Angular Momentum
&
Fixed Axis Rotation (contd)

Summary of rotational motion

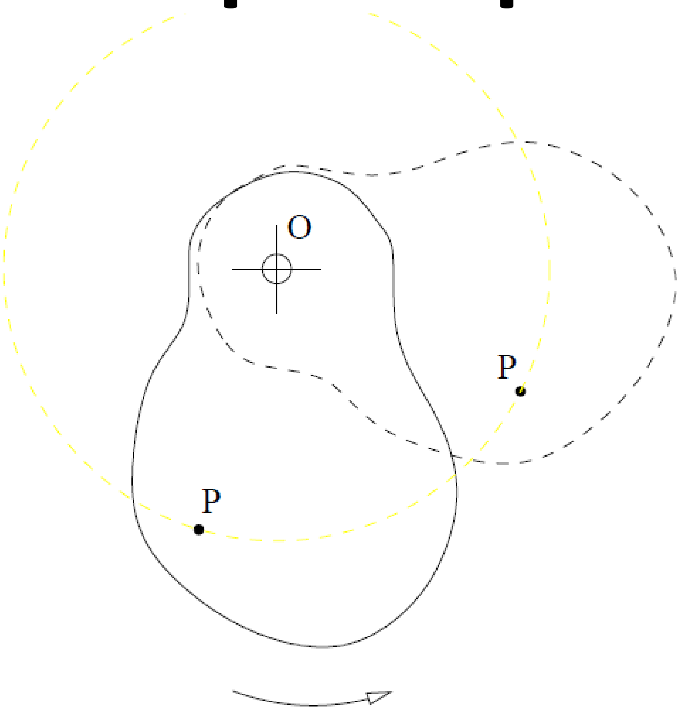
All rigid body motion can be split into:

- A translation of one point of rigid body
- Rotation of rigid body about that point

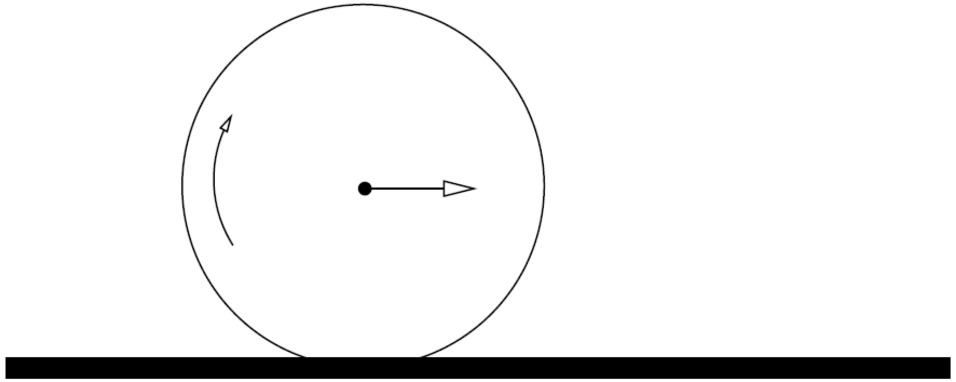
A special case in which rigid body motion is combination of fixed axis rotation + translation of fixed axis keeping it parallel to the some fixed axis in space.

A general motion can always be split into a rotation + a translation

Example of pure rotation and rotation plus translation



Pure Rotation

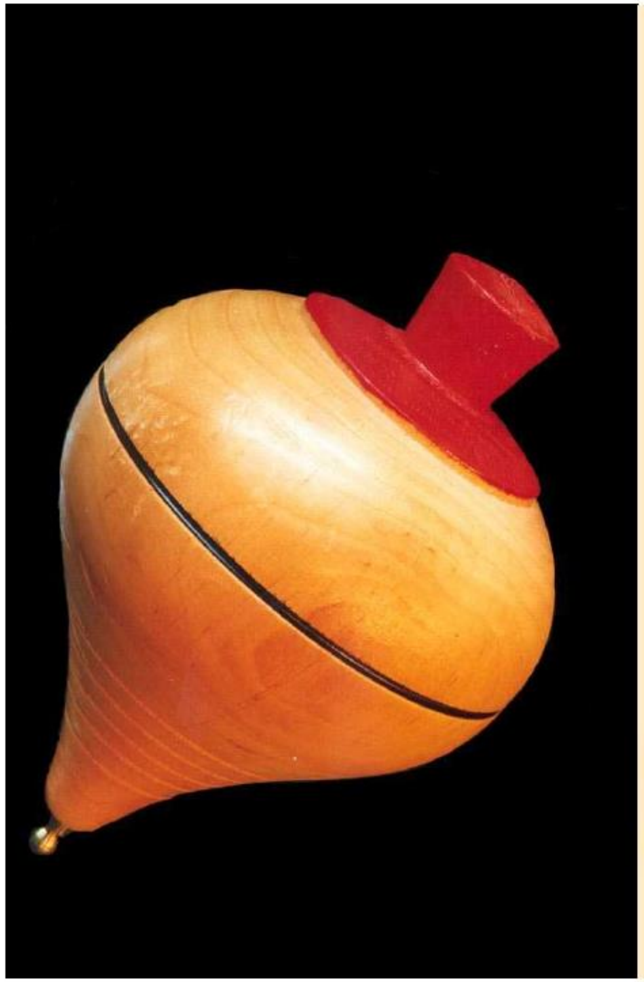


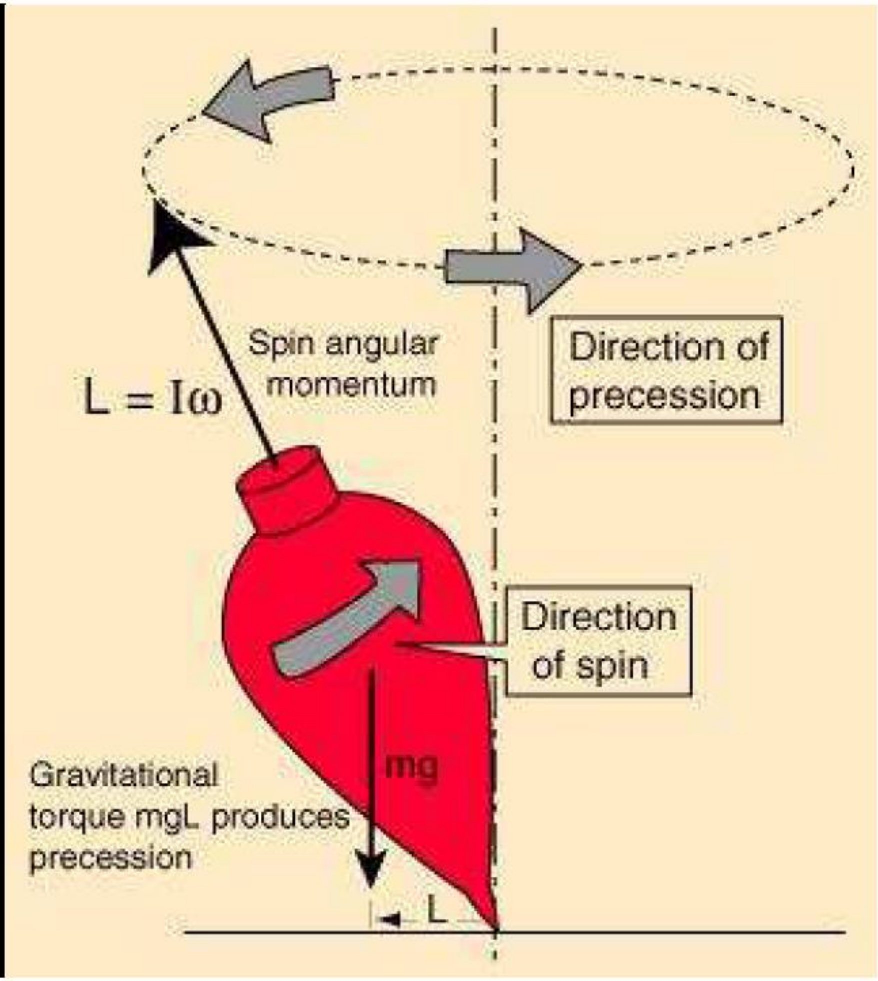
Rotation plus translation

Each point of the rigid body performs a circular motion about O.

The point shown moves long a straight line

Example of pure rotation: A rotating top

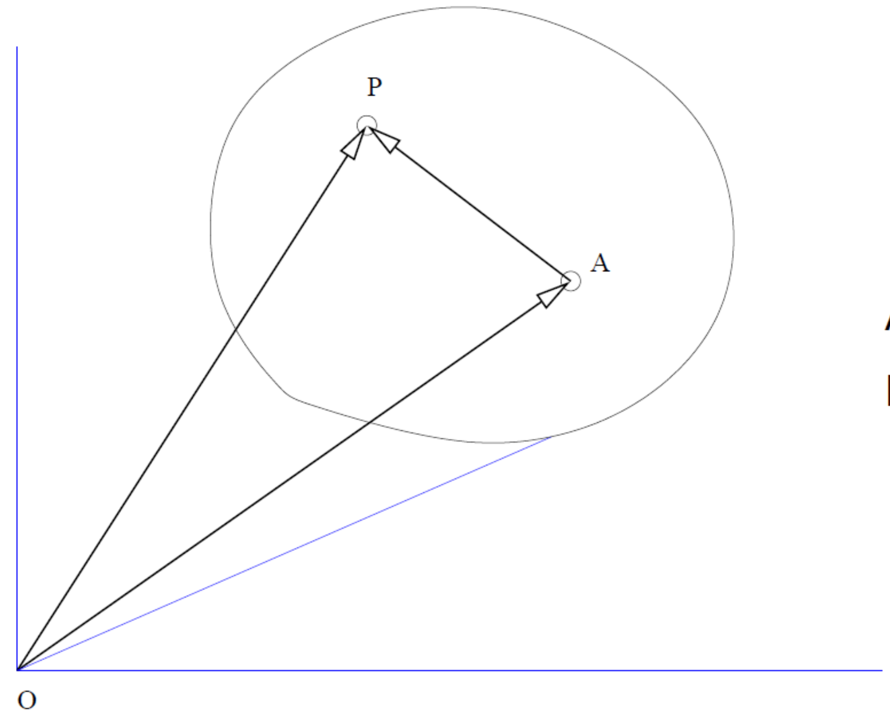




All points of the top are restricted to a spherical surface.

Example of rotation + Translation: Tyre Rolling

Calculating Angular Momentum for rotation + translation



A: CM of body R_A

P: A point of body $\vec{r}$

$$\begin{aligned}\vec{r} &= \vec{R}_A + \vec{r}'\\ \vec{v} &= \vec{V}_A + \vec{v}'\end{aligned}$$

Angular Momentum about O: $\vec{L} = \sum m_i\,\vec{r}_i \times \vec{v}_i$

Angular Momentum about A: $\vec{L}_0 = \sum m_i\,\vec{r'}_i \times \vec{v'}_i$

$$\begin{aligned}\vec{L} &= \sum m_i \vec{r}_i \times \vec{v}_i\\ &= \sum m_i \left(\vec{R}_A + \vec{r'}_i\right) \times \left(\vec{V}_A + \vec{v'}_i\right)\end{aligned}$$

$$\begin{aligned}\vec{L} &= \sum m_i \left(\vec{R}_A + \vec{r'}_i\right) \times \left(\vec{V}_A + \vec{v'}_i\right)\\ &= \sum m_i \vec{R}_A \times \vec{V}_A + \sum m_i \vec{r'}_i \times \vec{v'}_i\\ &\quad + \sum m_i \vec{R}_A \times \vec{v}_i + \sum m_i \vec{r'}_i \times \vec{V}_A\\ &= \left(\sum m_i\right) \vec{R}_A \times \vec{V}_A + \sum m_i \vec{r'}_i \times \vec{v'}_i\\ &\quad + \vec{R}_A \times \left(\sum m_i \vec{v}_i\right) + \left(\sum m_i \vec{r'}_i\right) \times \vec{V}_A\\ &= M \vec{R}_A \times \vec{V}_A + \vec{L}_0\\ &= \vec{L}_{cm} + \vec{L}_0\end{aligned}$$

Angular Momentum splits nicely into two terms

The angular momentum relative to the origin of a body can be found by treating the body like a point mass located at CM

and

finding the angular momentum of this point mass relative to the origin plus the angular momentum of the body relative to CM.

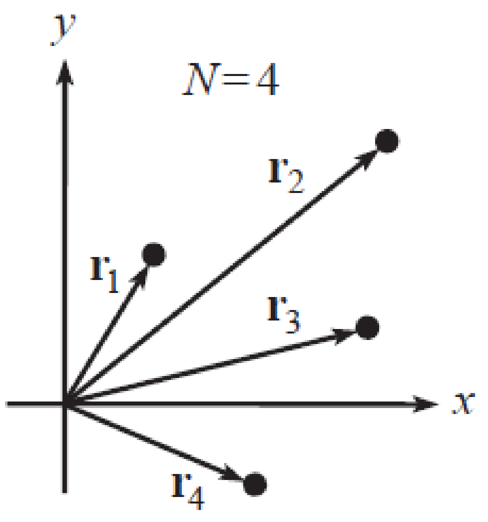
Kinetic energy

$$K = \frac{1}{2} M \, V^2_{\text{cm}} + \frac{1}{2} I^{\text{cm}}_{z} \omega^2$$

The Kinetic energy of the body can be found by treating the body like a point mass located at the CM, and the (rotational) kinetic energy of the body relative to CM.

Angular Momentum of a System of Particles

Consider a collection of *N* particles. The total angular momentum of the system is



$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i.$$

The force acting on each particle is $\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} = d\mathbf{p}_i/dt.$

The internal forces come from the adjacent particles which are usually central forces, so that the force between two particles is directed along the line between them.

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt}$$

$$= \sum_i \mathbf{v}_i \times (m\mathbf{v}_i) + \sum_i \mathbf{r}_i \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}})$$

$$= 0 + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} \equiv \sum_i \boldsymbol{\tau}_i^{\text{ext}}.$$

$$\mathbf{F}_i^{\text{int}} = \sum_i \mathbf{F}_{ij}^{\text{int}}.$$

$$\boldsymbol{\tau}^{\text{int}} \equiv \sum \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = \sum\sum \mathbf{r}_i \times \mathbf{F}_{ij}^{\text{int}}.$$

$$\boldsymbol{\tau}^{\text{int}} = \sum_j\sum_i \mathbf{r}_j \times \mathbf{F}_{ji}^{\text{int}} = -\sum_j\sum_i \mathbf{r}_j \times \mathbf{F}_{ij}^{\text{int}},$$

$$\mathbf{F}_{ij}^{\text{int}} = -\mathbf{F}_{ji}^{\text{int}}$$

$$2\boldsymbol{\tau}^{\text{int}} = \sum_i\sum_j (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{\text{int}}. = 0$$

- **The *total external torque* acting on the body, which may come from forces acting at many different points.**
- **The particles may not be rigidly connected to each other, they might have relative motion .**

In the continuous case, the sums need to be replaced with integrals.

The Torque

$$
\begin{aligned}
\vec{\tau} &= \sum \left(\vec{R}_A + \vec{r'}_i\right) \times \vec{F}_i \\
&= \sum \vec{R}_A \times \vec{F}_i + \sum \vec{r'}_i \times \vec{F}_i \\
&= \vec{R}_A \times \left(\sum \vec{F}_i\right) + \sum \vec{r'}_i \times \vec{F}_i \\
&= \vec{R}_A \times \vec{F} + \vec{\tau}_0
\end{aligned}
$$

Torque also appears as two terms. Compare with

$$
\begin{aligned}
\frac{d\vec{L}}{dt} &= M\vec{R}_A \times \frac{d\vec{V}_A}{dt} + \frac{d\vec{L}_0}{dt} \\
&= \vec{R}_A \times \vec{F} + \frac{d\vec{L}_0}{dt}
\end{aligned}
$$

Dynamical Equations

$$
\begin{aligned}
\frac{d\vec{P}_{cm}}{dt} &= F \\
\frac{d\vec{L}_0}{dt} &= \vec{\tau}_0
\end{aligned}
$$

If a body moves such that the axis of rotation moves parallel to a fixed axis then we need to consider only the z component of angular momentum.

$$
\begin{aligned}
\frac{d\vec{L}_{0z}}{dt} &= \vec{\tau}_{0z} \\
I^0_{zz}\alpha &= \tau_{0z}
\end{aligned}
$$

Conservation of *L* for a system of particles about a Point:

$$\vec{\tau}^{ext} = \frac{d\vec{L}}{dt}, \text{ where } \vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i = \sum_{i=1}^{N} \vec{L}_i \quad \text{and} \quad \vec{\tau}^{ext} = \sum_{i=1}^{N} \vec{\tau}_i^{ext}$$

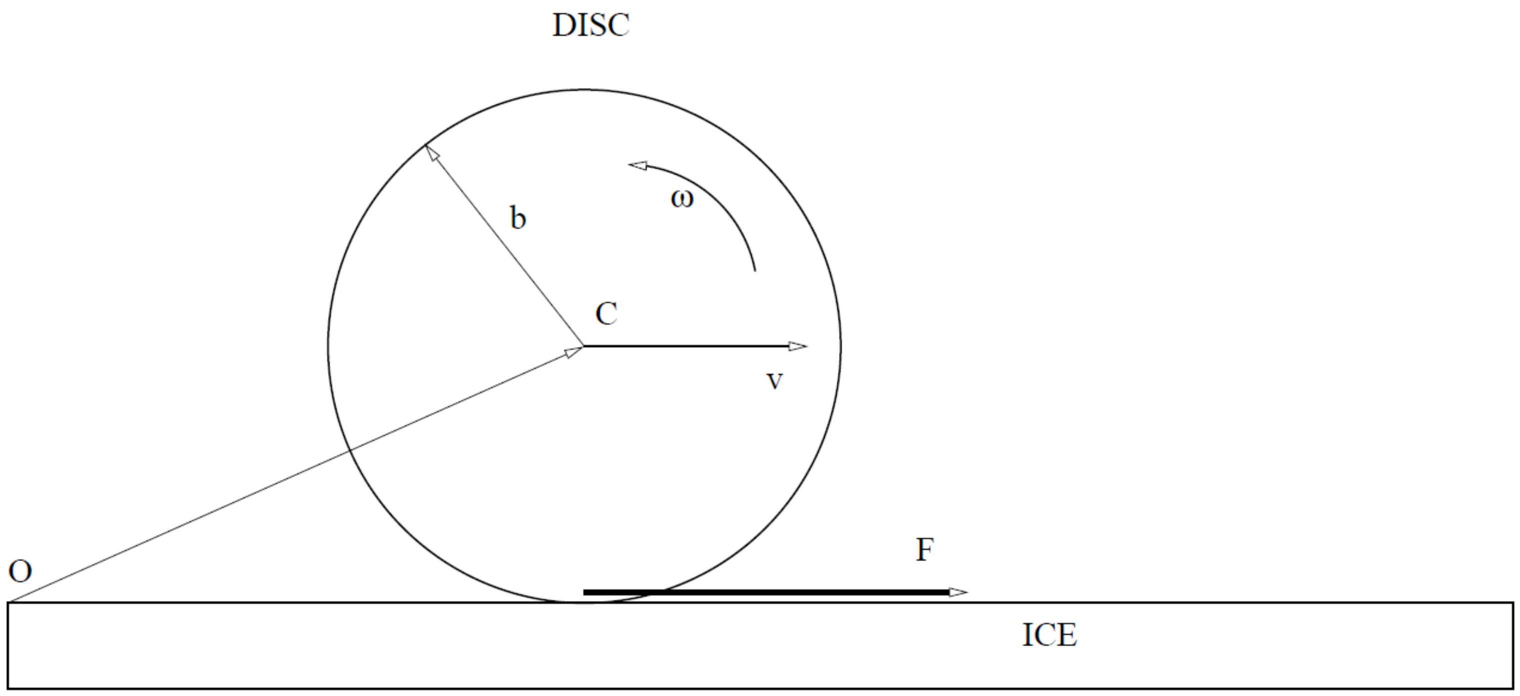
If $\vec{\tau}^{ext} = 0$, the net torque i.e. the sum of torque on individual particles is zero, the total angular momentum $\vec{L}$, the sum of angular momentum of individual particles will be constant.

That is

$$\vec{L}_{initial} = \vec{L}_{final}$$

$$\sum_{i=1}^{N} \left(\vec{r}_i \times \vec{p}_i\right)_{initial} = \sum_{i=1}^{N} \left(\vec{r}_i \times \vec{p}_i\right)_{final}$$

Example: Drum on Ice



No net force

$$V_{cm} = \text{const}$$

No Net torque about C

$$\omega = \text{const}$$

Ang Mom about C

$$L_0 = Mb^2\omega/2$$

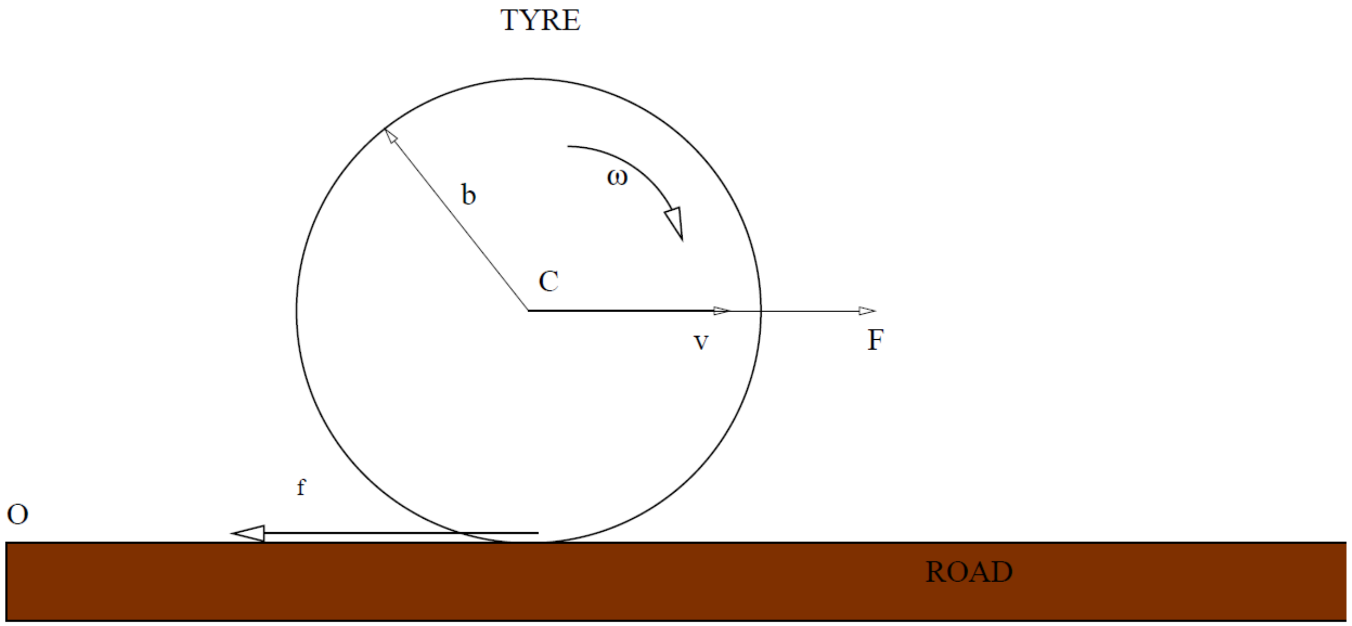
Ang Mom about O

$$L = MbV_{cm} + L_0$$

$$L = Fbt + L_0$$

Tyre on road will have a friction acting opposite to the direction of motion.

Example: Tyre on the road



Forces on the tyre

- Force F
- Frictional Force f

Torque about C

- $\tau_0 = bf$

The equations $(f < \mu N)$

$$\begin{aligned} Ma_{cm} &= F - f \\ I_0\alpha &= bf \end{aligned}$$

If $f < \mu N$ then there is no slipping, $a = b\alpha$.

$$\begin{aligned} Ma_{cm} &= F - I_0\alpha/b \\ Ma_{cm} + \frac{1}{2}Mb\alpha &= F \\ a_{cm} &= \frac{2F}{3M} \\ \alpha &= \frac{2F}{3Mb} \end{aligned}$$

and $f = F/3$. Clearly $F < 3\mu N$.

The equations $(f > \mu N)$ that is $F > 3\mu N$

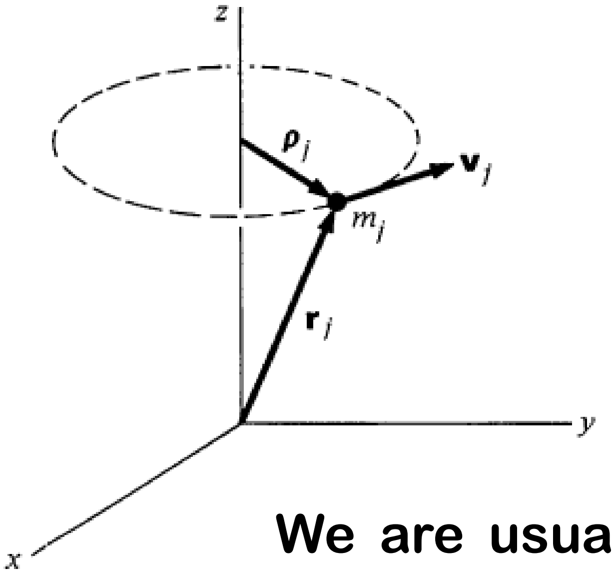
$$\begin{aligned} Ma_{cm} &= F - \mu N \\ I_0\alpha &= b\mu N \end{aligned}$$

In this case tyre slides on the road, there is no relationship between α and a_{cm}

Angular momentum for Fixed Axis Rotation

By fixed axis we mean that the direction of the axis of rotation is always along the same line; the axis itself may translate.

For example, a car wheel attached to an axle undergoes fixed axis rotation as long as the car drives straight ahead. If the car turns, the wheel must rotate about a vertical axis while simultaneously spinning on the axle; the motion is no longer fixed axis rotation.



$$\vec{L}_j = \vec{r}_j \times \vec{p}_j,$$

$$\vec{r}_j = \rho_j \hat{\rho} + z_j \hat{k}, \quad \vec{p}_j = m_j \vec{v}_j,$$

$$\vec{v}_j = \vec{\omega} \times \vec{r}_j = \omega \hat{k} \times \left(\rho_j \hat{\rho} + z_j \hat{k} \right) = \omega \rho_j \hat{\theta}$$

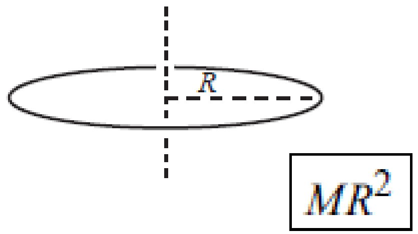
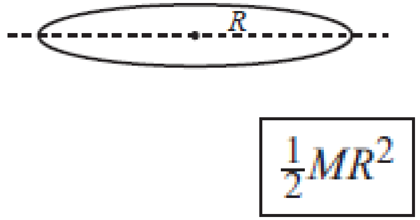
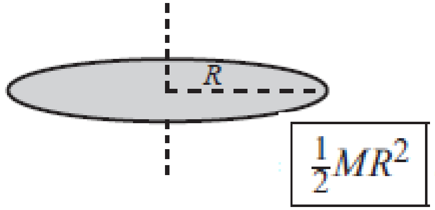
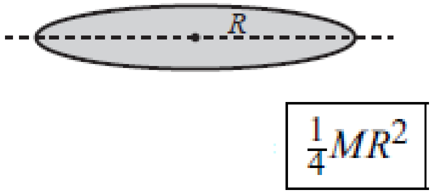
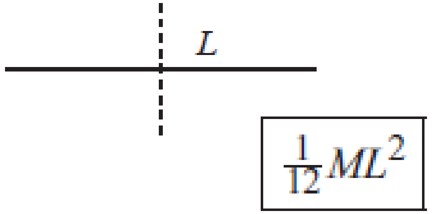
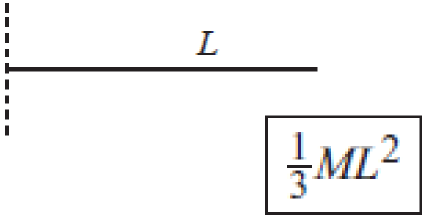
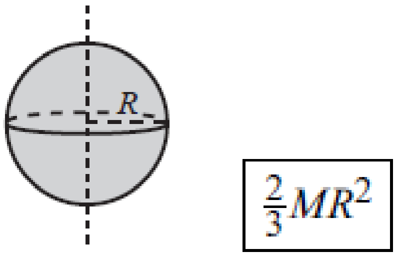
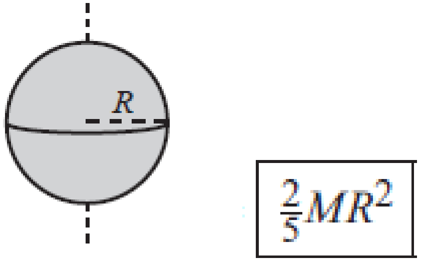
$$\vec{L}_j = \left(\rho_j \hat{\rho} + z_j \hat{k} \right) \times m_j \omega \rho_j \hat{\theta} = m_j \rho_j^2 \omega \hat{k} - m_j z_j \rho_j \omega \hat{\rho}$$

We are usually concerned only with L_z, the component of angular momentum along the axis of rotation.

$$\left(\vec{L}_j \right)_z = L_{jz} = m_j \rho_j^2 \omega, \quad L_z = \sum_i L_{jz} = \left(\sum_i m_j \rho_j^2 \right) \omega = I\omega, \quad I = \sum_j m_j \rho_j^2$$

For continuously distributed mass: $\sum_j m_j \rho_j^2 \to \int \rho^2 \, dm,$ I = **moment of inertia**

Moments of inertia of few symmetric objects:

A ring of mass *M* and radius *R,* axis through center, perpendicular to plane.	 MR^2	A ring of mass M and radius *R,* axis through center, in plane.	 $\frac{1}{2}MR^2$
A disk of mass *M* and radius *R,* axis through center, perpendicular to plane.	 $\frac{1}{2}MR^2$	A disk of mass *M* and radius *R,* axis through center, in plane.	 $\frac{1}{4}MR^2$
A thin uniform rod of mass *M* and length *L,* axis through center, perpendicular to rod.	 $\frac{1}{12}ML^2$	A thin uniform rod of mass *M* and length *L* axis through end, perpendicular to rod.	 $\frac{1}{3}ML^2$
A spherical shell of mass *M* and radius *R,* any axis through center.	 $\frac{2}{3}MR^2$	A solid sphere of mass *M* and radius *R,* any axis through center.	 $\frac{2}{5}MR^2$

The parallel-axis theorem:

$$I_z = MR^2 + I_z^{\mathrm{CM}}$$

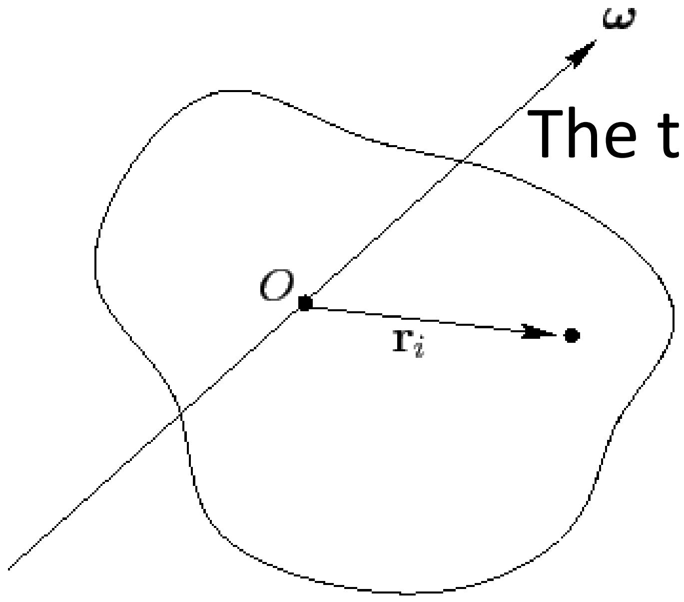
The perpendicular-axis theorem:

$$I_z = I_x + I_y$$

Moment of Inertia Tensor

- Consider a rigid body rotating with a constaht angular velocity **ω** about an axis passing thru' its origin.
- The velocity of the point i is given by

$$\frac{d\mathbf{r}_i}{dt} = \boldsymbol{\omega} \times \mathbf{r}_i.$$



The total angular momentum of the body about the origin is,

$$\mathbf{L} = \sum_{i=1,N} m_i\, \mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} = \sum_{i=1,N} m_i\, \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \sum_{i=1,N} m_i \left[r_i^2\, \boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\, \mathbf{r}_i \right],$$

The above formula has a matrix form

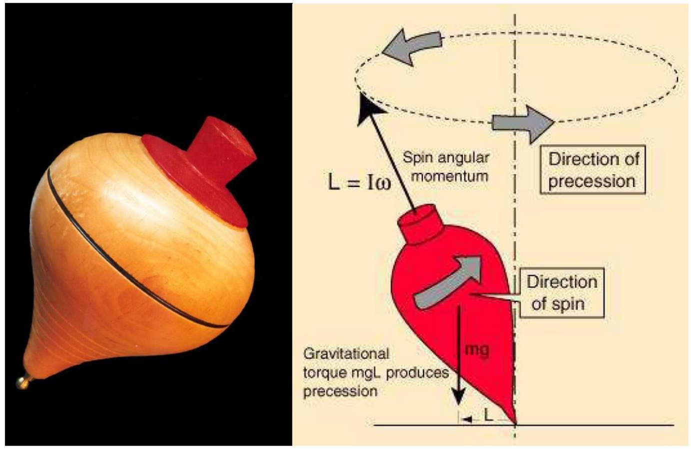
$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix},$$

where

$$I_{xx} = \sum_{i=1,N} (y_i^2 + z_i^2)\, m_i = \int (y^2 + z^2)\, dm, \quad I_{xy} = I_{yx} = -\sum_{i=1,N} x_i\, y_i\, m_i = -\int x\, y\, dm,$$

are the Moment of Inertia about the x-axis and the product of inertia respectively.

Rotational Kinetic energy



All points of the top are restricted to a spherical surface.

The Kinetic energy is written as, $K = \frac{1}{2}\sum_{i=1,N} m_i \left(\frac{d\mathbf{r}_i}{dt}\right)^2.$

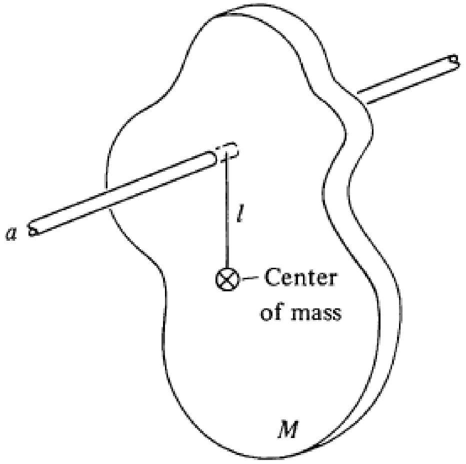
$$K = \frac{1}{2}\sum_{i=1,N} m_i\,(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot(\boldsymbol{\omega}\times\mathbf{r}_i) = \frac{1}{2}\,\boldsymbol{\omega}\cdot\sum_{i=1,N} m_i\,\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i).$$

It follows that $K = \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbf{L}.$

With **ω** having all the components, the kinetic energy is written as,

$$K = \frac{1}{2}\left(I_{xx}\,\omega_x^2 + I_{yy}\,\omega_y^2 + I_{zz}\,\omega_z^2 + 2\,I_{xy}\,\omega_x\,\omega_y + 2\,I_{yz}\,\omega_y\,\omega_z + 2\,I_{xz}\,\omega_x\,\omega_z\right).$$

The Physical Pendulum



$$\tau = I\alpha. \qquad -lW \sin\phi = I_a\ddot{\phi}.$$

Making the small angle approximation,

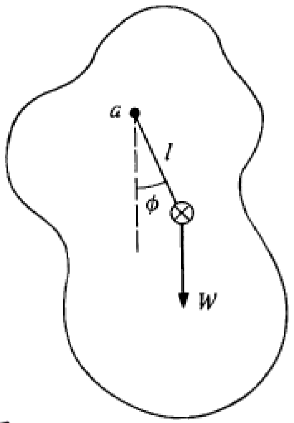
$$I_a\ddot{\phi} + Mlg\phi = 0.$$

$$\phi = A\cos\omega t + B\sin\omega t, \quad \text{where } \omega = \sqrt{Mlg/I_a}.$$

By the parallel axis theorem we have

$$I_a = I_0 + Ml^2 = M(k^2 + l^2),$$

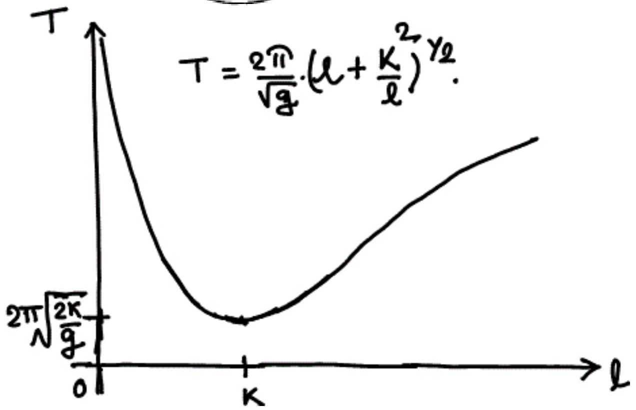
$$k = \sqrt{\frac{I_0}{M}} \qquad I_0 = Mk^2.$$



so that

$$\omega = \sqrt{\frac{gl}{k^2 + l^2}}.$$

The simple pendulum corresponds to $k = 0$, $\omega = \sqrt{g/l}$,



$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{(k^2 + l^2)}{gl}}$$

For a bar pendulum: $k = \sqrt{\dfrac{L^2 + b^2}{12}}$

Angular Impulse and Change in Angular Momentum

If there is a total applied torque $\vec{\boldsymbol{\tau}}_S$ about a point S over an interval of time $\Delta t = t_f - t_0$, then the torque applies an *angular impulse* about a point S, given by

$$\vec{\mathbf{J}}_S = \int_{t_0}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt .$$

Because $\vec{\boldsymbol{\tau}}_S = d\vec{\mathbf{L}}_S^{\text{total}} / dt$, the angular impulse about S is equal to the change in angular momentum about S,

$$\vec{\mathbf{J}}_S = \int_{t_0}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt = \int_{t_0}^{t_f} \frac{d\vec{\mathbf{L}}_S}{dt} dt = \Delta \vec{\mathbf{L}}_S = \vec{\mathbf{L}}_{S,f} - \vec{\mathbf{L}}_{S,0} .$$