DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA642: Real Analysis -1 | MidSem
Instructor: Rajesh Srivastava | February 27, 2023
Time duration: Two hours | Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) If O is a bounded open set in $\mathbb{R}$, does it imply that O must be the finite union of bounded open intervals? **[1]**
   (b) If A is a bounded set in $(C[0,1], \|\cdot\|_1)$, does it imply that A is necessarily a bounded subset in $(C[0,1], \|\cdot\|_2)$? **[1]**
   (c) If $f : \mathbb{R} \to \mathbb{R}$ is a function continuous, bounded and monotone function, does it imply that $\lim\limits_{x \to \pm\infty} f(x)$ are finite? **[1]**
   (d) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous and there exists $A > 0$ such that $f(x) \leq A|x|$ holds true for each $x \in \mathbb{R}$, does it imply that f is uniformly continuous on $\mathbb{R}$? **[1]**

2. Show that $\{(x_n) \in l^2 : |x_n| < \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$ is a convex set with empty interior. **[3]**

3. For $f \in C[0,1]$, define $\|f\| = \sup\limits_{0 \leq t \leq 1} |t^2 f(t)|$. Show that $(C[0,1], \|\cdot\|)$ is not a complete normed liner space. **[3]**

4. Let $\varphi_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$. Show that φ_n is uniformly convergent on each bounded open interval. Does φ_n converge uniformly on $\mathbb{R}$? **[4]**

5. Let $C(\mathbb{R})$ denote the space of all continuous function on $\mathbb{R}$. Let $p(f) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} p_n(f)$, where $p_n(f) = \sup\limits_{|t| \leq n} |f(t)|$. Find an infinite dimensional subspace M of $C(\mathbb{R})$ which satisfies (i) P is norm on M, and (ii) (M, p) is complete. **[4]**

6. Suppose $x \in l^p$ for some $p \geq 1$. Show that $\liminf\limits_{p \to \infty} \|x\|_p \geq \|x\|_\infty$. Prove/disprove that $\lim\limits_{p \to \infty} \|x\|_p = \|x\|_\infty$. **[4]**

7. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \sum\limits_{n=0}^{n_x} 2^{-n}$ if $x < 1$, where $n_x = [\frac{1}{1-x}]$ and $f(1) = 3$. Show that f is increasing and discontinuous on $\{1 - \frac{1}{k} : k \in \mathbb{N}\}$. **[4]**

8. Find a neighborhood of $x = 0$ in which initial value problem $y' = \frac{x}{1+y^2}$ with $y(0) = 0$ has a unique solution. **[4]**

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