DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA642: Real Analysis -1 | EndSem
Instructor: Rajesh Srivastava | May 7, 2023
Time duration: Three hours | Maximum Marks: 50

N.B. Answer without proper justification will attract zero mark.

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1. (a) If X is a finite metric space, does it imply that $C(X)$, the space of continuous functions on X, is a finite dimensional normed linear space? **[1]**

   (b) Let $f : (X, d) \to \mathbb{R}$ be such that $G_f = \{(x, f(x)) : x \in X\}$ is connected. Does it imply X is connected? **[1]**

   (c) Whether $\{x = (x_1, x_2, \ldots) \in l^2 : |x_n| \leq \frac{1}{n}\}$ is totally bounded in l^2? **[1]**

   (d) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be such that $f(tx) = t^2 f(x)$ for every $t > 0$ and $x \in \mathbb{R}^n$. Does it imply that f is differentiable at 0? **[1]**

   (e) If every countable closed set in a metric space (X, d) is complete, does it imply X is complete? **[1]**

2. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $A(x, y) = (2x + y, x + y)$. Find the norm of A. **[3]**

3. Let A be a connected subset of a metric space X, and let B be an open and closed set in X such that $A \cap B \neq \emptyset$. Show that $A \subset B$. **[3]**

4. Let $f : [1, \infty) \to \mathbb{R}$ be continuous and $\lim\limits_{x \to \infty} f(x) = 0$. For every $\epsilon > 0$, show that there exists a polynomial p satisfying $|f(x) - p(1/x)| < \epsilon$ for all $x \geq 1$. **[4]**

5. Show that the complement of any countable set E in $\mathbb{R}^2$ is path connected. **[3]**

6. Let $f : [a, b] \to \mathbb{R}$ be satisfying intermediate value property, and $f^{-1}(\{y\})$ is closed for every $y \in \mathbb{R}$. Show that f is continuous. **[4]**

7. Let $A \in GL_n(\mathbb{C})$. Show that the set $E = \left\{B \in L_n(\mathbb{C}) : \|B - A\| < \frac{1}{2\|A^{-1}\|}\right\}$ is open in $GL_n(\mathbb{C})$. And hence reduce that E is path connected in $L_n(\mathbb{C})$. **[4]**

8. Show that a subset A of a metric space X is closed if and only if $A \cap K$ is compact for every compact set K in X. **[3]**

9. Let $f_n \in C[0, 1]$ be satisfying $\|f_n\|_\infty \leq 1$. Let $F_n(x) = \int_0^x f_n(t)dt$. Show that F_n has a convergent subsequence. **[3]**

10. Give an example of sequence of function $f_n \in C[0,1]$, which decreases point wise to f but not uniformly. **3**

11. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function with $\|f'(x)\| \leq 1$. Show that f satisfies $\|f(x) - f(y)\| \leq |x - y|$ for every $x, y \in \mathbb{R}$. (Hint: use one dimensional MVT.) **4**

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Find an appropriate condition such that $f(x, (f(x,y)) = 0$ can be solved for x in some neighborhood of $(0,0)$. **3**

13. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $f'(0) \neq 0$. Show that $F(x,y) = (x - yf(y), f(y))$ is locally invertible in some neighborhood of $(0,0)$. Does there exists some f for which F is globally invertible? **4**

14. A map $f : (X,d) \to \mathbb{R}$ is called lower semi-continuous (LSC) if $\{x \in X : f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$. If f is LSC, show that for every $x \in X$, and every sequence $x_n \to x$, implies $f(x) \leq \lim_{n\to\infty} \inf f(x_n)$. **4**

END