DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA746: Fourier Analysis | End Semester Exam
Instructor: Rajesh Srivastava | May 9, 2019
Time duration: Three hours | Maximum Marks: 40

N.B. Answer without proper justification will attract zero mark.

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1. (a) Is it necessary that the Fourier transform of every compactly supported function in $L^1(\mathbb{R})$ is real analytic ? **[1]**

   (b) What is the distributional support of the function $\chi_{\mathbb{Q}}$, where $\mathbb{Q}$ is the set of rational numbers ? **[1]**

   (c) For $n \in \mathbb{N}$, let δ_n denote the Dirac delta distribution at n. Does $\delta_n \to 0$ in the weak* topology of $C_o(\mathbb{R})$ (the space of all continuous functions vanishing at infinity) ? **[1]**

   (d) What is the order of $\Lambda \in \mathcal{D}'(\mathbb{R})$ which is given by $\Lambda(\varphi) = \int\limits_{|x|>1} \log x \, \varphi(x) dx$? **[1]**

2. Find all those functions $f, g \in C^\infty(\mathbb{R})$ which are satisfying $f \, \delta_o + g \, \delta_o' = 0$. **[3]**

3. Suppose $f \in L^\infty(\mathbb{R})$ is satisfying $\int_{\mathbb{R}} f(y) e^{-y^2} e^{2xy} dy = 0$ for all $x \in \mathbb{R}$. Prove that $f = 0$. **[4]**

4. Let
$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 1 & \text{if } x < 0. \end{cases}$$
   Show that the Fourier transform of f satisfies $(1 - ix)\hat{f} = \hat{H}$ in the sense of tempered distribution, where $H = \chi_{(-\infty,0)}$. **[4]**

5. Find the distributional derivative of function $f(x) = e^{x^2} \chi_{[0,1]}(x)$. **[3]**

6. Let Λ be a distribution on $\mathbb{R}$ such that $x^2 \, \Lambda = 0$ for each $x \in \mathbb{R}$. Show that $\Lambda = c \, \delta_o + d \, \delta_o'$ for some constants c and d. **[4]**

7. For $n \in \mathbb{N}$, let $f_n = \chi_{[0,n)}$. Find $\lim\limits_{n \to \infty} f_n'$ in the weak* topology of $\mathcal{D}(\mathbb{R})$. **[2]**

8. Give an example of function $f \in L^\infty[(0, \infty)]$ whose derivative f' is a well defined function on $(0, \infty)$ but $f' \notin L^\infty[(0, \infty)]$. **[3]**

9. For $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 < p < 2$, show that $f * g \in L^p(\mathbb{R}^n)$. Further derive that $\widehat{f * g} = \hat{f}\hat{g}$. (Hint: use Hausdorff Young inequality). **[5]**

10. Suppose $f \in L^2(\mathbb{R}^n)$ is such that $\{\tau_x f : x \in \mathbb{R}^n\}$ is dense in $L^2(\mathbb{R}^n)$. Show that $\hat{f}$ cannot be zero on a set of positive measure in $\mathbb{R}^n$. **[4]**

11. Classify all those continuous functions on $\mathbb{R}$ which are tempered distributions on $\mathbb{R}$. **[4]**

END