ma^2/4\right)^2\right] = 0$$

$$\lambda_1 = \frac{7Ma^2}{12}, \lambda_2 = \frac{Ma^2}{12}, \lambda_3 = \frac{2Ma^2}{3}$$