

DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Guwahati

Topology

Lecture Notes

MA549

Point-set topology and foundational topological methods

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Preface

These lecture notes have been prepared for the course *MA549: Topology*. They are intended to provide a clear and self-contained introduction to point-set topology, with emphasis on the concepts, examples, and proof techniques that form the foundation of the subject. The exposition is organized so that definitions are followed by motivating examples, structural results, and proofs that illustrate the main methods of topological reasoning.

The notes are suitable for an advanced undergraduate or beginning graduate course. They begin with topological spaces, open and closed sets, neighborhoods, bases, subbases, and standard examples. The discussion then develops closure, interior, boundary, limit points, continuity, and homeomorphisms before moving to the fundamental constructions of subspace, product, and quotient topologies. Throughout the text, special attention is given to the transition from metric-space intuition to the more flexible language of general topology.

A central aim of the course is to explain how local information, expressed through open sets and neighborhoods, gives rise to global topological structure. For this reason, connectedness and compactness are treated as major organizing principles. The notes also discuss countability axioms and separation axioms, which clarify the extent to which familiar metric arguments remain valid in abstract spaces and where new phenomena appear.

The later chapters include classical results such as Urysohn's lemma, the Tietze extension theorem, and Urysohn's metrization theorem, together with further tools for studying convergence and structure in general topological spaces. Nets and filters are introduced to supplement sequential convergence, and the Baire category theorem is presented as an important connection between topology and analysis.

The manuscript has been written with two complementary purposes in mind. First, it is meant to support classroom lectures by presenting the main definitions and theorems in a systematic order. Second, it is meant to help the reader develop independent problem-solving ability by highlighting recurring proof patterns: reducing arguments to basis elements, using neighborhoods effectively, passing between open and closed formulations by complements, extracting finite subcovers in compactness arguments, and separating sets by continuous functions.

Learning objectives.

- Develop a precise understanding of topological spaces, continuity, and the standard constructions of the subject.
- Learn how closure, interior, boundary, convergence, and compactness encode global topological structure.
- Understand the countability and separation axioms and the role they play in metrization and function extension theorems.

- Study connectedness, compactness, quotient constructions, and product spaces as central organizing themes of point-set topology.
- Acquire the conceptual tools needed for later work in analysis, geometry, algebraic topology, and related areas.

Key ideas.

- The notes are designed to help the reader move from metric-space intuition to genuinely abstract topological reasoning.
- Standard examples and counterexamples are used throughout to show which phenomena depend only on open sets and which require additional structure.
- The chapter sequence is arranged so that logical dependencies are clearer and the main ideas of the subject emerge more naturally.

***How to read these notes.** These notes are best read in stages. On a first reading, focus on the definitions, the principal examples, and the statements of the main theorems. On a second reading, reconstruct the proofs without following the text line by line, paying special attention to the recurring patterns of argument that topology uses again and again. On a third reading, use the exercises and examples to test whether the abstract notions have become effective working tools. The notes are intended not only to present theorems but also to help the reader acquire the habits of thought by which topological arguments are actually built.*

Contents

Preface	i
1 Introduction	1
1.1 What topology abstracts	1
1.2 Recurring proof patterns	2
1.3 Roadmap of the notes	2
1.4 Prerequisites and conventions	2
2 Topological spaces, bases, and subbases	4
2.1 From metric spaces to topologies	4
Problem Set	11
3 Closed sets, subspaces, and set-theoretic operations	12
3.1 Closed sets and the subspace topology	12
3.2 Closure of a set	14
3.3 Interior of a set	14
3.4 Limit points, boundary, and related notions	15
Problem Set	20
4 Continuity and homeomorphisms	22
4.1 Continuous maps	22
Problem Set	31
5 Subspace, product, and quotient topologies	32
5.1 Finite products: the product topology on $X \times Y$	32
5.2 Products, box topology, and the universal property	34
5.3 Quotient spaces and quotient maps	36
Problem Set	41
6 Countability and separation axioms	43
6.1 Separation axioms	43
6.2 Countability axioms	45
Problem Set	47
7 Connectedness	48
7.1 Connected spaces and components	48
7.2 Path connectedness and local connectedness	50

Problem Set	51
8 Compactness and compactifications	52
8.1 Compact sets	52
8.2 Compactifications	66
Problem Set	69
9 Urysohn's lemma and Tietze's extension theorem	70
9.1 Urysohn's lemma	70
9.2 Tietze's extension theorem	71
Problem Set	73
10 Urysohn's metrization theorem	74
10.1 Statement and proof	74
Problem Set	76
11 Convergence: sequences, nets, and filters	78
11.1 Why sequences are not enough	78
11.2 Closure and cluster points via nets	79
11.3 Continuity in terms of nets	80
11.4 Filters and ultrafilters (optional but standard)	80
11.5 Countability and the sequential viewpoint	81
Problem Set	82
12 Baire category	84
12.1 The Baire category theorem	84
Notation and Glossary	86
Problem Sets	88
Problem Set 1: Topological spaces and bases	88
Problem Set 2: Continuous maps and homeomorphisms	95
Problem Set 3: Quotient spaces and compact spaces	99
Problem Set 4: Connected spaces	102
Problem Set 5: Countability and separation axioms	104
Problem Set 6: Nets, filters, and general convergence	108
Problem Set 7: Compactness, products, and compactifications	108
Problem Set 8: Baire category and applications	109
Problem Set 9: Urysohn–Tietze, metrization, and separation	110
References	110

Introduction

Topology is the language of “continuous” phenomena. It abstracts the parts of metric geometry that are genuinely needed to speak about continuity, convergence, and qualitative shape, while discarding the quantitative structure of distances. In these notes we develop the foundational definitions and tools of point-set topology (bases, closure and interior, continuous maps, and the standard constructions), and then study the structural axioms and global properties that organize the subject (separation, countability, connectedness, compactness). We conclude with classical theorems that illustrate the power of the abstract framework.

Roadmap.

- The opening sections explain what topology retains from metric space theory and why open sets rather than distances are the fundamental primitives.
- Several proof patterns are highlighted in advance so that later chapters can be read structurally: basis reduction, complement duality, and finite-subcover arguments.
- The roadmap section places the main theorems of the course in logical order, which is essential because later chapters repeatedly build on the constructions introduced early on.

1.1 What topology abstracts

The subject begins by asking which parts of metric reasoning remain once the metric itself is forgotten. The answer is that openness, continuity, and convergence can all be reformulated purely in terms of neighborhoods. This observation is the conceptual starting point of general topology.

In a metric space (X, d) , openness is encoded by open balls, and continuity can be expressed without ε - δ estimates: a function $f : X \rightarrow Y$ is continuous if and only if the inverse image of every open set in Y is open in X . Likewise, convergence can be described in terms of neighborhoods. Both ideas make sense without a metric once we decide which subsets of X are to be declared open.

A *topological space* is therefore a set X together with a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of subsets, called *open sets*, satisfying axioms modeled on the behavior of open subsets in metric spaces. Once the axioms are in place, the guiding philosophy is:

Two spaces are considered the same from the topological viewpoint if they are related by a homeomorphism (a bijective continuous map with continuous inverse).

Thus topology studies properties and constructions that are invariant under homeomorphisms. This includes both “local” information (neighborhood structure, separation and countability axioms) and “global” information (connectedness and compactness).

1.2 Recurring proof patterns

A first course in topology can seem definition-heavy, but the proofs themselves are built from a relatively small repertoire of recurring ideas. Recognizing these patterns early makes the subject much easier to learn and allows later arguments to be read structurally rather than line by line.

Point-set topology has a small number of argument patterns that appear repeatedly. Keeping them in mind makes proofs shorter and more conceptual.

Key ideas.

- **Basis arguments:** to prove a set is open (or a map is continuous), it is often enough to test the claim on a basis or subbasis.
- **Closure–interior dualities:** many statements come in pairs related by complements, e.g. closure vs. interior, closed vs. open maps.
- **Compactness via finite subcovers:** the typical compactness proof reduces an infinite condition to a finite one, often by a minimal-counterexample or refinement argument.

1.3 Roadmap of the notes

Topology is cumulative: later results depend not only on earlier definitions but also on a growing stock of standard constructions and proof techniques. A brief roadmap is therefore useful, both to indicate the logical order of the material and to explain why particular chapters appear where they do.

The development follows the standard logical progression of the subject. After establishing the language of topologies and the basic set-theoretic operations, we study continuous maps and homeomorphisms, then the three fundamental constructions (subspaces, products, and quotients). We next introduce the principal hierarchies of hypotheses (countability and separation axioms), and then the two central global properties (connectedness and compactness). The later chapters highlight cornerstone theorems (Urysohn–Tietze and metrization) that show how these notions interact.

1.4 Prerequisites and conventions

Because the course moves quickly from familiar metric-space examples to genuinely abstract spaces, it is helpful to make the standing conventions explicit at the outset. Doing so keeps the exposition uniform and avoids repeated interruptions later in the notes.

Basic set theory and familiarity with metric spaces (open and closed sets, convergence, continuity) are assumed. Unless stated otherwise, all spaces are nonempty. We use $\overline{A}$ for closure, $\text{int}(A)$ for interior, and ∂A for boundary. For maps $f : X \rightarrow Y$, the fiber over $y \in Y$ is $f^{-1}(\{y\})$ and the graph is $\{(x, f(x)) : x \in X\}$.

Topological spaces, bases, and subbases

We introduce topologies and the intuition of continuity coming from metric spaces. After basic examples (discrete, indiscrete, cofinite), we discuss comparison of topologies (finer/coarser) and develop the indispensable tools of bases and subbases for generating and working with topologies.

Roadmap.

- The main structural result is that a basis (respectively, subbasis) determines the smallest topology containing it; this is the mechanism by which most natural topologies are built.
- Basis comparison criteria are emphasized because they let one prove equality or inclusion of topologies locally, point by point.
- The lower-limit topology and the K -topology are included to show early that small changes in the basic neighborhoods can produce genuinely different spaces.

2.1 From metric spaces to topologies

The first conceptual step in topology is to separate the genuinely topological content of metric space theory from the numerical apparatus of distance. What survives is the behavior of open sets under arbitrary unions and finite intersections. Once this passage from geometry to axioms is understood, we can treat continuity and convergence in settings where no metric is available.

Let (X, d) be a metric space. A subset $O \subseteq X$ is *open* if for every $x \in O$ there exists $r > 0$ such that the open ball

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

is contained in O . In particular, every open set is a union of open balls.

Let $\mathcal{T}_d$ denote the collection of all open sets in (X, d) . Then:

- (i) $\emptyset \in \mathcal{T}_d$ and $X \in \mathcal{T}_d$;
- (ii) if $\{O_i\}_{i \in I} \subseteq \mathcal{T}_d$, then $\bigcup_{i \in I} O_i \in \mathcal{T}_d$;
- (iii) if $O_1, \dots, O_n \in \mathcal{T}_d$, then $\bigcap_{k=1}^n O_k \in \mathcal{T}_d$.

These three axioms do not mention the metric d explicitly; they depend only on the distinguished family $\mathcal{T}_d \subseteq \mathcal{P}(X)$. This motivates the abstract notion of a topology.

Definition 2.1. Let X be a set and let $\mathcal{T} \subseteq \mathcal{P}(X)$. We say that $\mathcal{T}$ is a *topology* on X if:

- (i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
- (ii) for every family $\{U_i\}_{i \in I} \subseteq \mathcal{T}$, the union $\bigcup_{i \in I} U_i$ belongs to $\mathcal{T}$;
- (iii) for every finite subfamily $U_1, \dots, U_n \in \mathcal{T}$, the intersection $\bigcap_{k=1}^n U_k$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*, and the members of $\mathcal{T}$ are called *open sets*.

Example 2.2. Two extremal examples occur on every nonempty set X :

- (a) $\mathcal{T}_{\text{ind}} = \{\emptyset, X\}$, the *indiscrete* topology;
- (b) $\mathcal{T}_{\text{disc}} = \mathcal{P}(X)$, the *discrete* topology.

If $|X| \geq 3$, there are many other topologies in between these two extremes.

If $\mathcal{T}_1, \mathcal{T}_2$ are topologies on X , we say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ (and $\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$) if $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Thus $\mathcal{T}_{\text{disc}}$ is the finest topology on X , and $\mathcal{T}_{\text{ind}}$ is the coarsest.

Example 2.3. Let $X = \{a, b, c\}$. Besides the indiscrete and discrete topologies, there are many intermediate topologies on X . We record a few typical examples. (Counting the number of distinct topologies on an n -point set is a difficult combinatorial problem; closed formulas are not known.)

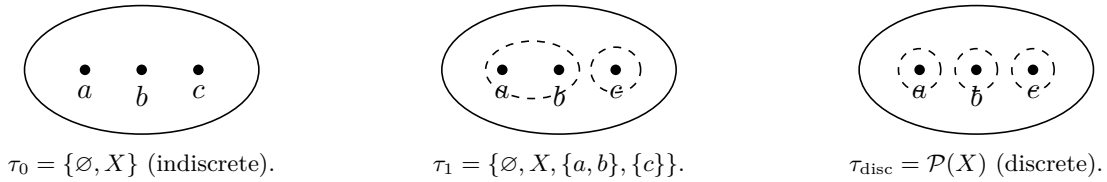


Figure 2.1: Schematic depiction of some topologies on $X = \{a, b, c\}$.

It is also instructive to write down collections that *fail* to be topologies. For instance,

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{b\}\} \quad \text{is not a topology, since } \{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1,$$

and

$$\mathcal{T}_2 = \{\emptyset, X, \{a, b\}, \{b, c\}\} \quad \text{is not a topology, since } \{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_2.$$

Example 2.4 (Cofinite topology). Let X be a set. The *cofinite topology* on X is

$$\mathcal{T}_{\text{cof}} = \{\emptyset\} \cup \{O \subseteq X : X \setminus O \text{ is finite}\}.$$

A direct verification shows that $\mathcal{T}_{\text{cof}}$ is a topology:

- $\emptyset \in \mathcal{T}_{\text{cof}}$ by definition and $X \in \mathcal{T}_{\text{cof}}$ since $X \setminus X = \emptyset$ is finite;
- if $\{O_i\}_{i \in I} \subseteq \mathcal{T}_{\text{cof}}$ and $O = \bigcup_{i \in I} O_i \neq \emptyset$, then $X \setminus O = \bigcap_{i \in I} (X \setminus O_i)$, an intersection of finite sets, hence finite;

- if $O_1, \dots, O_m \in \mathcal{T}_{\text{cof}}$ are nonempty, then $X \setminus \bigcap_{k=1}^m O_k = \bigcup_{k=1}^m (X \setminus O_k)$, a finite union of finite sets, hence finite.

Example 2.5 (Cocountable topology). Let X be a set. The *cocountable topology* on X is

$$\mathcal{T}_{\text{coc}} = \{\emptyset\} \cup \{O \subseteq X : X \setminus O \text{ is countable}\}.$$

The verification that $\mathcal{T}_{\text{coc}}$ is a topology is identical to the cofinite case, using that arbitrary intersections of countable sets are countable and finite unions of countable sets are countable.

Remark 2.6. If X is countable, then every subset of X has countable complement, hence $\mathcal{T}_{\text{coc}} = \mathcal{P}(X)$ is the discrete topology.

Definition 2.7 (Basis). Let X be a set. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* on X if

- (B1) for every $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$;
- (B2) whenever $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Proposition 2.8 (Topology generated by a basis). *Given a basis $\mathcal{B}$ on X , define*

$$\mathcal{T}(\mathcal{B}) = \left\{ O \subseteq X : \text{for every } x \in O \text{ there exists } B \in \mathcal{B} \text{ with } x \in B \subseteq O \right\}.$$

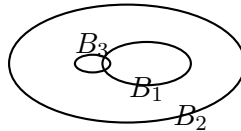
Then $\mathcal{T}(\mathcal{B})$ is a topology on X . Moreover, $\mathcal{B} \subseteq \mathcal{T}(\mathcal{B})$ and $\mathcal{T}(\mathcal{B})$ is the smallest topology on X containing $\mathcal{B}$.

Proof. By (B1), $X \in \mathcal{T}(\mathcal{B})$, and $\emptyset \in \mathcal{T}(\mathcal{B})$ holds vacuously. If $\{O_i\}_{i \in I} \subseteq \mathcal{T}(\mathcal{B})$ and $O = \bigcup_{i \in I} O_i$, then for each $x \in O$ we have $x \in O_{i_0}$ for some i_0 , and hence $x \in B \subseteq O_{i_0} \subseteq O$ for some $B \in \mathcal{B}$. Thus $O \in \mathcal{T}(\mathcal{B})$. If $O_1, \dots, O_n \in \mathcal{T}(\mathcal{B})$ and $x \in \bigcap_{k=1}^n O_k$, choose $B_k \in \mathcal{B}$ with $x \in B_k \subseteq O_k$. By repeated use of (B2), there exists $B \in \mathcal{B}$ with $x \in B \subseteq \bigcap_{k=1}^n B_k \subseteq \bigcap_{k=1}^n O_k$, so $\bigcap_{k=1}^n O_k \in \mathcal{T}(\mathcal{B})$.

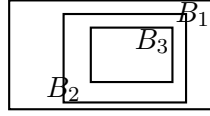
Finally, if $B \in \mathcal{B}$ and $x \in B$, then $x \in B \subseteq B$, so $B \in \mathcal{T}(\mathcal{B})$. If $\mathcal{T}$ is any topology on X with $\mathcal{B} \subseteq \mathcal{T}$, then every $O \in \mathcal{T}(\mathcal{B})$ is a union of sets in $\mathcal{B}$, hence a union of open sets in $\mathcal{T}$, and therefore $O \in \mathcal{T}$. This establishes the minimality assertion. $\square$

Remark 2.9. Equivalently, $\mathcal{T}(\mathcal{B})$ consists of all unions of basis elements: $O \subseteq X$ is open if and only if it can be written as $O = \bigcup_{\alpha \in A} B_\alpha$ with $B_\alpha \in \mathcal{B}$.

Example 2.10. Let $\mathcal{B}$ be the collection of all open discs in $\mathbb{R}^2$. Then $\mathcal{B}$ is a basis for the usual topology on $\mathbb{R}^2$.



Example 2.11. Let $\mathcal{B}'$ be the collection of all open rectangles in the plane $\mathbb{R}^2$. Then $\mathcal{B}'$ is a basis. Indeed, condition (B2) is immediate.



We will see later that both $\mathcal{B}$ and $\mathcal{B}'$ generate the same topology on $\mathbb{R}^2$, namely the usual topology.

Lemma 2.12. *Let X be a set, and let $\mathcal{B} \subset \mathcal{P}(X)$ be a basis for a topology $\mathcal{T}$ on X . Then*

$$\mathcal{T} = \left\{ \bigcup_{\alpha \in A} B_{\alpha} : B_{\alpha} \in \mathcal{B} \right\},$$

that is, every open set is a union of basis elements, and conversely every union of basis elements is open.

Proof. Since $\mathcal{T} = \mathcal{T}(\mathcal{B})$, each basis element belongs to $\mathcal{T}$; hence every union of members of $\mathcal{B}$ is open.

Conversely, let $O \in \mathcal{T}$. By the defining property of a basis, for each $x \in O$ there exists $B_x \in \mathcal{B}$ such that

$$x \in B_x \subset O.$$

Therefore

$$O = \bigcup_{x \in O} B_x.$$

Thus every open set is a union of basis elements. The representation is generally not unique. $\square$

Remark 2.13. The two basis axioms are purely local: one asks first that every point lie in at least one basis element, and then that intersections of basis elements can be refined by smaller basis elements around each common point. Once these conditions hold, the topology is recovered simply by taking unions of basis elements.

In practice, one often uses this criterion in reverse: rather than starting with a topology and extracting a basis, one first specifies a convenient family of sets and then checks that it satisfies the basis axioms.

Lemma 2.14. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be such that for every open set $O \in \mathcal{T}$ and every point $x \in O$, there exists $C \in \mathcal{C}$ with*

$$x \in C \subset O.$$

Then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Proof. We first verify the basis axioms. Since $X \in \mathcal{T}$, for every $x \in X$ there exists $C \in \mathcal{C}$ such that $x \in C \subset X$. Thus every point of X lies in some member of $\mathcal{C}$.

Next, let $C_1, C_2 \in \mathcal{C}$ and let $x \in C_1 \cap C_2$. Because $C_1 \cap C_2$ is open in X and contains x , the hypothesis yields $C_3 \in \mathcal{C}$ such that

$$x \in C_3 \subset C_1 \cap C_2.$$

Hence $\mathcal{C}$ is a basis for some topology on X , which we denote by $\mathcal{T}'$.

It remains to show that $\mathcal{T}' = \mathcal{T}$. If $O \in \mathcal{T}$, then by hypothesis each $x \in O$ belongs to some $C_x \in \mathcal{C}$ with $C_x \subset O$, so

$$O = \bigcup_{x \in O} C_x.$$

Therefore $O \in \mathcal{T}'$, and hence $\mathcal{T} \subseteq \mathcal{T}'$. Conversely, every member of $\mathcal{C}$ already belongs to $\mathcal{T}$, so every union of members of $\mathcal{C}$ is open in $\mathcal{T}$. Thus $\mathcal{T}' \subseteq \mathcal{T}$, as required. $\square$

If we know bases for two topologies on the same set, it is useful to have a basis-level criterion for comparing them.

Lemma 2.15. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X . Then the following are equivalent:*

- (i) $\mathcal{T}'$ is finer than $\mathcal{T}$, that is, $\mathcal{T} \subseteq \mathcal{T}'$.
- (ii) For every $x \in X$ and every $B \in \mathcal{B}$ with $x \in B$, there exists $B' \in \mathcal{B}'$ such that

$$x \in B' \subset B.$$

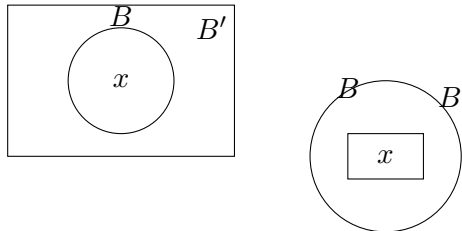
Proof. Assume (ii), and let $O \in \mathcal{T}$. For each $x \in O$, choose $B \in \mathcal{B}$ such that $x \in B \subset O$. By (ii) there exists $B' \in \mathcal{B}'$ with $x \in B' \subset B \subset O$. Hence every point of O lies in a $\mathcal{B}'$ -element contained in O , so $O \in \mathcal{T}'$. Therefore $\mathcal{T} \subseteq \mathcal{T}'$, proving (i).

Conversely, assume (i), and let $x \in B \in \mathcal{B}$. Since $B \in \mathcal{T} \subseteq \mathcal{T}'$, and $\mathcal{B}'$ is a basis for $\mathcal{T}'$, there exists $B' \in \mathcal{B}'$ such that

$$x \in B' \subset B.$$

Thus (ii) holds. $\square$

The lemma immediately shows that the topology generated by open discs in $\mathbb{R}^2$ agrees with the topology generated by open rectangles. Indeed, each open disc contains a sufficiently small open rectangle around any of its points, and each open rectangle contains a sufficiently small open disc around any of its points.



Consequently, the two families define the same topology on $\mathbb{R}^2$.

Example 2.16. Let

$$\mathcal{B} = \{(a, b) : a < b\}, \quad \mathcal{B}' = \{[a, b) : a < b\},$$

and let

$$K = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}, \quad \mathcal{B}'' = \mathcal{B} \cup \{(c, d) \setminus K : c < d\}.$$

Denote by $\mathcal{T} = \mathcal{T}(\mathcal{B})$ the usual topology on $\mathbb{R}$, by $\mathcal{T}' = \mathcal{T}(\mathcal{B}')$ the lower limit topology, and by $\mathcal{T}'' = \mathcal{T}(\mathcal{B}'')$ the K -topology. Then $\mathcal{T}'$ and $\mathcal{T}''$ are both strictly finer than $\mathcal{T}$, but $\mathcal{T}'$ and $\mathcal{T}''$ are not comparable.

Indeed, every Euclidean open interval belongs to $\mathcal{T}'$ because

$$(a, b) = \bigcup_{x \in (a, b)} [x, b).$$

Hence $\mathcal{T} \subseteq \mathcal{T}'$. The inclusion is strict because $[a, b) \in \mathcal{T}'$ whereas $[a, b) \notin \mathcal{T}$.

Likewise $\mathcal{T} \subseteq \mathcal{T}''$ because $\mathcal{B} \subseteq \mathcal{B}''$. The inclusion is strict since $(-1, 1) \setminus K \in \mathcal{T}''$, but this set is not Euclidean open: it contains 0, and every Euclidean open interval about 0 contains some $1/n \in K$.

Finally, the two finer topologies are not comparable. The set $[0, 1)$ is open in $\mathcal{T}'$ but not in $\mathcal{T}''$, because every $\mathcal{T}''$ -basic neighborhood of 0 contains either negative numbers or points of K . On the other hand, $(-1, 1) \setminus K$ is open in $\mathcal{T}''$ but not in $\mathcal{T}'$, since every $\mathcal{T}'$ -basic neighborhood $[0, \varepsilon)$ of 0 contains points of K .

Definition 2.17 (Subbasis). Let X be a set and let $\mathcal{S} \subseteq \mathcal{P}(X)$. Define

$$\mathcal{B}(\mathcal{S}) := \left\{ \bigcap_{k=1}^n S_k : n \in \mathbb{N}, S_k \in \mathcal{S} \right\} \cup \{X\},$$

where X is interpreted as the intersection of the empty family. The collection $\mathcal{S}$ is called a *subbasis* on X if $\mathcal{B}(\mathcal{S})$ is a basis on X . The topology generated by $\mathcal{S}$ is, by definition, the topology generated by the basis $\mathcal{B}(\mathcal{S})$.

Proposition 2.18 (Topology generated by a subbasis). *Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a subbasis on X . Then the topology generated by $\mathcal{S}$ is the smallest topology on X containing $\mathcal{S}$. In particular, when $\mathcal{S} = \emptyset$, the generated topology is the indiscrete topology $\{\emptyset, X\}$.*

Proof. By construction, every member of $\mathcal{S}$ belongs to $\mathcal{B}(\mathcal{S})$, hence is open in the topology generated by $\mathcal{B}(\mathcal{S})$. Thus the generated topology contains $\mathcal{S}$. Conversely, if $\mathcal{T}$ is any topology on X with $\mathcal{S} \subseteq \mathcal{T}$, then every finite intersection of members of $\mathcal{S}$ is in $\mathcal{T}$, because topologies are closed under finite intersections. Therefore $\mathcal{B}(\mathcal{S}) \subseteq \mathcal{T}$, and since every set open in the topology generated by $\mathcal{S}$ is a union of elements of $\mathcal{B}(\mathcal{S})$, it follows that the generated topology is contained in $\mathcal{T}$. This proves the minimality assertion. $\square$

Example 2.19. We know that

$$\mathcal{B} = \{(a, b) : a < b, a, b \in \mathbb{R}\}$$

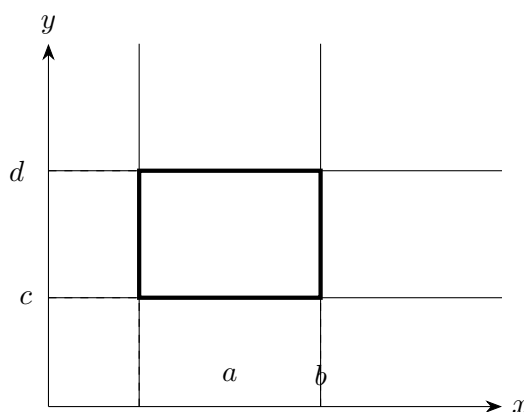
is a basis for the usual topology on $\mathbb{R}$.

$$\mathcal{J} = \{(a, +\infty), (-\infty, b) : a < b, a, b \in \mathbb{R}\}$$

is an open subbasis for the usual topology $\mathcal{U}$ on $\mathbb{R}$.

Example 2.20. We know that open rectangles are a basis for the usual topology on $\mathbb{R}^2$.

But any open rectangle can be written as intersection of two strips.



Thus, open strips form a subbasis for the usual topology on $\mathbb{R}^2$.

$$(a, b) \times \mathbb{R} \cap \mathbb{R} \times (c, d) = (a, b) \times (c, d).$$

$$\mathcal{J} = \{(a, b) \times \mathbb{R} : a, b \in \mathbb{R}, a < b\} \cup \{\mathbb{R} \times (c, d) : c, d \in \mathbb{R}, c < d\}$$

is a subbasis for the usual topology on $\mathbb{R}^2$.

Supplementary remarks and examples

Remark 2.21. A basis is local data: it tells us what neighborhoods look like near each point. Different bases may generate exactly the same topology, and in practice one often chooses a basis that makes computations easier rather than a basis that is somehow canonical.

Example 2.22. Let

$$\mathcal{B}_{\mathbb{Q}} = \{(a, b) : a, b \in \mathbb{Q}, a < b\}.$$

Then $\mathcal{B}_{\mathbb{Q}}$ is countable and is still a basis for the usual topology on $\mathbb{R}$. Thus the Euclidean topology can be recovered from a very small amount of local data.

Problem Set

Problem guide. *The problems in this chapter are best approached in the same order as the exposition. Start with basis and subbasis recognition problems, then move to comparison of generated topologies, and finally test whether you can build or reconstruct examples from minimal local information. A good check on understanding is whether you can move fluently between “open-set language” and “basis language.”*

Exercise 2.23. Let $\mathcal{B} \subseteq \mathcal{P}(X)$. Prove that $\mathcal{B}$ is a basis for some topology on X if and only if (i) $\bigcup \mathcal{B} = X$ and (ii) for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Exercise 2.24. Let $\mathcal{S} \subseteq \mathcal{P}(X)$. Describe explicitly the topology generated by $\mathcal{S}$ as a subbasis, and prove that it is the smallest topology containing $\mathcal{S}$.

Exercise 2.25. On an infinite set X , show that the cofinite topology is T_1 but not Hausdorff.

Exercise 2.26. On an uncountable set X , show that the cocountable topology is T_1 but not Hausdorff. What changes if X is countable?

Exercise 2.27. Let $\mathcal{T}_1 \subseteq \mathcal{T}_2$ be topologies on X . Prove that the identity map $\text{id}_X : (X, \mathcal{T}_2) \rightarrow (X, \mathcal{T}_1)$ is continuous, and that the converse holds.

Exercise 2.28. Let (X, d) be a metric space. Show that the family of all open balls with rational radii is a basis for $\mathcal{T}_d$.

Closed sets, subspaces, and set-theoretic operations

We study closed sets and the subspace topology, and then develop the fundamental set-theoretic operations in a topological space: closure, interior, boundary, and limit points. These notions govern density and nowhere density and provide a convenient language for formulating many later arguments.

Roadmap.

- The central theorem identifies closure through neighborhoods and then reinterprets it in terms of limit points, giving several equivalent ways to recognize when a point lies in $\overline{A}$.
- Subspace formulas are recorded carefully so that later arguments can move cleanly between ambient spaces and subspaces.
- Boundary, derived set, and isolated-point decompositions are treated as structural consequences of the closure-interior formalism rather than as isolated definitions.

3.1 Closed sets and the subspace topology

Open sets encode the local structure of a space, but many arguments are more naturally phrased in terms of their complements. Closed sets are therefore not secondary objects: they provide an equivalent language in which limit phenomena, compactness arguments, and restriction to subsets can be expressed cleanly. The subspace topology explains how a subset inherits exactly the open sets needed to regard it as a topological space in its own right.

Definition 3.1. A subset $A \subseteq X$ of a topological space X is said to be *closed* if its complement $X \setminus A$ is open.

Example 3.2. In $(\mathbb{R}, \mathcal{U})$, the interval $[a, b]$ is closed because

$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty),$$

which is open in the usual topology.

Example 3.3. If X carries the cofinite topology, then the closed sets are precisely X and the finite subsets of X .

Example 3.4. If X carries the discrete topology, then every subset of X is both open and closed.

Proposition 3.5. *Let $(X, \mathcal{T})$ be a topological space. Then the family of closed subsets of X satisfies the dual axioms to those of the open sets:*

- (i) $\emptyset$ and X are closed;
- (ii) arbitrary intersections of closed sets are closed;
- (iii) finite unions of closed sets are closed.

Proof. Each assertion follows immediately by taking complements in the corresponding axioms for open sets together with De Morgan's laws. $\square$

Definition 3.6. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. The collection

$$\mathcal{T}_Y := \{O \cap Y : O \in \mathcal{T}\}$$

defines a topology on Y , called the *subspace topology*. Thus the open sets of Y are precisely the traces on Y of open sets in the ambient space X .

Remark 3.7. The subspace topology is the natural mechanism for restricting a topology to a subset. Many arguments in topology proceed by passing from a space to a subspace and comparing the corresponding notions of openness, closedness, closure, and continuity.

Example 3.8. Let $Y = [0, 1] \cup (2, 3)$, regarded as a subspace of $\mathbb{R}$. Then $[0, 1]$ is open in Y , since

$$[0, 1] = \left(-\frac{1}{2}, \frac{3}{2}\right) \cap Y.$$

Similarly, $(2, 3)$ is open in Y . Because these two subsets are complements of one another in Y , each of them is both open and closed in the subspace topology.

Proposition 3.9. *Let Y be a subspace of a topological space X . A subset $A \subseteq Y$ is closed in Y if and only if there exists a closed set $C \subseteq X$ such that*

$$A = C \cap Y.$$

Proof. Assume first that A is closed in Y . Then $Y \setminus A$ is open in the subspace topology, so there exists an open set $O \subseteq X$ such that

$$Y \setminus A = Y \cap O.$$

Let $C := X \setminus O$, which is closed in X . Then

$$A = Y \setminus (Y \cap O) = Y \cap (X \setminus O) = Y \cap C.$$

Conversely, if $A = Y \cap C$ for some closed set $C \subseteq X$, then

$$Y \setminus A = Y \setminus (Y \cap C) = Y \cap (X \setminus C),$$

and $X \setminus C$ is open in X . Hence $Y \setminus A$ is open in Y , so A is closed in the subspace topology. $\square$

3.2 Closure of a set

The closure of a set records the points that cannot be separated from it by open neighborhoods. It is the smallest closed enlargement of the set and thus formalizes the idea of adjoining all possible limit points. This operation appears throughout topology, analysis, and geometry, so it is important to understand both its abstract definition and its behavior under passage to subspaces.

The closure of a subset $A \subseteq X$ is the smallest closed subset of X containing A ; equivalently,

$$\overline{A} = \bigcap \{F \subseteq X : F \text{ is closed and } A \subseteq F\}.$$

The next theorem explains how closure behaves when one passes to a subspace.

Theorem 3.10. *Let Y be a subspace of X , and let $A \subseteq Y$. If $\overline{A}^X$ denotes the closure of A in X and $\overline{A}^Y$ denotes the closure of A in Y , then*

$$\overline{A}^Y = \overline{A}^X \cap Y.$$

Proof. Since $\overline{A}^X$ is closed in X , the set $\overline{A}^X \cap Y$ is closed in the subspace Y . Moreover, because $A \subseteq \overline{A}^X$ and $A \subseteq Y$, we have

$$A \subseteq \overline{A}^X \cap Y.$$

Hence $\overline{A}^Y \subseteq \overline{A}^X \cap Y$, by the minimality of $\overline{A}^Y$ among closed subsets of Y containing A .

Conversely, let $B \subseteq Y$ be any closed subset of Y with $A \subseteq B$. By the definition of the subspace topology, there exists a closed set $C \subseteq X$ such that $B = C \cap Y$. Since $A \subseteq B \subseteq C$, the minimality of $\overline{A}^X$ gives $\overline{A}^X \subseteq C$. Intersecting with Y , we obtain

$$\overline{A}^X \cap Y \subseteq C \cap Y = B.$$

This inclusion holds for every closed subset B of Y containing A ; in particular, it holds for $B = \overline{A}^Y$. Therefore $\overline{A}^X \cap Y \subseteq \overline{A}^Y$, and the proof is complete. $\square$

3.3 Interior of a set

Whereas closure enlarges a set by adding limiting behavior, interior extracts the part of the set that is topologically robust. A point lies in the interior precisely when the set contains an entire neighborhood of that point. The duality between closure and interior, mediated by complements, is one of the most useful formal patterns in elementary topology.

The interior of a subset $A \subseteq X$ is the largest open subset of X contained in A . Equivalently,

$$A^\circ = \bigcup \{O \subseteq X : O \text{ is open and } O \subseteq A\}.$$

Example 3.11. In $(\mathbb{R}, \mathcal{U})$, the set $\mathbb{Q}$ has empty interior. Indeed, every nonempty open interval contains irrational numbers, so no open interval is contained in $\mathbb{Q}$. Hence

$$\text{int}(\mathbb{Q}) = \emptyset.$$

Example 3.12. The Cantor set $C \subseteq [0, 1]$ has empty interior. At each stage of the construction one removes a middle open interval from every surviving interval, so no nonempty open interval can remain inside C . Therefore

$$\text{int}(C) = \emptyset.$$

Example 3.13. Let

$$A = (\mathbb{R} \setminus \mathbb{Q}) \cup [0, 1].$$

Since $\mathbb{R} \setminus \mathbb{Q}$ has empty interior in the usual topology, while $[0, 1]^\circ = (0, 1)$, it follows that

$$\text{int}(A) = (0, 1).$$

3.4 Limit points, boundary, and related notions

Closure and interior become especially effective once they are reinterpreted through neighborhoods, accumulation, and boundary behavior. These notions allow one to describe where a set is locally persistent, where it is thin, and where it changes from inside to outside. They also provide the correct vocabulary for density, nowhere density, perfection, and several later constructions.

The set-theoretic formulas for closure and interior are conceptually important, but they are often inconvenient for computation. The next results recast these notions in terms of neighborhoods and basis elements, which is the form most frequently used in practice.

Theorem 3.14. *Let A be a subset of a topological space (X, τ) . Then:*

1. *$x \in \overline{A}$ if and only if every open set $O \ni x$ intersects A .*
2. *If $\mathcal{B}$ is a basis for τ , then $x \in \overline{A}$ if and only if every basis element $B \in \mathcal{B}$ with $x \in B$ intersects A .*

Proof. For (1), note that $x \notin \overline{A}$ if and only if $x \in X \setminus \overline{A}$. Since $X \setminus \overline{A}$ is open, this is equivalent to the existence of an open set $O \ni x$ such that $O \cap A = \emptyset$. Taking complements gives the stated criterion.

Statement (2) is immediate from (1) in one direction, because every basis element is open. Conversely, assume every basis element containing x intersects A , and let O be any open set containing x . Since $\mathcal{B}$ is a basis, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq O$. By hypothesis, $B \cap A \neq \emptyset$, hence also $O \cap A \neq \emptyset$. Therefore $x \in \overline{A}$ by (1). $\square$

Example 3.15. Let A be a subset of a metric space (X, d) . Then

$$x \in \overline{A} \iff B_\varepsilon(x) \cap A \neq \emptyset \text{ for every } \varepsilon > 0.$$

This is exactly Theorem 3.14(2) applied to the standard basis of open balls. In first-countable spaces, this is also equivalent to the existence of a sequence $(x_n) \subseteq A$ with $x_n \rightarrow x$.

Corollary 3.16. Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every neighborhood of x intersects A .

A point $x \in X$ is called a *limit point* (or *accumulation point*) of A if every neighborhood of x meets $A \setminus \{x\}$. The set of all limit points of A is denoted by A' and is called the *derived set* of A .

Example 3.17. For $A \subseteq X$ and $x \in X$,

$$x \in A' \iff x \in \overline{A \setminus \{x\}}.$$

Proof. This is a direct reformulation of the definition: every neighborhood of x meets $A \setminus \{x\}$ if and only if every neighborhood of x intersects $A \setminus \{x\}$, which is equivalent to $x \in \overline{A \setminus \{x\}}$ by Corollary 3.16. $\square$

Theorem 3.18. Let $A \subseteq X$. Then

$$\overline{A} = A \cup A'.$$

Proof. If $x \in A$, then certainly $x \in \overline{A}$, so $A \subseteq \overline{A}$. If $x \in A'$, every neighborhood of x meets $A \setminus \{x\} \subseteq A$; hence $x \in \overline{A}$ by Corollary 3.16. Thus $A \cup A' \subseteq \overline{A}$.

Conversely, let $x \in \overline{A}$. If $x \in A$, there is nothing to prove. If $x \notin A$, then every neighborhood of x meets A , and since $x \notin A$ this is equivalent to meeting $A \setminus \{x\}$. Hence $x \in A'$. Therefore $\overline{A} \subseteq A \cup A'$, and the result follows. $\square$

Example 3.19. For any subsets $A, B \subseteq X$,

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Indeed, $\overline{A \cup B}$ is a closed set containing $A \cup B$, so by minimality of closure,

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$

On the other hand, $A \subseteq A \cup B$ and $B \subseteq A \cup B$, hence

$$\overline{A} \subseteq \overline{A \cup B} \quad \text{and} \quad \overline{B} \subseteq \overline{A \cup B}.$$

Therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$, proving the equality.

By contrast, the inclusion

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

may be strict. For example, if $A = [0, 1]$ and $B = [1, 2]$, then $A \cap B = \emptyset$, whereas

$$\overline{A} \cap \overline{B} = \{1\}.$$

Notice also that the equality $\overline{A} = A \cup A'$ need not be disjoint. This motivates the separate notion of an isolated point, which allows one to decompose a set into isolated and accumulation behavior more cleanly.

Definition 3.20. Let $A \subseteq X$, where X is a topological space. A point $x \in A$ is called an *isolated point* of A if there exists a neighborhood N_x of x such that

$$N_x \cap A = \{x\}.$$

The set of all isolated points of A is denoted by $\text{iso}(A)$.

Theorem 3.21. Let X be a topological space and let $A \subseteq X$. Then

$$\overline{A} = \text{iso}(A) \sqcup A',$$

where $\text{iso}(A)$ is the set of isolated points of A and A' is the derived set of A .

Proof. First note that $\text{iso}(A) \subseteq A \subseteq \overline{A}$, and every limit point of A lies in $\overline{A}$, so $A' \subseteq \overline{A}$.

Conversely, let $x \in \overline{A}$. If $x \in A'$, there is nothing to prove. Suppose $x \notin A'$. Then there exists a neighborhood U of x such that

$$U \cap (A \setminus \{x\}) = \emptyset.$$

Since $x \in \overline{A}$, every neighborhood of x meets A ; in particular, $U \cap A \neq \emptyset$. Hence $x \in A$ and in fact $U \cap A = \{x\}$. Therefore $x \in \text{iso}(A)$. This proves $\overline{A} \subseteq \text{iso}(A) \cup A'$.

Finally, $\text{iso}(A) \cap A' = \emptyset$: if $x \in \text{iso}(A)$, then some neighborhood U of x satisfies $U \cap (A \setminus \{x\}) = \emptyset$, so $x \notin A'$. $\square$

Definition 3.22 (Boundary). For $A \subseteq X$, the *boundary* of A is

$$\partial(A) = \overline{A} \cap \overline{X \setminus A}.$$

Proposition 3.23. The boundary $\partial(A)$ is closed. Moreover, a point $x \in X$ lies in $\partial(A)$ if and only if every neighborhood of x meets both A and $X \setminus A$.

Proof. Closedness is immediate since $\partial(A)$ is an intersection of closed sets. For the neighborhood characterization, note that $x \in \overline{A}$ means every neighborhood of x intersects A , and $x \in \overline{X \setminus A}$ means every neighborhood of x intersects $X \setminus A$. Combining the two conditions gives the claim. $\square$

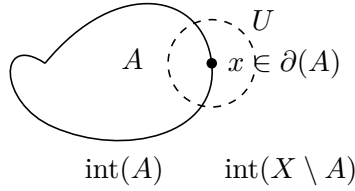


Figure 3.1: A boundary point x is characterized by neighborhoods U that meet both A and $X \setminus A$.

Example 3.24. In $(\mathbb{R}, \mathcal{U})$ with the usual topology,

$$\partial(\mathbb{N}) = \mathbb{N}, \quad \partial(\mathbb{Q}) = \mathbb{R}, \quad \partial(\mathbb{R}) = \emptyset.$$

Thus the boundary may be strictly larger than A (for example, $A = \mathbb{Q}$).

Example 3.25. In the usual topology on $\mathbb{R}$, the singleton $\{0\}$ is closed and its complement is dense. Hence

$$\partial(\{0\}) = \overline{\{0\}} \cap \overline{\mathbb{R} \setminus \{0\}} = \{0\} \cap \mathbb{R} = \{0\}.$$

Example 3.26. Similarly, the finite set $\{0, 1\}$ is closed in $\mathbb{R}$, and its complement is dense. Therefore

$$\partial(\{0, 1\}) = \{0, 1\}.$$

Example 3.27. A useful criterion is that $A \subseteq X$ is closed if and only if $\partial(A) \subseteq A$. Indeed, if A is closed, then $\overline{A} = A$, so

$$\partial(A) = \overline{A} \cap \overline{X \setminus A} = A \cap \overline{X \setminus A} \subseteq A.$$

Conversely, if $\partial(A) \subseteq A$, then every point of $\overline{A} \setminus A$ would lie in $\partial(A)$, which is impossible. Thus $\overline{A} \subseteq A$, and therefore A is closed.

Example 3.28. If A is both open and closed, then its boundary is empty. Indeed,

$$\partial(A) = \overline{A} \cap \overline{X \setminus A} = A \cap (X \setminus A) = \emptyset.$$

Example 3.29. For every subset $A \subseteq X$,

$$\overline{A} = A^\circ \cup \partial(A).$$

Indeed, both A° and $\partial(A)$ are contained in $\overline{A}$. Conversely, if $x \in \overline{A}$ but $x \notin A^\circ$, then every neighborhood of x meets $X \setminus A$; since $x \in \overline{A}$, every neighborhood also meets A . Hence $x \in \partial(A)$.

Definition 3.30. A subset $A \subseteq X$ is called *perfect* if $A = A'$, equivalently, if A is closed and every point of A is a limit point of A .

Example 3.31. The Cantor set is a standard example of a perfect set: it is closed, and every point of it is an accumulation point of the set.

Definition 3.32. A subset $A \subseteq X$ is said to be *dense* in X if $\overline{A} = X$. Equivalently, every nonempty open subset of X meets A .

Example 3.33. In $(\mathbb{R}, \mathcal{U})$, both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense.

Example 3.34. Another standard identity is

$$\text{int}(A^c) = (\overline{A})^c.$$

Indeed, a point lies outside $\overline{A}$ exactly when it has an open neighborhood disjoint from A , which is equivalent to lying in the interior of A^c .

Definition 3.35. A subset $A \subseteq X$ is called *nowhere dense* if

$$\text{int}(\overline{A}) = \emptyset.$$

Example 3.36. The Cantor set is nowhere dense in $[0, 1]$: it is closed and has empty interior.

Example 3.37. If $A \subseteq X$ is closed, then A is nowhere dense if and only if $\text{int}(A) = \emptyset$. Indeed, for a closed set one has $\overline{A} = A$, so the defining condition $\text{int}(\overline{A}) = \emptyset$ becomes simply $\text{int}(A) = \emptyset$.

Example 3.38. If $F \subseteq X$ is closed, then $\partial(F)$ is nowhere dense. Indeed,

$$\partial(F) = F \cap \overline{X \setminus F} = F \setminus \text{int}(F),$$

so $\partial(F)$ is closed. Moreover, if $U \subseteq \partial(F)$ is open, then $U \subseteq F$; since F is closed, this implies $U \subseteq \text{int}(F)$, which is impossible because $\partial(F) \cap \text{int}(F) = \emptyset$. Hence $\partial(F)$ has empty interior and is therefore nowhere dense. This fails for arbitrary sets: in $(\mathbb{R}, \mathcal{U})$, the set $\mathbb{Q}$ has

$$\partial(\mathbb{Q}) = \mathbb{R},$$

which is certainly not nowhere dense.

Example 3.39. In a metric space (X, d) , closure and boundary admit useful distance descriptions:

$$\overline{A} = \{x \in X : d(x, A) = 0\}$$

and

$$\partial(A) = \{x \in X : d(x, A) = 0 \text{ and } d(x, X \setminus A) = 0\}.$$

These formulas are just metric versions of the neighborhood characterizations of closure and boundary.

Example 3.40. Two useful special cases are

$$\partial(X) = \partial(\emptyset) = \emptyset$$

and

$$\partial(X \setminus A) = \partial(A).$$

The first identity is immediate. For the second,

$$\partial(X \setminus A) = \overline{X \setminus A} \cap \overline{X \setminus (X \setminus A)} = \overline{X \setminus A} \cap \overline{A} = \partial(A).$$

Example 3.41. The boundary identities may be summarized as follows:

- (i) $\overline{A} = A^\circ \cup \partial(A)$,
- (ii) $A^\circ = A \setminus \partial(A)$,
- (iii) $X = A^\circ \sqcup \partial(A) \sqcup (X \setminus A)^\circ$.

In particular, every point of X is either an interior point of A , a boundary point of A , or an interior point of the complement, and these three possibilities are mutually exclusive.

Supplementary remarks and examples

Remark 3.42. The operations of closure and interior are dual under complements:

$$\text{int}(A) = X \setminus \overline{(X \setminus A)}, \quad \overline{(A)} = X \setminus \text{int}(X \setminus A).$$

Many formal identities are most quickly proved by passing from one formula to the other through this duality.

Example 3.43. In $\mathbb{R}$ with the usual topology, the set $\mathbb{Q}$ satisfies

$$\overline{(\mathbb{Q})} = \mathbb{R}, \quad \text{int}(\mathbb{Q}) = \emptyset, \quad \partial(\mathbb{Q}) = \mathbb{R}.$$

This single example is a useful stress test for almost every statement about closure, interior, and boundary.

Problem Set

Problem guide. Begin with formal identities for closure, interior, and boundary, since these become reusable lemmas later. Then work through subspace examples carefully: many mistakes in topology come from forgetting which ambient space is controlling openness or closedness. The final problems are most valuable if you solve them by proving inclusions both directly and by using complement dualities.

Exercise 3.44. Prove that $\overline{A}$ is the smallest closed subset of X containing A , and that $\text{int}(A)$ is the largest open subset of X contained in A .

Exercise 3.45. Show that for every $A \subseteq X$,

$$\partial A = \overline{A} \setminus \text{int}(A) = \overline{A} \cap \overline{X \setminus A}.$$

Exercise 3.46. Let $Y \subseteq X$ with the subspace topology. Prove that for any $A \subseteq Y$,

$$\overline{A}^Y = \overline{A}^X \cap Y \quad \text{and} \quad \text{int}_Y(A) = Y \cap \text{int}_X(A).$$

Exercise 3.47. Let $A \subseteq X$. Prove that A is dense in X if and only if every nonempty open subset of X meets A .

Exercise 3.48. Show that $A \subseteq X$ is nowhere dense if and only if $\text{int}(\overline{A}) = \emptyset$.

Exercise 3.49. Let X be a T_1 space. Show that the derived set A' is closed for every $A \subseteq X$.

Continuity and homeomorphisms

We define continuity for maps between topological spaces and collect standard equivalent characterizations (via open sets, closed sets, neighborhoods, and bases/subbases). We then discuss homeomorphisms and basic tools such as the pasting lemma and restrictions to subspaces.

Roadmap.

- The equivalence of the open-set, closed-set, closure, and neighborhood formulations of continuity is the chapter's conceptual core.
- Restriction lemmas and pasting principles are proved early because they recur throughout the later construction chapters.
- The final results characterize homeomorphisms, open maps, and closed maps in ways that make later quotient and compactness arguments much more efficient.

4.1 Continuous maps

Topology is organized around the class of maps that preserve neighborhood structure. The topological definition of continuity isolates exactly the part of the metric notion that remains meaningful in arbitrary spaces. Once formulated in this way, continuity admits several equivalent characterizations, each of which is useful in a different type of argument.

Continuity is the central organizing principle of topology. In metric spaces it is first encountered through the ε - δ definition; however, that formulation is only one manifestation of a more structural fact, namely that continuity is exactly the preservation of openness under inverse images. This observation survives unchanged in arbitrary topological spaces and therefore becomes the correct abstract definition.

Thus, for topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a map

$$f : X \rightarrow Y$$

is continuous if and only if $f^{-1}(O)$ is open in X for every open set $O \subseteq Y$. Equivalently, f is continuous at $x \in X$ if every neighborhood of $f(x)$ has inverse image containing a neighborhood of x . In symbols, for each neighborhood $N_{f(x)}$ of $f(x)$ there exists a neighborhood N_x of x such that

$$f(N_x) \subseteq N_{f(x)}, \quad \text{or equivalently} \quad N_x \subseteq f^{-1}(N_{f(x)}).$$

A major advantage of the topological definition is that one need not test continuity against all open sets. It is enough to test inverse images on a basis, or even on a subbasis, for the topology of the codomain. Indeed, if $\mathcal{B}_Y$ is a basis for Y , every open set $O \in \mathcal{T}_Y$ can be written as a union of members of $\mathcal{B}_Y$, so

$$f^{-1}(O) = \bigcup_{B \in \mathcal{B}_Y, B \subseteq O} f^{-1}(B).$$

Likewise, if $\mathcal{S}_Y$ is a subbasis, then finite intersections of members of $\mathcal{S}_Y$ form a basis, and inverse images commute with finite intersections. The following theorem records the standard equivalent formulations of continuity in a form that will be used repeatedly throughout the notes.

Theorem 4.1. *Let $f : X \rightarrow Y$ be a map between topological spaces $X = (X, \mathcal{T}_X)$ and $Y = (Y, \mathcal{T}_Y)$. The following statements are equivalent:*

1. *f is continuous.*
2. *For every basis $\mathcal{B}$ of Y and every $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X .*
3. *For every subbasis $\mathcal{S}$ of Y and every $S \in \mathcal{S}$, the set $f^{-1}(S)$ is open in X .*
4. *For every closed set $F \subseteq Y$, the preimage $f^{-1}(F)$ is closed in X .*
5. *For every $x \in X$ and every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.*
6. *For every $A \subseteq X$, one has $f(\overline{A}) \subseteq \overline{f(A)}$.*
7. *For every $B \subseteq Y$, one has $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.*

Proof. The equivalence of (1), (2), and (3) is the usual basis/subbasis criterion: every open set is a union of basis elements, and every basis element generated by a subbasis is a finite intersection of subbasic sets.

The equivalence of (1) and (4) is immediate from taking complements.

To see that (1) implies (5), let V be a neighborhood of $f(x)$. Choose an open set $O \subseteq V$ with $f(x) \in O$. Then $f^{-1}(O)$ is an open neighborhood of x , and $f(f^{-1}(O)) \subseteq O \subseteq V$. Conversely, assume (5). Let $O \subseteq Y$ be open. For each $x \in f^{-1}(O)$, apply (5) with $V = O$ to obtain a neighborhood U_x of x such that $f(U_x) \subseteq O$. Then $U_x \subseteq f^{-1}(O)$, so $f^{-1}(O)$ is open. Thus (5) implies (1).

Assume (1), and let $x \in \overline{A}$. If W is any neighborhood of $f(x)$, then $f^{-1}(W)$ is a neighborhood of x , so $f^{-1}(W) \cap A \neq \emptyset$. Hence $W \cap f(A) \neq \emptyset$, which shows that $f(x) \in \overline{f(A)}$. Therefore (1) implies (6).

Assume (6), and let $B \subseteq Y$. Applying (6) to $A = f^{-1}(B)$, we obtain

$$f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}.$$

Taking preimages under f yields

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}),$$

so (6) implies (7).

Finally, if (7) holds and $F \subseteq Y$ is closed, then

$$\overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) = f^{-1}(F).$$

Hence $f^{-1}(F)$ is closed, so (4) holds. Hence all seven statements are equivalent. $\square$

Lemma 4.2. *If $f : X \rightarrow Y$ is continuous and $A \subseteq X$, then the restriction $f|_A : A \rightarrow Y$ is continuous with respect to the subspace topology on A .*

Proof. Let $O \subseteq Y$ be open. Then

$$(f|_A)^{-1}(O) = A \cap f^{-1}(O),$$

which is open in the subspace topology on A because $f^{-1}(O)$ is open in X . $\square$

Theorem 4.3 (Pasting Lemma). *Let $X = A \cup B$, where either both A and B are open in X , or both are closed in X . Suppose $f : X \rightarrow Y$ is a map such that $f|_A$ and $f|_B$ are continuous and agree on $A \cap B$. Then f is continuous.*

Proof. Because the two restrictions agree on $A \cap B$, the map f is well defined. Let $O \subseteq Y$ be open. Then

$$f^{-1}(O) = ((f|_A)^{-1}(O)) \cup ((f|_B)^{-1}(O)).$$

If A and B are open, each term on the right is open in the corresponding subspace and therefore open in X ; hence $f^{-1}(O)$ is open. If A and B are closed, let $F \subseteq Y$ be closed. Then

$$f^{-1}(F) = ((f|_A)^{-1}(F)) \cup ((f|_B)^{-1}(F))$$

is a union of closed subsets of the closed sets A and B , hence is closed in X . Thus f is continuous in either case. $\square$

Theorem 4.4. *Let X , Y , and Z be topological spaces. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then $g \circ f : X \rightarrow Z$ is continuous.*

Proof. Let $O \subseteq Z$ be open. Since g is continuous, $g^{-1}(O)$ is open in Y . Since f is continuous, the preimage of this set under f is open in X . But

$$(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O)),$$

so $g \circ f$ is continuous. $\square$

Definition 4.5. A family $\{A_i : i \in I\}$ of subsets of X is called *locally finite* if for every $x \in X$ there exists a neighborhood U_x of x that meets only finitely many of the sets A_i .

Lemma 4.6. *Let $\{A_i\}_{i \in I}$ be a locally finite family of subsets of X .*

1. For every $B \subseteq X$, the family $\{A_i \cap B\}_{i \in I}$ is locally finite in the subspace B .
2. If each A_i is closed in X , then $\bigcup_{i \in I} A_i$ is closed in X .

Proof. For (1), if $x \in B$, choose a neighborhood U_x of x in X meeting only finitely many A_i . Then $U_x \cap B$ is a neighborhood of x in B meeting only finitely many $A_i \cap B$.

For (2), let $x \notin \bigcup_{i \in I} A_i$. Choose a neighborhood U_x meeting only $A_{i_1}, \dots, A_{i_n}$. Since each A_{i_k} is closed and does not contain x , the set

$$V_x = U_x \cap \bigcap_{k=1}^n (X \setminus A_{i_k})$$

is an open neighborhood of x disjoint from every A_i . Thus every point of the complement has an open neighborhood contained in the complement, so the union is closed. $\square$

Theorem 4.7. Let $\{A_i\}_{i \in I}$ be a cover of X . Assume either:

1. each A_i is open, or
2. each A_i is closed and the family is locally finite.

Then a set $B \subseteq X$ is open (respectively closed) in X if and only if each $B \cap A_i$ is open (respectively closed) in the subspace A_i .

Proof. If B is open (respectively closed) in X , then each $B \cap A_i$ is open (respectively closed) in the subspace topology on A_i .

Conversely, suppose each $B \cap A_i$ is open in A_i . If the A_i are open, then each $B \cap A_i$ is open in X , and hence

$$B = \bigcup_{i \in I} (B \cap A_i)$$

is open in X .

Now suppose each $B \cap A_i$ is closed in A_i , with the family $\{A_i\}$ locally finite and closed. Then each $B \cap A_i$ is closed in X , and by Lemma 4.6 the family $\{B \cap A_i\}$ is locally finite. Therefore

$$B = \bigcup_{i \in I} (B \cap A_i)$$

is closed in X . $\square$

Theorem 4.8. Let $\{A_\lambda\}_{\lambda \in I}$ be a cover of a topological space X such that either all A_λ are open, or all A_λ are closed and the family is locally finite. Suppose $f_\lambda : A_\lambda \rightarrow Y$ is continuous for each $\lambda \in I$, and that the maps agree on overlaps:

$$f_\lambda|_{A_\lambda \cap A_\mu} = f_\mu|_{A_\lambda \cap A_\mu} \quad (\lambda, \mu \in I).$$

Then there exists a unique continuous map $f : X \rightarrow Y$ such that $f|_{A_\lambda} = f_\lambda$ for every λ .

Proof. Define $f(x) = f_\lambda(x)$ whenever $x \in A_\lambda$. This is well defined because the maps agree on overlaps, and uniqueness is immediate from the definition.

Let $O \subseteq Y$ be open. For each λ ,

$$f^{-1}(O) \cap A_\lambda = f_\lambda^{-1}(O),$$

which is open in the subspace A_λ because f_λ is continuous. By Theorem 4.7, it follows that $f^{-1}(O)$ is open in X . Hence f is continuous. $\square$

Theorem 4.9. *Let $f : X \rightarrow Y$ be a bijection. The following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and open.*
3. *f is continuous and closed.*
4. *For every $A \subseteq X$, one has $f(\overline{A}) = \overline{f(A)}$.*

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are immediate: if f^{-1} is continuous, then images of open sets (respectively closed sets) under f are open (respectively closed). Conversely, if f is continuous and open, then for every open set $O \subseteq X$ we have

$$(f^{-1})^{-1}(O) = f(O),$$

which is open in Y ; hence f^{-1} is continuous. This proves (1) $\Leftrightarrow$ (2). The proof of (1) $\Leftrightarrow$ (3) is identical, using closed sets instead of open sets.

Assume (1). Since continuous maps satisfy $f(\overline{A}) \subseteq \overline{f(A)}$, it remains to prove the reverse inclusion. Apply the same inclusion to the continuous map f^{-1} and the set $f(A)$:

$$f^{-1}(\overline{f(A)}) \subseteq \overline{f^{-1}(f(A))} = \overline{A}.$$

Applying f gives $\overline{f(A)} \subseteq f(\overline{A})$, and therefore $f(\overline{A}) = \overline{f(A)}$.

Finally, assume (4). If $F \subseteq X$ is closed, then

$$f(F) = f(\overline{F}) = \overline{f(F)},$$

so $f(F)$ is closed in Y . Thus f is closed. Since (3) implies (1), we conclude that (4) also implies (1). Hence all four statements are equivalent. $\square$

Theorem 4.10. *Let $f : X \rightarrow Y$. The following are equivalent:*

1. *f is an open map.*
2. *For every $A \subseteq X$, one has $f(A^\circ) \subseteq (f(A))^\circ$.*
3. *If $\mathcal{B}$ is a basis for X , then $f(B)$ is open in Y for every $B \in \mathcal{B}$.*

4. For every $x \in X$ and every neighborhood N_x of x , there exists a neighborhood W of $f(x)$ such that $W \subseteq f(N_x)$.

Proof. (1) $\Rightarrow$ (2): Since A° is open and $A^\circ \subseteq A$, the set $f(A^\circ)$ is open and contained in $f(A)$, hence $f(A^\circ) \subseteq (f(A))^\circ$.

(2) $\Rightarrow$ (3): If $B \in \mathcal{B}$, then $B = B^\circ$, so

$$f(B) = f(B^\circ) \subseteq (f(B))^\circ \subseteq f(B).$$

Thus $f(B) = (f(B))^\circ$ is open.

(3) $\Rightarrow$ (4): Let N_x be a neighborhood of x . Choose $B \in \mathcal{B}$ with $x \in B \subseteq N_x$. Then $f(B)$ is an open neighborhood of $f(x)$ contained in $f(N_x)$.

(4) $\Rightarrow$ (1): Let $O \subseteq X$ be open and let $y \in f(O)$. Choose $x \in O$ with $f(x) = y$. Applying (4) to the neighborhood O of x , we obtain a neighborhood W_y of y such that $W_y \subseteq f(O)$. Therefore

$$f(O) = \bigcup_{y \in f(O)} W_y,$$

so $f(O)$ is open.

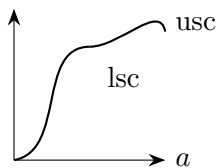
For maps into $\mathbb{R}$, it is often enough to test inverse images of rays. Indeed,

$$(a, b) = (-\infty, b) \cap (a, \infty), \quad (-\infty, b) = \bigcup_{n=1}^{\infty} (-n, b), \quad (a, \infty) = \bigcup_{n=1}^{\infty} (a, n).$$

Hence a map $f : X \rightarrow \mathbb{R}$ is continuous if and only if the sets $\{x : f(x) < b\}$ and $\{x : f(x) > a\}$ are open for all $a, b \in \mathbb{R}$. This leads naturally to lower and upper semicontinuity. $\square$

Definition 4.11. A function $f : X \rightarrow \mathbb{R}$ is called *lower semicontinuous* if $\{x : f(x) > a\}$ is open for each $a \in \mathbb{R}$. Similarly, f is called *upper semicontinuous* if $\{x : f(x) < a\}$ is open for each $a \in \mathbb{R}$.

Accordingly, f is continuous if and only if it is both lower semicontinuous (lsc) and upper semicontinuous (usc).



Example 4.12. A sequential characterization of lower semicontinuity on $\mathbb{R}$ is the following: a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is lower semicontinuous if and only if

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$$

whenever $x_n \rightarrow x$.

Example 4.13. Similarly, $f : \mathbb{R} \rightarrow \mathbb{R}$ is upper semicontinuous if and only if

$$\limsup_{n \rightarrow \infty} f(x_n) \leq f(x)$$

for every sequence $x_n \rightarrow x$. Consequently, f is continuous if and only if

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

for every sequence $x_n \rightarrow x$; equivalently, f is continuous precisely when it is both lower semicontinuous and upper semicontinuous.

The next theorem is stated in a form that extends naturally to other settings, for example in measure theory and in the study of measurable functions.

Theorem 4.14. *Let $f, g : X \rightarrow \mathbb{R}$ be continuous. Then:*

1. $|f|$ is continuous;
2. $af + bg$ is continuous for all $a, b \in \mathbb{R}$;
3. fg is continuous;
4. if $g(x) \neq 0$ for all $x \in X$, then f/g is continuous (in particular, $1/f$ is continuous whenever f is nowhere zero).

Proof. The maps $t \mapsto |t|$, $(s, t) \mapsto as + bt$, $(s, t) \mapsto st$, and $t \mapsto 1/t$ on $\mathbb{R} \setminus \{0\}$ are continuous. Therefore each assertion follows by composing these continuous maps with f and g . If one prefers explicit formulas, note that

$$fg = \frac{1}{4}((f+g)^2 - (f-g)^2), \quad \frac{f}{g} = f \cdot \frac{1}{g}.$$

This proves the result. □

Example 4.15. The arc-length functional

$$L(f) = \int_0^1 \sqrt{1 + (f'(t))^2} dt$$

on $C^1([0, 1])$ is a standard example of a lower semicontinuous map with values in $\mathbb{R}$.

Example 4.16. Let $\{f_\alpha\}_{\alpha \in I}$ be a family of lower semicontinuous functions on X such that for each $x \in X$ the set $\{f_\alpha(x) : \alpha \in I\}$ is bounded above. Define

$$g(x) = \sup_{\alpha \in I} f_\alpha(x).$$

Then g is lower semicontinuous. Indeed, for every $a \in \mathbb{R}$ we have

$$\{x : g(x) > a\} = \bigcup_{\alpha \in I} \{x : f_\alpha(x) > a\},$$

and the right-hand side is open as a union of open sets.

Example 4.17. There is a completely analogous statement for upper semicontinuous functions: the pointwise infimum of any family of upper semicontinuous functions is upper semicontinuous, provided it is finite everywhere.

Example 4.18. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is right-continuous in the ordinary sense, that is,

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{for every } a \in \mathbb{R}.$$

Then f , viewed as a map

$$f : (\mathbb{R}, \tau_r) \rightarrow (\mathbb{R}, \nu),$$

where τ_r is the topology generated by intervals of the form $(a, b]$ and ν is the usual topology, is continuous.

Indeed, let $a \in \mathbb{R}$ and let V be a usual open neighborhood of $f(a)$. Choose $\varepsilon > 0$ such that $(f(a) - \varepsilon, f(a) + \varepsilon) \subset V$. By right-continuity there exists $\delta > 0$ with

$$f((a, a + \delta)) \subset (f(a) - \varepsilon, f(a) + \varepsilon) \subset V.$$

Thus every neighborhood of $f(a)$ contains the image of a τ_r -neighborhood of a , which is precisely the continuity criterion expressed in topological language.

Example 4.19. Let $f : X \rightarrow Y$ be a map. The following are equivalent:

- (i) f is continuous;
- (ii) for every $B \subseteq Y$, one has

$$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ;$$

- (iii) for every $B \subseteq Y$, one has

$$\partial(f^{-1}(B)) \subseteq f^{-1}(\partial B).$$

Proof. (i) $\Rightarrow$ (ii). Let $x \in f^{-1}(B^\circ)$. Then $f(x) \in B^\circ$, so there exists an open set $V \subseteq Y$ such that

$$f(x) \in V \subseteq B.$$

By continuity of f , the set $f^{-1}(V)$ is open in X , contains x , and satisfies

$$f^{-1}(V) \subseteq f^{-1}(B).$$

Hence $x \in (f^{-1}(B))^\circ$.

(ii) $\Rightarrow$ (iii). Let $x \in \partial(f^{-1}(B))$. Then $x \notin (f^{-1}(B))^\circ$. If $f(x) \in B^\circ$, then by (ii) we would have

$$x \in f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ,$$

a contradiction. Thus $f(x) \notin B^\circ$. Applying (ii) to $Y \setminus B$, we similarly obtain

$$f^{-1}((Y \setminus B)^\circ) \subseteq (X \setminus f^{-1}(B))^\circ.$$

Since $x \in \partial(f^{-1}(B))$, the point x does not belong to the interior of $X \setminus f^{-1}(B)$ either; hence $f(x) \notin (Y \setminus B)^\circ$. Therefore $f(x) \in \partial B$, which proves (iii).

(iii) $\Rightarrow$ (i). Let $O \subseteq Y$ be open. Then $Y \setminus O$ is closed, so $\partial(Y \setminus O) \subseteq Y \setminus O$. Applying (iii) with $B = Y \setminus O$, we get

$$\partial(f^{-1}(Y \setminus O)) \subseteq f^{-1}(\partial(Y \setminus O)) \subseteq f^{-1}(Y \setminus O).$$

Recall that a subset $A \subseteq X$ is closed if and only if $\partial A \subseteq A$. Hence $f^{-1}(Y \setminus O)$ is closed, and therefore $f^{-1}(O)$ is open. So f is continuous. $\square$

Example 4.20. Let $f, g : X \rightarrow \mathbb{R}$ be continuous.

(a) The set

$$A = \{x \in X : f(x) \leq g(x)\}$$

is closed. Indeed,

$$X \setminus A = \{x \in X : f(x) - g(x) > 0\},$$

and since $f - g$ is continuous, the right-hand side is open.

(b) The function

$$h(x) = \min\{f(x), g(x)\}$$

is continuous. One convenient formula is

$$h(x) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2},$$

and the right-hand side is continuous because sums, differences, absolute values, and scalar multiples preserve continuity.

Supplementary remarks and examples

Remark 4.21. Continuity becomes easier to verify when one chooses the right test family. Instead of checking preimages of all open sets, it is often enough to check preimages of basis elements in the codomain or neighborhood conditions at each point of the domain.

Example 4.22. Consider the identity map

$$\text{id} : (\mathbb{R}, \tau_\ell) \rightarrow (\mathbb{R}, \tau_u),$$

where τ_ℓ is the lower-limit topology generated by half-open intervals $[a, b)$, and τ_u is the usual topology. This map is continuous because every usual open interval is τ_ℓ -open. The reverse identity $(\mathbb{R}, \tau_u) \rightarrow (\mathbb{R}, \tau_\ell)$ is not continuous. The example is a convenient way to remember how finer and coarser topologies interact with continuity.

Problem Set

Problem guide. Treat this problem set as a catalogue of equivalent continuity tests. For each problem, ask first which language is most efficient: open sets, closed sets, neighborhoods, bases, restrictions, or compositions. The strongest solutions are the ones that explicitly justify why a chosen test family is sufficient.

Exercise 4.23. Let $f : X \rightarrow Y$ be a map between topological spaces. Prove that f is continuous if and only if $f^{-1}(C)$ is closed in X whenever C is closed in Y .

Exercise 4.24. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous. Prove that $g \circ f$ is continuous.

Exercise 4.25. Let $f : X \rightarrow Y$ be a bijection. Prove that f is a homeomorphism if and only if f is continuous and open (equivalently, continuous and closed).

Exercise 4.26 (Pasting lemma). Let $X = A \cup B$ where A, B are closed in X . If $f : A \rightarrow Y$ and $g : B \rightarrow Y$ are continuous and agree on $A \cap B$, show that the induced map $h : X \rightarrow Y$ is continuous.

Exercise 4.27. Give an example of a continuous bijection $f : X \rightarrow Y$ that is not a homeomorphism. (Hint: compare different topologies on the same set.)

Subspace, product, and quotient topologies

We discuss the three central ways to build new spaces from old ones: subspaces, products, and quotients. For products we emphasize the role of projections and contrast the product topology with the box topology in infinite products. For quotients we highlight the universal property and the role of quotient maps in constructing identification spaces.

Roadmap.

- For products, the main results are the basis description, the universal property with respect to coordinate projections, and the coordinatewise description of closure.
- For quotients, the key ideas are saturation, restriction criteria, and the factorization theorem that characterizes maps descending to the quotient.
- This chapter is foundational for geometry: later compactification and gluing arguments use these universal properties repeatedly.

5.1 Finite products: the product topology on $X \times Y$

Once subspaces have been understood, the next basic construction is to place spaces side by side and study them simultaneously. The product topology is designed so that the coordinate projections are continuous and so that local information in the product is encoded by local information in each factor.

Alongside the subspace construction, the product construction is one of the two most basic mechanisms for producing new topological spaces from old ones. For two spaces, the guiding principle is simple: a neighborhood of (x, y) should record independent neighborhood information about x in X and about y in Y . This leads to the family of open rectangles.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and define

$$\mathcal{B} := \{O \times W : O \in \mathcal{T}_X, W \in \mathcal{T}_Y\}.$$

We claim that $\mathcal{B}$ is a basis on $X \times Y$. Indeed:

- (i) for every $(x, y) \in X \times Y$, the rectangle $X \times Y$ itself belongs to $\mathcal{B}$ and contains (x, y) ;
- (ii) if $(x, y) \in (O_1 \times W_1) \cap (O_2 \times W_2)$, then

$$(x, y) \in (O_1 \cap O_2) \times (W_1 \cap W_2) \subseteq (O_1 \times W_1) \cap (O_2 \times W_2),$$

and the set $(O_1 \cap O_2) \times (W_1 \cap W_2)$ again belongs to $\mathcal{B}$.

Hence $\mathcal{B}$ is a basis. The topology generated by $\mathcal{B}$ is called the *product topology* on $X \times Y$.

It is often convenient to describe this topology in terms of the coordinate projections

$$\pi_1 : X \times Y \rightarrow X, \quad \pi_1(x, y) = x,$$

$$\pi_2 : X \times Y \rightarrow Y, \quad \pi_2(x, y) = y.$$

These maps are surjective whenever both X and Y are nonempty. Moreover,

$$\pi_1^{-1}(O) = O \times Y \quad (O \in \mathcal{T}_X),$$

$$\pi_2^{-1}(W) = X \times W \quad (W \in \mathcal{T}_Y),$$

and therefore

$$\pi_1^{-1}(O) \cap \pi_2^{-1}(W) = O \times W.$$

This observation shows that the product topology may equally well be described from a natural subbasis.

Theorem 5.1. *Let*

$$\mathcal{J} = \{\pi_1^{-1}(O) : O \in \mathcal{T}_X\} \cup \{\pi_2^{-1}(W) : W \in \mathcal{T}_Y\}.$$

Then $\mathcal{J}$ is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by the subbasis $\mathcal{J}$. Since each member of $\mathcal{J}$ is open in the product topology, we immediately obtain $\mathcal{T}' \subseteq \mathcal{T}$.

Conversely, every basic open rectangle in the product topology has the form

$$O \times W = \pi_1^{-1}(O) \cap \pi_2^{-1}(W), \quad O \in \mathcal{T}_X, W \in \mathcal{T}_Y.$$

Thus each basis rectangle is a finite intersection of members of $\mathcal{J}$ and therefore belongs to $\mathcal{T}'$. Since the product topology is generated by such rectangles, it follows that $\mathcal{T} \subseteq \mathcal{T}'$. Hence $\mathcal{T} = \mathcal{T}'$, proving that $\mathcal{J}$ is indeed a subbasis. □

Example 5.2. Each coordinate projection on a product space is an open map. Indeed, if $O \times W \subseteq X \times Y$ is a basic open rectangle, then

$$\pi_1(O \times W) = O \quad \text{and} \quad \pi_2(O \times W) = W,$$

which are open. Since arbitrary open sets are unions of basis rectangles, both projections send open sets to open sets.

$\pi_1^{-1}(O)$ is a vertical strip, $\pi_2^{-1}(W)$ is a horizontal strip

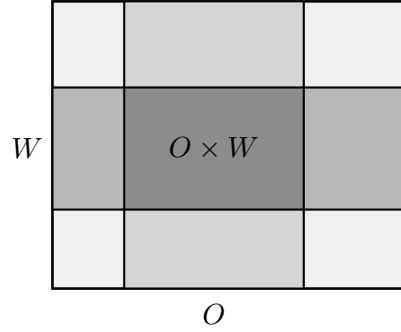


Figure 5.1: A basis rectangle $O \times W$ as the intersection of two subbasic sets.

5.2 Products, box topology, and the universal property

For infinite families of spaces, one must decide how much openness should be demanded in each coordinate. The product topology makes only finitely many coordinates active at a time, and this finiteness is precisely what preserves the correct continuity properties and leads to the major compactness theorems of the subject. The comparison with the box topology clarifies why the usual product topology is the natural choice.

For a finite product $X_1 \times \cdots \times X_n$, the product topology may be described in two equivalent ways. One may take as a basis the family of all products

$$\prod_{i=1}^n O_i, \quad O_i \in \mathcal{T}_i,$$

or, equivalently, one may take as a subbasis the inverse images of open sets under the coordinate projections. In the two-factor case this amounts to the observation that

$$\pi_1^{-1}(O_X) = O_X \times Y, \quad \pi_2^{-1}(O_Y) = X \times O_Y,$$

and hence

$$\pi_1^{-1}(O_X) \cap \pi_2^{-1}(O_Y) = O_X \times O_Y.$$

Thus basic open rectangles are precisely finite intersections of subbasic strips.

For an arbitrary family $\{X_i\}_{i \in I}$ the distinction between the product and box topologies becomes essential. The *product topology* on

$$X = \prod_{i \in I} X_i$$

is generated by the subbasis

$$\mathcal{J} = \{\pi_i^{-1}(O_i) : i \in I, O_i \subseteq X_i \text{ open}\},$$

so a basis consists of all sets of the form $\prod_{i \in I} O_i$, where each O_i is open in X_i and $O_i = X_i$ for all but finitely many indices. By contrast, the *box topology* is generated by the larger basis consisting of all products $\prod_{i \in I} O_i$ with no finiteness restriction. Consequently, the box topology is always finer than the product topology, and in infinite products the two topologies are usually different.

The product topology is the more natural choice for continuity theory and universal constructions; the next theorem records its universal property.

Theorem 5.3. *Let $f : A \rightarrow \prod_{\alpha \in I} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in I}$, where $f_\alpha : A \rightarrow X_\alpha$. Let $\prod X_\alpha$ be given the product topology. Then f is continuous if and only if each f_α is continuous.*

Proof. Suppose f is continuous on A .

Notice that each projection π_α is continuous. For this, let $Q_\alpha \in \mathcal{U}_\alpha$. The set $\pi_\alpha^{-1}(Q_\alpha)$ is a subbasis element for the product topology on $\prod X_\alpha$. Hence $f^{-1}(\pi_\alpha^{-1}(Q_\alpha))$ is open in A .

Define $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ by

$$f(t) = (t, t, t, \dots).$$

Then each coordinate function $f_n(t) = t$ is continuous, so f is continuous in the product topology on $\mathbb{R}^\omega$. However, f is not continuous when $\mathbb{R}^\omega$ is endowed with box topology. For this, let us consider a basis open set

$$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots \times (-\frac{1}{n}, \frac{1}{n}) \times \cdots$$

for the box topology on $\mathbb{R}^\omega$. Then $f^{-1}(B)$ is not open in $\mathbb{R}$.

If $f^{-1}(B)$ is open in $\mathbb{R}$, then as $0 \in f^{-1}(B)$, there exists a small interval (open) in $f^{-1}(B)$, i.e.,

$$(-\varepsilon, \varepsilon) \subset f^{-1}(B).$$

Therefore,

$$A_\varepsilon \subset B$$

where

$$A_\varepsilon = f((-\varepsilon, \varepsilon)).$$

Thus,

$$\pi_n(A_\varepsilon) \subset \pi_n(B)$$

which gives

$$(-\varepsilon, \varepsilon) \subset (-\frac{1}{n}, \frac{1}{n}), \quad \text{for all } n \in \mathbb{N}.$$

Which is not possible for large n .

□

Example 5.4. Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be real sequences with $a_n \neq 0$ for all n . Define $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ by

$$h(x_1, x_2, \dots) = (a_1x_1 + b_1, a_2x_2 + b_2, \dots).$$

Then h is a homeomorphism of $\mathbb{R}^\omega$ onto itself for the product topology (and, in fact, also for the box topology).

Proof. For each n , the n th coordinate map $h_n : \mathbb{R} \rightarrow \mathbb{R}$, $h_n(t) = a_nt + b_n$, is a homeomorphism of $\mathbb{R}$. If $\mathbb{R}^\omega$ carries the product topology, the universal property of products (Theorem 5.3) implies that h is continuous. Its inverse is given by

$$h^{-1}(y_1, y_2, \dots) = \left(\frac{y_1 - b_1}{a_1}, \frac{y_2 - b_2}{a_2}, \dots \right),$$

which is again continuous by the same argument; hence h is a homeomorphism.

If $\mathbb{R}^\omega$ carries the box topology, the same formulas still define a homeomorphism: for a basic box-open set $U = \prod_{n \in \mathbb{N}} U_n$,

$$h^{-1}(U) = \prod_{n \in \mathbb{N}} h_n^{-1}(U_n),$$

which is box-open since each $h_n^{-1}(U_n)$ is open in $\mathbb{R}$, and similarly for $(h^{-1})^{-1} = h$. $\square$

Theorem 5.5. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of topological spaces and let $A_\alpha \subseteq X_\alpha$ for each $\alpha \in I$. Equip $X = \prod_{\alpha \in I} X_\alpha$ with either the product topology or the box topology. Then

$$\overline{\prod_{\alpha \in I} A_\alpha} = \prod_{\alpha \in I} \overline{A_\alpha}.$$

Proof. Write $A := \prod_{\alpha \in I} A_\alpha$.

($\subseteq$) Let $x = (x_\alpha) \in \overline{A}$. Fix $\beta \in I$ and let $V \subseteq X_\beta$ be open with $x_\beta \in V$. For both the product and the box topology, the projection $\pi_\beta : X \rightarrow X_\beta$ is continuous, hence $\pi_\beta^{-1}(V)$ is open in X and contains x . Thus $\pi_\beta^{-1}(V) \cap A \neq \emptyset$, so there exists $a = (a_\alpha) \in A$ with $a_\beta \in V \cap A_\beta$. Therefore $V \cap A_\beta \neq \emptyset$, proving $x_\beta \in \overline{A_\beta}$. Since β was arbitrary, $x \in \prod_{\alpha \in I} \overline{A_\alpha}$.

($\supseteq$) Conversely, assume $x = (x_\alpha) \in \prod_{\alpha \in I} \overline{A_\alpha}$. If $A = \emptyset$, then some $A_\alpha = \emptyset$ and both sides are empty, so assume $A \neq \emptyset$. Let $U = \prod_{\alpha \in I} U_\alpha$ be a basic open neighborhood of x in the chosen topology; thus each U_α is open in X_α , $x_\alpha \in U_\alpha$, and in the product topology we additionally have $U_\alpha = X_\alpha$ for all but finitely many α . For each α with $U_\alpha \neq X_\alpha$ choose $a_\alpha \in A_\alpha \cap U_\alpha$, which is nonempty because $x_\alpha \in \overline{A_\alpha}$. For the remaining coordinates choose any $a_\alpha \in A_\alpha$. Then $a := (a_\alpha) \in A \cap U$, so $U \cap A \neq \emptyset$, and hence $x \in \overline{A}$. $\square$

5.3 Quotient spaces and quotient maps

If product spaces assemble information coordinatewise, quotient spaces do the opposite: they collapse specified subsets or equivalence classes and thereby encode geometric identifications.

Circles from intervals, spheres from disks, and many other standard spaces arise in this way. The quotient topology is therefore the basic formal device for gluing points together while retaining a workable notion of continuity.

Quotient constructions formalize the “cut-and-paste” procedures that produce new spaces by identifying points. A basic example is the circle S^1 , obtained from $\mathbb{R}$ by identifying $x \sim y$ whenever $x - y \in \mathbb{Z}$. More generally, one may obtain a cylinder from a rectangle by identifying two opposite sides, and then a torus by identifying the remaining pair of opposite sides.

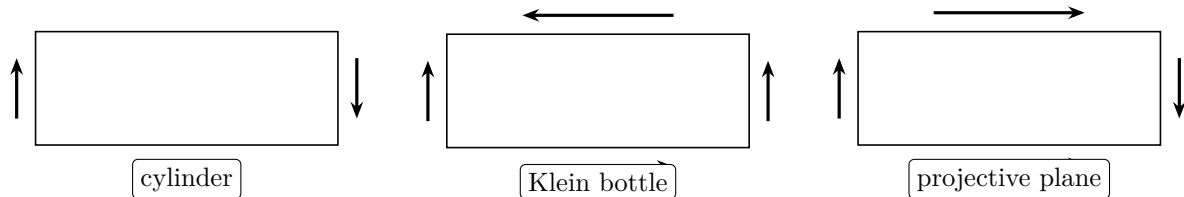


Figure 5.2: Edge-identification pictures for standard quotients. Arrows indicate the direction in which boundary points are identified.

Definition 5.6 (Quotient topology and quotient map). Let X be a topological space, let Y be a set, and let $q : X \rightarrow Y$ be a surjection. The *quotient topology* on Y (induced by q) is the collection

$$\mathcal{T}_q := \{ U \subseteq Y : q^{-1}(U) \text{ is open in } X \}.$$

Equipped with $\mathcal{T}_q$, the map q is continuous by construction and is called a *quotient map*. Equivalently, a surjection $q : X \rightarrow Y$ is a quotient map if it is continuous and a set $U \subseteq Y$ is open whenever $q^{-1}(U)$ is open.

Remark 5.7. If $q : X \rightarrow Y$ is continuous, surjective, and either open or closed, then q is a quotient map. In general, quotient maps need be neither open nor closed.

Example 5.8. Let $X = [0, 1] \cup [2, 3]$ and $Y = [0, 2]$, and define $q : X \rightarrow Y$ by $q(x) = x$ on $[0, 1]$ and $q(x) = x - 1$ on $[2, 3]$. Then q is a quotient map, but it is neither open nor closed.

Definition 5.9. Let $q : X \rightarrow Y$ be a surjective map. A subset $S \subseteq X$ is said to be *saturated* (with respect to q) if it is a union of fibers of q ; equivalently,

$$S \text{ is saturated} \iff (q^{-1}(\{y\}) \cap S \neq \emptyset \Rightarrow q^{-1}(\{y\}) \subseteq S) \text{ for all } y \in Y.$$

Equivalently, S is saturated if and only if $S = q^{-1}(q(S))$.

Proposition 5.10. Let $q : X \rightarrow Y$ be a continuous surjection. Then q is a quotient map if and only if $q(U)$ is open in Y for every open saturated set $U \subseteq X$.

Proof. Assume q is a quotient map and let $U \subseteq X$ be open and saturated. Then $U = q^{-1}(q(U))$. Since $q^{-1}(q(U))$ is open, the defining property of the quotient topology implies that $q(U)$ is open in Y .

Conversely, assume q is continuous and that $q(U)$ is open for every open saturated $U \subseteq X$. Let $V \subseteq Y$ satisfy $q^{-1}(V)$ open in X . Then $q^{-1}(V)$ is saturated and $q(q^{-1}(V)) = V$ because q is surjective. Hence V is open by hypothesis, so q is a quotient map. $\square$

Example 5.11. Let $X = [0, 1] \cup [2, 3]$ and $Y = [0, 2]$. Define $\beta : X \rightarrow Y$ by

$$\beta(x) = \begin{cases} x, & x \in [0, 1], \\ x - 1, & x \in [2, 3]. \end{cases}$$

Then β is a quotient map. Now let

$$A = [0, 1) \cup [2, 3], \quad g := \beta|_A : A \rightarrow Y.$$

The map g is continuous and surjective, but it is not a quotient map: the set $[2, 3]$ is open and saturated in A , whereas

$$g([2, 3]) = [1, 2]$$

is not open in Y . This shows that restrictions of quotient maps need not remain quotient maps.

Example 5.12. Let $\pi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the projection $\pi(x, y) = x$. Then π is open, continuous, and surjective; hence it is a quotient map. However, it is not closed, because

$$\pi(\{(x, y) : xy = 1\}) = \mathbb{R} \setminus \{0\},$$

which is not closed in $\mathbb{R}$.

Now let

$$A = \{(x, y) : xy = 1\} \cup \{(0, 0)\}, \quad g := \pi|_A : A \rightarrow \mathbb{R}.$$

Then g is continuous and surjective, but again it is not a quotient map, because $\{(0, 0)\}$ is open and saturated in A while

$$g(\{(0, 0)\}) = \{0\}$$

is not open in $\mathbb{R}$.

Example 5.13. (*Quotient topologies from surjections and partitions.*) Let $f : X \rightarrow A$ be a surjection from a topological space X to a set A . There is a unique topology on A that makes f a quotient map, namely

$$\tau_f := \{U \subseteq A : f^{-1}(U) \text{ is open in } X\}.$$

This is precisely the quotient topology from Definition 5.6.

For instance, let $A = \{a, b, c\}$ and define $f : \mathbb{R} \rightarrow A$ by

$$f(x) = \begin{cases} a, & x > 0, \\ b, & x < 0, \\ c, & x = 0. \end{cases}$$

Then $\tau_f = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, A\}$.

A particularly important case arises from partitions (or, equivalently, equivalence relations). If $\sim$ is an equivalence relation on X , let $X/\sim$ denote the set of equivalence classes and let $q : X \rightarrow X/\sim$ be the canonical surjection. Endowing $X/\sim$ with the quotient topology induced by q produces the *quotient space* $X/\sim$.

Theorem 5.14. *Let $p : X \rightarrow Y$ be a quotient map, and let $A \subseteq X$ be saturated with respect to p . Set $q := p|_A : A \rightarrow p(A)$.*

(i) *If A is open in X , then q is a quotient map.*

(ii) *If p is an open map, then q is a quotient map.*

Proof. Because A is saturated, for every $V \subseteq p(A)$ one has

$$q^{-1}(V) = A \cap p^{-1}(V) = p^{-1}(V),$$

since $p^{-1}(V) \subseteq p^{-1}(p(A)) = A$. Likewise, for every $U \subseteq X$,

$$p(U \cap A) = p(U) \cap p(A).$$

For (i), let $V \subseteq p(A)$ and assume that $q^{-1}(V)$ is open in A . Then there exists an open set $U \subseteq X$ such that $q^{-1}(V) = A \cap U$. Since A is open in X , the set $A \cap U$ is open in X ; by the first identity above,

$$p^{-1}(V) = q^{-1}(V)$$

is therefore open in X . Because p is a quotient map, it follows that V is open in Y , hence also open in the subspace $p(A)$. Thus q is a quotient map.

For (ii), again let $V \subseteq p(A)$ and suppose that $q^{-1}(V)$ is open in A . Choose an open set $U \subseteq X$ with $q^{-1}(V) = A \cap U$. Using the second identity,

$$V = q(q^{-1}(V)) = p(A \cap U) = p(A) \cap p(U).$$

Since p is open, the set $p(U)$ is open in Y , and therefore V is open in the subspace $p(A)$. Hence q is a quotient map. $\square$

Corollary 5.15. *The same argument shows that if $p : X \rightarrow Y$ is a quotient map, $A \subseteq X$ is saturated, A is closed in X , and p is closed, then the restriction $p|_A : A \rightarrow p(A)$ is also a quotient map.*

Example 5.16. The composition of two quotient maps is again a quotient map. By contrast, the product of two quotient maps need not be a quotient map in general; however, if both maps are open, then their product is open, continuous, and surjective, and hence is a quotient map.

One of the main uses of quotient maps is the construction of functions on quotient spaces by factoring through equivalence classes. The next theorem records the precise mechanism.

Theorem 5.17. *Let $\pi : X \rightarrow Y$ be a quotient map, and let $g : X \rightarrow Z$ be a map that is constant on each fiber $\pi^{-1}(\{y\})$, $y \in Y$. Then there exists a unique map*

$$f : Y \rightarrow Z$$

such that $f \circ \pi = g$. Moreover:

- (i) *f is continuous if and only if g is continuous;*
- (ii) *f is a quotient map if and only if g is a quotient map.*

Proof. Since g is constant on each fiber of π , the formula

$$f(\pi(x)) := g(x)$$

defines a map $f : Y \rightarrow Z$. It is well defined because if $\pi(x) = \pi(x')$, then x and x' lie in the same fiber, and hence $g(x) = g(x')$. The identity $f \circ \pi = g$ is immediate, and uniqueness follows because π is surjective.

For (i), if f is continuous, then $g = f \circ \pi$ is continuous as a composition of continuous maps. Conversely, assume that g is continuous. Let $V \subseteq Z$ be open. Then

$$\pi^{-1}(f^{-1}(V)) = g^{-1}(V)$$

is open in X . Since π is a quotient map, $f^{-1}(V)$ is open in Y . Hence f is continuous.

For (ii), if f is a quotient map, then $g = f \circ \pi$ is a composition of quotient maps and therefore a quotient map. Conversely, assume that g is a quotient map. By part (i), f is continuous. Now let $V \subseteq Z$ and assume that $f^{-1}(V)$ is open in Y . Then

$$g^{-1}(V) = \pi^{-1}(f^{-1}(V))$$

is open in X . Since g is a quotient map, this implies that V is open in Z . Thus f is a quotient map. □

Corollary 5.18. *Let $g : X \xrightarrow{\text{onto, cont}} Z$, and set*

$$X^* := \{g^{-1}(\{z\}) : z \in Z\}.$$

Let $\pi : X \rightarrow X^*$ be the canonical surjection, and equip X^* with the quotient topology induced by π . Then:

(i) there exists a unique continuous bijection $f : X^* \rightarrow Z$ such that $g = f \circ \pi$;

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ & \searrow \pi \quad \nearrow f & \\ & X^* & \end{array}$$

(ii) if g is a quotient map, then f is a homeomorphism;

(iii) in particular, if g is a quotient map and Z is Hausdorff, then X^* is Hausdorff.

Proof. Because g is constant on each fiber of π , Theorem 5.17 yields a unique map $f : X^* \rightarrow Z$ satisfying $g = f \circ \pi$. The map f is bijective: surjectivity follows from the surjectivity of g , and injectivity holds because distinct points of X^* are distinct fibers of g . Part (i) of Theorem 5.17 shows that f is continuous.

If g is a quotient map, then part (ii) of Theorem 5.17 implies that f is a quotient map. A continuous bijective quotient map is a homeomorphism, proving (ii). Finally, (iii) is immediate from (ii), since Hausdorffness is preserved under homeomorphism. $\square$

Remark 5.19. Some properties, such as compactness, connectedness, and path connectedness, do pass to quotient spaces, but many others do not. In general, quotient spaces are often best understood through a careful study of basic examples rather than through overly broad heuristic principles. For that reason, it is important to keep concrete models in mind throughout this discussion.

Supplementary remarks and examples

Remark 5.20. Subspace, product, and quotient topologies are the three basic structural constructions in topology. Conceptually, the subspace and product topologies are initial constructions, while the quotient topology is a final construction. Their universal properties explain why the same definitions recur throughout geometry and analysis.

Example 5.21. Identify the endpoints of the interval $[0, 1]$ by imposing the relation $0 \sim 1$. The quotient space $[0, 1]/\sim$ is homeomorphic to the circle S^1 . This is the model example for quotient topology: one first prescribes an identification, and only then asks for the strongest topology making the quotient map continuous.

Problem Set

Problem guide. Read these problems as training in universal properties. For products, try to prove statements by composing with projections. For quotients, try to factor maps through the quotient map whenever possible. A strong sign of mastery is that you can tell immediately whether a claim should be attacked from the definition of the topology or from its universal characterization.

Exercise 5.22. Let $X = \prod_{\lambda \in I} X_\lambda$ with the product topology. Show that each coordinate projection $\pi_\lambda : X \rightarrow X_\lambda$ is continuous, and that the product topology is the smallest topology making all π_λ continuous.

Exercise 5.23. Show that a subbasis for the product topology on $\prod_{\lambda \in I} X_\lambda$ is given by sets of the form $\pi_\lambda^{-1}(U)$ where $U \subseteq X_\lambda$ is open.

Exercise 5.24. Give an explicit example where the box topology on $\mathbb{R}^{\mathbb{N}}$ is strictly finer than the product topology.

Exercise 5.25. Let $q : X \rightarrow Y$ be a surjection. Prove that the quotient topology on Y is the largest topology making q continuous.

Exercise 5.26. Let $q : X \rightarrow Y$ be a quotient map and $f : Y \rightarrow Z$ be a map. Prove that f is continuous if and only if $f \circ q : X \rightarrow Z$ is continuous.

Exercise 5.27. Let X be a topological space and $A \subseteq X$. Show that collapsing A to a point produces a quotient space, and describe a neighborhood basis at the collapsed point.

Countability and separation axioms

We study two families of structural hypotheses. The countability axioms (first and second countability, separability, Lindelöfness) control the size and complexity of the topology. The separation axioms (T_0 , T_1 , Hausdorff, regularity, normality) control how well points and closed sets can be separated by open neighborhoods.

Roadmap.

- The diagonal criterion for Hausdorffness and the normality results are included because they provide robust geometric tests for separation.
- The countability side culminates in the chain second countable $\Rightarrow$ first countable, Lindelöf, separable, together with the converse theorem for separable metric spaces.
- These results are not merely classificatory: they are the hypotheses that drive the Urysohn–Tietze and metrization theorems later in the notes.

6.1 Separation axioms

Not all topological spaces distinguish points and closed sets equally well. The separation axioms measure how effectively the topology can separate geometric objects by neighborhoods or by continuous functions. These conditions form a hierarchy, and many central theorems in topology become available only once the appropriate level of separation is assumed.

Definition 6.1. A topological space X is said to satisfy:

- (i) T_0 if for any distinct $x, y \in X$ there exists an open set containing one of x, y but not the other;
- (ii) T_1 if every singleton $\{x\}$ is closed (equivalently, for any distinct x, y there exists an open set containing x but not y);
- (iii) T_2 (or *Hausdorff*) if for any distinct $x, y \in X$ there exist disjoint open sets $U \ni x$ and $V \ni y$.

Remark 6.2. In these notes we reserve the symbols T_0, T_1, T_2 for the separation axioms. Earlier examples of distinct *topologies* on a finite set are denoted by τ .

Definition 6.3. A T_1 space X is *regular* if for every closed set $F \subset X$ and every point $x \notin F$, there exist disjoint open sets U, V with $x \in U$ and $F \subset V$. A space X is *normal* if for any two disjoint closed sets $F, G \subset X$ there exist disjoint open sets U, V with $F \subset U$ and $G \subset V$.

Proposition 6.4. *Every metric space is Hausdorff and normal.*

Proof. Hausdorffness follows by taking disjoint open balls around distinct points. Normality follows by separating disjoint closed sets using the distance function; one standard argument uses the open sets

$$U = \{x \in X : d(x, F) < d(x, G)\}, \quad V = \{x \in X : d(x, G) < d(x, F)\}.$$

These are open, disjoint, and contain F and G , respectively. □

Proposition 6.5. *A topological space X is Hausdorff if and only if the diagonal*

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in the product space $X \times X$.

Proof. Assume first that X is Hausdorff and let $(x, y) \in (X \times X) \setminus \Delta_X$. Then $x \neq y$, so there exist disjoint open sets $U \ni x$ and $V \ni y$. Hence $U \times V$ is an open neighborhood of (x, y) disjoint from Δ_X . Therefore $(X \times X) \setminus \Delta_X$ is open, so Δ_X is closed.

Conversely, assume Δ_X is closed. Let $x \neq y$. Then $(x, y) \notin \Delta_X$, so because $(X \times X) \setminus \Delta_X$ is open, there exist open sets $U \ni x$ and $V \ni y$ such that

$$(x, y) \in U \times V \subseteq (X \times X) \setminus \Delta_X.$$

If $U \cap V \neq \emptyset$, choose $z \in U \cap V$. Then $(z, z) \in U \times V \subseteq (X \times X) \setminus \Delta_X$, a contradiction. Thus $U \cap V = \emptyset$, and X is Hausdorff. □

Corollary 6.6. *Let X and Y be topological spaces, with Y Hausdorff, and let $f : X \rightarrow Y$ be continuous. Then the graph*

$$G_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$$

is closed.

Proof. Consider the map $\Phi : X \times Y \rightarrow Y \times Y$ given by $\Phi(x, y) = (f(x), y)$. This map is continuous. Since Y is Hausdorff, Δ_Y is closed in $Y \times Y$ by Proposition 6.5. But

$$G_f = \Phi^{-1}(\Delta_Y),$$

hence G_f is closed in $X \times Y$. □

Proposition 6.7. *If X is compact and Hausdorff, then X is normal.*

Proof. Let $F, G \subset X$ be disjoint closed sets. For each $x \in F$ and $y \in G$, Hausdorffness yields disjoint open neighborhoods $U_{x,y} \ni x$ and $V_{x,y} \ni y$. Fix $x \in F$. Then $\{V_{x,y} : y \in G\}$ is

an open cover of G , hence admits a finite subcover $V_{x,y_1}, \dots, V_{x,y_n}$. Set $U_x = \bigcap_{k=1}^n U_{x,y_k}$ and $V_x = \bigcup_{k=1}^n V_{x,y_k}$. Then U_x is open, $x \in U_x$, and $U_x \cap V_x = \emptyset$ while $G \subset V_x$. Now $\{U_x : x \in F\}$ is an open cover of the compact set F ; choose $x_1, \dots, x_m$ with $F \subset \bigcup_{j=1}^m U_{x_j}$. Let $U = \bigcup_{j=1}^m U_{x_j}$ and $V = \bigcap_{j=1}^m V_{x_j}$. Then U, V are open, disjoint, $F \subset U$, and $G \subset V$, proving normality. $\square$

6.2 Countability axioms

Although topological definitions are generally infinitary, many important spaces are controlled by countable data. Countability axioms identify precisely when this happens. They connect abstract topology with sequences, separability, and metrization, and they frequently determine whether local arguments can be globalized in a manageable way.

Definition 6.8. A topological space X is *first countable* if each point $x \in X$ admits a countable neighborhood basis, i.e. there exists a sequence of neighborhoods $(U_n)_{n \in \mathbb{N}}$ of x such that for every neighborhood U of x there is n with $U_n \subseteq U$. The space X is *second countable* if there exists a countable base $\mathcal{B}$ for the topology of X .

Remark 6.9. Second countability is a strong global hypothesis. Among its consequences are the Lindelöf property and (in the presence of a metric) separability; it also plays a central role in metrization theory.

Proposition 6.10. *Every second countable space is first countable.*

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable base. Fix $x \in X$ and consider $\mathcal{B}_x := \{B_n \in \mathcal{B} : x \in B_n\}$, which is countable. If U is any neighborhood of x , choose $B_n \in \mathcal{B}$ with $x \in B_n \subseteq U$; then $B_n \in \mathcal{B}_x$, so $\mathcal{B}_x$ is a countable neighborhood basis at x . $\square$

Definition 6.11. A topological space X is *Lindelöf* if every open cover of X admits a countable subcover.

Theorem 6.12 (Lindelöf theorem). *Every second countable space is Lindelöf.*

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable base for X , and let $\mathcal{U}$ be an open cover of X . Consider the index set

$$I := \{n \in \mathbb{N} : \text{there exists } U \in \mathcal{U} \text{ with } B_n \subseteq U\}.$$

For each $n \in I$, choose one such set $U_n \in \mathcal{U}$. Then

$$\mathcal{U}_0 := \{U_n : n \in I\}$$

is countable.

We claim that $\mathcal{U}_0$ covers X . Let $x \in X$. Since $\mathcal{U}$ covers X , there exists $U \in \mathcal{U}$ with $x \in U$. Because $\mathcal{B}$ is a base, there exists $B_n \in \mathcal{B}$ such that

$$x \in B_n \subseteq U.$$

Thus $n \in I$, and by construction $B_n \subseteq U_n$ for some $U_n \in \mathcal{U}_0$. Hence $x \in U_n$, proving that $\mathcal{U}_0$ is a countable subcover of $\mathcal{U}$. $\square$

Corollary 6.13. *Let X be second countable and let $\mathcal{C}$ be any basis for the topology of X . Then $\mathcal{C}$ contains a countable subfamily which is still a basis.*

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a fixed countable base for X . For each n , the family $\mathcal{C}_n := \{C \in \mathcal{C} : C \subseteq B_n\}$ is an open cover of B_n in the subspace topology. Since B_n is open in X and X is Lindelöf by Theorem 6.12, the subspace B_n is itself Lindelöf: indeed, any open cover of B_n by subspace-open sets can be enlarged to an open cover of X by adjoining $X \setminus B_n$. Hence $\mathcal{C}_n$ admits a countable subcover $\mathcal{C}'_n \subseteq \mathcal{C}_n$. Now set $\mathcal{C}' := \bigcup_{n \in \mathbb{N}} \mathcal{C}'_n$, a countable union of countable sets, hence countable. If U is open in X and $x \in U$, choose $B_n \in \mathcal{B}$ with $x \in B_n \subseteq U$. Since $\mathcal{C}'_n$ covers B_n , there exists $C \in \mathcal{C}'_n \subseteq \mathcal{C}'$ with $x \in C \subseteq B_n \subseteq U$, so $\mathcal{C}'$ is a basis. $\square$

Definition 6.14. A topological space X is *separable* if it contains a countable dense subset.

Corollary 6.15. *Every second countable space is separable.*

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable base. For each n with $B_n \neq \emptyset$, choose a point $x_n \in B_n$, and set $D := \{x_n : n \in \mathbb{N}\}$. Then D is countable. If $U \subseteq X$ is nonempty open, choose $B_n \subseteq U$ with $B_n \neq \emptyset$; then $x_n \in D \cap U$, so D meets every nonempty open set and hence is dense. $\square$

Theorem 6.16. *Every separable metric space is second countable.*

Proof. Let (X, d) be a metric space and let $D \subseteq X$ be a countable dense subset. Set $\mathbb{Q}_{>0} := \mathbb{Q} \cap (0, \infty)$, and consider the family

$$\mathcal{B} := \{B(a, q) : a \in D, q \in \mathbb{Q}_{>0}\}.$$

The family $\mathcal{B}$ is countable because it is indexed by the countable set $D \times \mathbb{Q}_{>0}$.

We show that $\mathcal{B}$ is a base for the metric topology. Let $U \subseteq X$ be open and let $x \in U$. Choose $r > 0$ such that $B(x, r) \subseteq U$. By density of D , there exists $a \in D \cap B(x, r/3)$. Since $d(x, a) < r/3 < r - d(x, a)$, the interval $(d(x, a), r - d(x, a))$ is nonempty; by density of $\mathbb{Q}$ in $\mathbb{R}$, choose

$$q \in \mathbb{Q}_{>0} \cap (d(x, a), r - d(x, a)).$$

Then $x \in B(a, q)$. Moreover, if $y \in B(a, q)$, then

$$d(x, y) \leq d(x, a) + d(a, y) < d(x, a) + q < r,$$

so $y \in B(x, r)$. Hence

$$x \in B(a, q) \subseteq B(x, r) \subseteq U.$$

Therefore every open set containing x contains a member of $\mathcal{B}$ that still contains x , and $\mathcal{B}$ is a countable base for X . Thus X is second countable. $\square$

Supplementary remarks and examples

Remark 6.17. The countability axioms and the separation axioms measure different kinds of regularity. Countability controls how much local information is needed to describe the topology, while separation controls how sharply the space can distinguish points and closed sets. Many classical theorems live precisely at the interaction of these two hierarchies.

Example 6.18. The Sorgenfrey line is first countable and separable, but it is not second countable. On the other hand, the cofinite topology on an infinite set is T_1 but not Hausdorff. Keeping these examples in mind prevents one from silently adding nonexistent implications to the standard diagrams of axioms.

Problem Set

Problem guide. This chapter repays careful organization. Make your own implication diagram for the separation axioms and a separate one for the countability axioms, then use the problems to test which arrows are reversible. Counterexamples matter as much as proofs here: whenever a converse fails, try to locate exactly where the forward proof breaks down.

Exercise 6.19. Prove that second countability implies separability.

Exercise 6.20. Show that in a first countable space, sequential continuity (preserving limits of sequences) is equivalent to continuity.

Exercise 6.21. Prove that a subspace of a second countable space is second countable.

Exercise 6.22. Prove that every metric space is normal.

Exercise 6.23. Give an example of a T_1 space that is not Hausdorff.

Exercise 6.24. Show that if X is compact and Hausdorff, then it is normal.

Connectedness

Connectedness formalizes the idea that a space cannot be decomposed into two disjoint nontrivial open pieces. We develop connectedness, components, and path connectedness, and briefly discuss local connectedness and totally disconnected spaces.

Roadmap.

- Equivalent formulations of connectedness are established first so that later proofs can pass freely between separations, clopen sets, and continuous images.
- The classification of connected subsets of $\mathbb{R}$ by intervals is the chapter's central concrete theorem and is used to derive the intermediate value property.
- Components and path connectedness provide two complementary ways of decomposing spaces into maximally connected pieces.

7.1 Connected spaces and components

Connectedness formalizes the idea that a space should remain in one piece. It is one of the first genuinely global properties in topology, because it cannot be checked by examining neighborhoods point by point. Components then decompose an arbitrary space into its maximal connected pieces and reveal how connectedness interacts with subspaces and continuous images.

Definition 7.1. A topological space X is *connected* if it cannot be written as $X = U \cup V$ where U, V are disjoint nonempty open sets. Such a decomposition (if it exists) is called a *separation* of X .

Proposition 7.2. *The following are equivalent for a space X :*

- (i) X is connected.
- (ii) The only subsets of X that are both open and closed are $\emptyset$ and X .
- (iii) There does not exist a continuous surjection $f : X \rightarrow \{0, 1\}$ (with $\{0, 1\}$ discrete).

Proof. (i) $\Rightarrow$ (ii). If $A \subseteq X$ is clopen and nontrivial, then A and $X \setminus A$ are nonempty disjoint open sets whose union is X . Thus X is disconnected. Hence a connected space has no nontrivial clopen subset.

(ii) $\Rightarrow$ (iii). If there were a continuous surjection $f : X \rightarrow \{0, 1\}$ onto the discrete two-point space, then $f^{-1}(\{0\})$ would be a nonempty proper subset of X that is both open and closed, contradicting (ii).

(iii) $\Rightarrow$ (i). If $X = U \cup V$ were a separation, define $f : X \rightarrow \{0, 1\}$ by $f(x) = 0$ on U and $f(x) = 1$ on V . The sets U and V are open, so f is continuous, and it is clearly surjective. This contradicts (iii). $\square$

Proposition 7.3. *The continuous image of a connected space is connected.*

Proof. Let $f : X \rightarrow Y$ be continuous and X connected. If $f(X) = U \cup V$ is a separation in Y , then $X = f^{-1}(U) \cup f^{-1}(V)$ is a separation of X , a contradiction. $\square$

Proposition 7.4. *Let $\{C_i\}_{i \in I}$ be a family of connected subsets of X such that*

$$\bigcap_{i \in I} C_i \neq \emptyset.$$

Then $\bigcup_{i \in I} C_i$ is connected.

Proof. Suppose $\bigcup_{i \in I} C_i = U \cup V$ is a separation. Choose $p \in \bigcap_{i \in I} C_i$. Then $p \in U$ or $p \in V$; assume $p \in U$. Since each C_i is connected and meets U , it cannot meet V . Hence $C_i \subseteq U$ for every i , and therefore $\bigcup_{i \in I} C_i \subseteq U$, contradicting that $V \neq \emptyset$. $\square$

Proposition 7.5. *If $A \subseteq X$ is connected, then its closure $\bar{A}$ is connected.*

Proof. Suppose $\bar{A} = U \cup V$ is a separation. Since $A \subseteq \bar{A}$, we have

$$A = (A \cap U) \cup (A \cap V),$$

where $A \cap U$ and $A \cap V$ are disjoint open subsets of A . Because A is connected, one of them must be empty; assume $A \cap V = \emptyset$. Then V is a nonempty open subset of $\bar{A}$ disjoint from A , contradicting the fact that every nonempty open subset of $\bar{A}$ meets A . Hence $\bar{A}$ is connected. $\square$

Theorem 7.6. *Every interval in $\mathbb{R}$ is connected.*

Proof. Let $I \subseteq \mathbb{R}$ be an interval and suppose $I = U \cup V$ is a separation in the subspace topology. Choose $u \in U$ and $v \in V$ with $u < v$. Set

$$E = U \cap [u, v].$$

Then E is nonempty and bounded above by v , so $s = \sup E$ exists. Since I is an interval and $u, v \in I$, we have $[u, v] \subseteq I$, hence $s \in I$.

If $s \in U$, then because U is open in I , there exists $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \cap I \subseteq U$. By definition of supremum there exists $x \in E$ with $s - \varepsilon < x \leq s$, and then any point of I between x and $s + \varepsilon$ also lies in U . In particular one finds a point of E larger than s , a contradiction.

If $s \in V$, then because V is open in I , there exists $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \cap I \subseteq V$. Again by the definition of supremum, there exists $x \in E$ with $s - \varepsilon < x \leq s$. But then $x \in U \cap V$, impossible.

Both cases lead to contradictions. Therefore I is connected. $\square$

Theorem 7.7. *A subset $A \subseteq \mathbb{R}$ is connected if and only if it is an interval.*

Proof. One implication is Theorem 7.6. Conversely, suppose A is connected and let $a, b \in A$ with $a < b$. If there were $c \in (a, b) \setminus A$, then

$$A \cap (-\infty, c) \quad \text{and} \quad A \cap (c, \infty)$$

would be disjoint nonempty open subsets of A whose union is A , contradicting connectedness. Hence every point between a and b lies in A , so A is an interval. $\square$

Corollary 7.8 (Intermediate value property). *If X is connected and $f : X \rightarrow \mathbb{R}$ is continuous, then $f(X)$ is an interval. In particular, if $a, b \in X$ and t lies between $f(a)$ and $f(b)$, there exists $x \in X$ such that $f(x) = t$.*

Proof. By Proposition 7.3, $f(X)$ is connected, and by Theorem 7.7 every connected subset of $\mathbb{R}$ is an interval. $\square$

Definition 7.9. A *component* of a space X is a maximal connected subset of X .

Proposition 7.10. *Components form a partition of X , and each component is closed.*

Proof. By Proposition 7.4, two connected sets with nonempty intersection have connected union, so maximality forces components to be either equal or disjoint; hence they partition X . If C is a component, then $\overline{C}$ is connected by Proposition 7.5; maximality therefore gives $\overline{C} = C$, so C is closed. $\square$

7.2 Path connectedness and local connectedness

Connectedness excludes a decomposition into two disjoint open parts, but it does not by itself provide an explicit geometric route from one point to another. Path connectedness strengthens connectedness by requiring such routes, while local connectedness ensures that connected structure is visible already at small scales. The distinction between these notions is subtle and leads to some of the most instructive examples in point-set topology.

Definition 7.11. A space X is *path connected* if for any $x, y \in X$ there exists a continuous map $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 7.12. *Path connected spaces are connected.*

Proof. If $X = U \cup V$ is a separation and $x \in U, y \in V$, any path γ from x to y yields a separation of $[0, 1]$ by $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$, contradicting connectedness of $[0, 1]$. $\square$

Definition 7.13. A space X is *totally disconnected* if its connected components are singletons. It is *locally connected* if every point has a neighborhood basis consisting of connected open sets.

Example 7.14. The rational numbers $\mathbb{Q} \subset \mathbb{R}$ (with the subspace topology) are totally disconnected.

Proof. Given distinct $p < q \in \mathbb{Q}$, choose an irrational number $\alpha \in (p, q)$. Then

$$\mathbb{Q} \cap (-\infty, \alpha) \quad \text{and} \quad \mathbb{Q} \cap (\alpha, \infty)$$

form a separation of $\mathbb{Q}$ into disjoint nonempty open sets. □

Example 7.15. The Cantor set is compact, perfect, and totally disconnected.

Supplementary remarks and examples

Remark 7.16. Path connectedness is stronger than connectedness because it asks not merely that the space stay in one piece, but that points can be joined by continuous paths. Many geometric spaces satisfy both properties, but the distinction becomes important in wild subspaces of Euclidean space.

Example 7.17. The topologist's sine curve is connected but not path connected. It is therefore the standard warning that connectedness alone does not guarantee the existence of paths. Whenever an argument implicitly draws a curve between two points, one should verify that path connectedness is actually available.

Problem Set

Problem guide. Solve the first problems by translating the definitions of separation and connectedness into statements about continuous maps to $\{0, 1\}$ or to intervals. Later problems should be used to build a library of permanence properties: subspaces, products, images, and unions behave differently, and the point of the chapter is to learn those differences precisely.

Exercise 7.18. Prove that the continuous image of a connected space is connected.

Exercise 7.19. Show that X is disconnected if and only if there exists a continuous surjection $f : X \rightarrow \{0, 1\}$ (with $\{0, 1\}$ discrete).

Exercise 7.20. Prove that if X is path connected then X is connected, and give an example of a connected space that is not path connected.

Exercise 7.21. Let $\{C_i\}_{i \in I}$ be a family of connected subsets of X with $\bigcap_i C_i \neq \emptyset$. Prove that $\bigcup_i C_i$ is connected.

Exercise 7.22. Show that the connected component of a point is closed, and that components form a partition of X .

Exercise 7.23. Prove that a space X is totally disconnected if and only if every connected subset of X is a singleton.

Compactness and compactifications

Compactness is the topological analog of closed and bounded subsets in Euclidean spaces, and it controls finiteness phenomena such as the existence of convergent subsequences and finite subcovers. We develop basic permanence properties, the Tube Lemma and Tychonoff's theorem, and then discuss locally compact spaces and the one-point compactification.

Roadmap.

- The first half of the chapter develops the basic permanence theory: closed subspaces, continuous images, finite products, and closedness of projections over compact factors.
- The finite-intersection-property theorem, Alexander subbase theorem, Heine–Borel, and Tychonoff together show how compactness can be recognized from several complementary viewpoints.
- The final section introduces local compactness and the one-point compactification, which convert noncompact spaces into compact ones by adding ideal boundary points.

8.1 Compact sets

Compactness is the topological substitute for finiteness. It allows one to extract finite information from arbitrary open covers and lies behind many fundamental theorems in analysis, including the extreme value theorem and various existence arguments. In topology, compactness interacts powerfully with continuity, products, and separation axioms, so it merits sustained attention.

Compactness is one of the central finiteness principles of topology. Finite spaces automatically enjoy many useful properties, and compactness is the correct topological substitute for that kind of finite behavior. In Euclidean spaces, the Heine–Borel theorem identifies compact sets with closed and bounded sets; in general topology, however, the defining feature is the finite subcover property for open covers. This formulation is robust, behaves well under continuous maps and products, and lies behind many of the deepest existence theorems in analysis and topology.

Definition 8.1. A topological space X is said to be *compact* if every open cover of X admits a finite subcover. Equivalently, whenever

$$X = \bigcup_{i \in I} U_i$$

with each U_i open in X , there exist indices $i_1, \dots, i_n \in I$ such that

$$X = U_{i_1} \cup \dots \cup U_{i_n}.$$

Example 8.2. The real line $(\mathbb{R}, \mathcal{U})$ is not compact, because the open cover

$$\mathcal{G} = \{(n, n+2) : n \in \mathbb{Z}\}$$

has no finite subcover.

Example 8.3. The set

$$K := \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \mathbb{R}$$

is compact in the usual topology. Indeed, any open cover of K contains an open set U with $0 \in U$; such a set contains all but finitely many points of the sequence $1/n$. The remaining finitely many points can then be covered one by one.

Example 8.4. The interval $(0, 1]$ is not compact in the usual topology on $\mathbb{R}$, because the family

$$\mathcal{G}_0 = \left\{ \left(\frac{1}{n}, 2 \right) : n \in \mathbb{N} \right\}$$

is an open cover of $(0, 1]$ with no finite subcover. Indeed, if one chooses finitely many members of $\mathcal{G}_0$, the smallest value of n among them determines a left endpoint $1/n > 0$, so points of $(0, 1/n]$ remain uncovered. This example already suggests that, in Hausdorff spaces, compact subsets often behave like closed sets.

Lemma 8.5. *Let Y be a subset of a topological space X . Then Y is compact if and only if every open cover of Y by sets open in X admits a finite subcover.*

Proof. Assume first that Y is compact in the subspace topology and let $\{O_i : i \in I\}$ be a family of open subsets of X with

$$Y \subseteq \bigcup_{i \in I} O_i.$$

Then $\{O_i \cap Y : i \in I\}$ is an open cover of Y in the subspace topology, so there are indices $i_1, \dots, i_n$ such that

$$Y \subseteq (O_{i_1} \cap Y) \cup \dots \cup (O_{i_n} \cap Y) \subseteq O_{i_1} \cup \dots \cup O_{i_n}.$$

Thus the original cover has a finite subcover.

Conversely, let $\{U_j : j \in J\}$ be an open cover of Y in the subspace topology. For each j choose an open set $O_j \subseteq X$ such that $U_j = O_j \cap Y$. Then $\{O_j : j \in J\}$ is an open cover of Y in X , so by hypothesis it has a finite subcover. Intersecting with Y shows that the corresponding U_j also form a finite subcover of Y . Hence Y is compact. $\square$

Theorem 8.6 (Closed subspaces of compact spaces). *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of a compact space X . If $\{U'_i : i \in I\}$ is an open cover of Y by sets open in Y , choose open sets $U_i \subseteq X$ such that $U'_i = U_i \cap Y$. Then

$$X = (X \setminus Y) \cup \bigcup_{i \in I} U_i$$

is an open cover of X . By compactness, finitely many of these sets cover X ; intersecting with Y yields a finite subcover of Y . Thus Y is compact. $\square$

Example 8.7. In an indiscrete space $(X, \{\emptyset, X\})$, every subset is compact, because every open cover already contains the whole space X . This shows that compact subsets need not be closed unless one imposes additional separation axioms.

Theorem 8.8. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of a Hausdorff space X . Fix $x_0 \in X \setminus Y$. For each $y \in Y$, choose disjoint open sets U_y and V_y with $x_0 \in U_y$ and $y \in V_y$. Then $\{V_y : y \in Y\}$ is an open cover of Y , so compactness gives points $y_1, \dots, y_n \in Y$ such that

$$Y \subseteq V_{y_1} \cup \dots \cup V_{y_n}.$$

Set

$$U := U_{y_1} \cap \dots \cap U_{y_n}, \quad V := V_{y_1} \cup \dots \cup V_{y_n}.$$

Then U is an open neighborhood of x_0 , the set V is an open neighborhood of Y , and $U \cap V = \emptyset$. In particular, $U \cap Y = \emptyset$. Thus every point of $X \setminus Y$ has an open neighborhood contained in $X \setminus Y$, so $X \setminus Y$ is open and Y is closed. $\square$

Lemma 8.9. *If Y is a compact subspace of a Hausdorff space X , then for every $x_0 \in X \setminus Y$ there exist disjoint open sets $U, V \subseteq X$ such that $x_0 \in U$ and $Y \subseteq V$.*

Proof. The construction in the proof of Theorem 8.8 already yields the desired sets U and V . $\square$

Example 8.10. Every closed subset of a compact interval $[a, b]$ is compact. In particular, the Cantor set is compact.

Example 8.11. The half-open intervals $[a, b)$ and $(a, b]$ are not compact in the usual topology on $\mathbb{R}$, because they are not closed subsets of the Hausdorff space $\mathbb{R}$.

Remark 8.12. The Hausdorff hypothesis in Theorem 8.8 is essential. In the cofinite topology on an infinite set, every subset is compact: given an open cover, any nonempty open set misses only finitely many points, and those remaining points can be covered by finitely many further members of the cover. However, in that topology the only proper closed sets are finite sets.

Remark 8.13. The cocountable topology on $\mathbb{R}$ is not compact. Indeed, the family

$$\mathcal{U} = \{(\mathbb{R} \setminus \mathbb{N}) \cup \{n\} : n \in \mathbb{N}\}$$

consists of open sets and covers $\mathbb{R}$, but no finite subfamily covers all of $\mathbb{R}$.

Theorem 8.14. *The continuous image of a compact space is compact.*

Proof. Let $f : X \rightarrow Y$ be continuous and assume that X is compact. If $f(X) \subseteq \bigcup_{i \in I} V_i$ with each V_i open in the subspace $f(X)$, then

$$X \subseteq \bigcup_{i \in I} f^{-1}(V_i),$$

and each $f^{-1}(V_i)$ is open in X . Compactness of X yields indices $i_1, \dots, i_n$ such that

$$X \subseteq f^{-1}(V_{i_1}) \cup \dots \cup f^{-1}(V_{i_n}),$$

whence

$$f(X) \subseteq V_{i_1} \cup \dots \cup V_{i_n}.$$

Thus $f(X)$ is compact. □

Theorem 8.15. *Let $f : X \rightarrow Y$ be continuous, where Y is Hausdorff.*

- (a) *If X is compact, then f is a closed map.*
- (b) *If X is compact and f is bijective, then f is a homeomorphism.*

Proof. (a) Let $F \subseteq X$ be closed. Since F is closed in a compact space, Theorem 8.6 shows that F is compact. Hence $f(F)$ is compact by Theorem 8.14, and therefore closed in the Hausdorff space Y by Theorem 8.8. Thus f is a closed map.

(b) Assume in addition that f is bijective. By part (a), f is closed. A bijective closed map has continuous inverse: indeed, if $C \subseteq X$ is closed, then $f(C)$ is closed in Y , so for every closed set $C \subseteq X$ the inverse image under f^{-1} satisfies

$$(f^{-1})^{-1}(C) = f(C),$$

which is closed in Y . Hence f^{-1} is continuous, and therefore f is a homeomorphism. □

Lemma 8.16 (Tube Lemma). *Let X be a topological space, let Y be compact, and let $N \subseteq X \times Y$ be open. If $\{x_0\} \times Y \subseteq N$, then there exists an open neighborhood W of x_0 in X such that*

$$W \times Y \subseteq N.$$

Proof. For each $y \in Y$, choose open sets $U_y \subseteq X$ and $V_y \subseteq Y$ such that

$$(x_0, y) \in U_y \times V_y \subseteq N.$$

The sets V_y form an open cover of Y , so by compactness there are $y_1, \dots, y_n \in Y$ with

$$Y = V_{y_1} \cup \dots \cup V_{y_n}.$$

Set

$$W := U_{y_1} \cap \dots \cap U_{y_n}.$$

Then W is an open neighborhood of x_0 . If $x \in W$ and $y \in Y$, choose k with $y \in V_{y_k}$. Since $x \in U_{y_k}$, we obtain

$$(x, y) \in U_{y_k} \times V_{y_k} \subseteq N.$$

Hence $W \times Y \subseteq N$. □

Theorem 8.17. *The product of finitely many compact spaces is compact.*

Proof. It suffices to treat the case of two spaces. Let X and Y be compact, and let $\{O_i : i \in I\}$ be an open cover of $X \times Y$. Fix $x \in X$. Since $\{x\} \times Y$ is homeomorphic to Y , it is compact, so finitely many of the O_i cover $\{x\} \times Y$. Because the union of finitely many open sets is open, there is an open set $N_x \subseteq X \times Y$ such that

$$\{x\} \times Y \subseteq N_x \subseteq \bigcup_{k=1}^{m_x} O_{i_k}.$$

By the argument of the Tube Lemma, there exists an open neighborhood W_x of x such that

$$W_x \times Y \subseteq N_x.$$

The family $\{W_x : x \in X\}$ is an open cover of X , so compactness gives points $x_1, \dots, x_n \in X$ such that

$$X = W_{x_1} \cup \dots \cup W_{x_n}.$$

Therefore

$$X \times Y = (W_{x_1} \times Y) \cup \dots \cup (W_{x_n} \times Y),$$

and each $W_{x_j} \times Y$ is covered by finitely many members of the original cover. Collecting those finitely many sets yields a finite subcover of $X \times Y$. □

Proposition 8.18. *If Y is compact, then the projection*

$$p_X : X \times Y \rightarrow X, \quad p_X(x, y) = x,$$

is a closed map.

Proof. Let $F \subseteq X \times Y$ be closed. We show that $X \setminus p_X(F)$ is open. Take $x_0 \in X \setminus p_X(F)$. Then

$$\{x_0\} \times Y \subseteq (X \times Y) \setminus F,$$

and the right-hand side is open. By the Tube Lemma, there exists an open neighborhood $W \ni x_0$ such that

$$W \times Y \subseteq (X \times Y) \setminus F.$$

Thus $W \cap p_X(F) = \emptyset$, so $W \subseteq X \setminus p_X(F)$. Hence $X \setminus p_X(F)$ is open and $p_X(F)$ is closed. $\square$

Example 8.19. The Tube Lemma fails if the second factor is not compact. For example, in $\mathbb{R}^2$ let

$$N' = \{(x, y) : |x| < \frac{1}{1+|y|}\}.$$

Then N' is an open set containing $\{0\} \times \mathbb{R}$, but it contains no tube $W \times \mathbb{R}$ about $\{0\} \times \mathbb{R}$, because every neighborhood W of 0 contains some $x \neq 0$, and for sufficiently large $|y|$ one has $|x| \geq \frac{1}{1+|y|}$.

Definition 8.20. A family $\mathcal{C} \subseteq \mathcal{P}(X)$ is said to have the *finite intersection property* (FIP) if every finite subfamily of $\mathcal{C}$ has nonempty intersection.

Example 8.21. Any nested decreasing family

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

has the finite intersection property, provided each A_n is nonempty.

Theorem 8.22. A topological space X is compact if and only if every family of closed subsets of X having the finite intersection property has nonempty intersection.

Proof. Assume first that X is compact, and let $\{F_\alpha\}_{\alpha \in A}$ be a family of closed subsets of X with the finite intersection property. If $\bigcap_{\alpha \in A} F_\alpha = \emptyset$, then the open sets $U_\alpha := X \setminus F_\alpha$ form an open cover of X . By compactness, finitely many of them cover X , say

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Taking complements yields

$$F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} = \emptyset,$$

contrary to the finite intersection property. Thus $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$.

Conversely, assume that every family of closed sets with the finite intersection property has nonempty intersection. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X . If it had no finite subcover, then for every finite subset $\{\alpha_1, \dots, \alpha_n\} \subseteq A$ one would have

$$X \neq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Hence the closed sets $F_\alpha := X \setminus U_\alpha$ would satisfy the finite intersection property. By hypothesis,

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset,$$

which contradicts the assumption that the U_α cover X . Therefore every open cover has a finite subcover, so X is compact. $\square$

Example 8.23. If $G_1 \supseteq G_2 \supseteq \cdots$ is a decreasing sequence of nonempty closed subsets of a compact space, then

$$\bigcap_{n=1}^{\infty} G_n \neq \emptyset.$$

This is an immediate consequence of the finite intersection property.

Example 8.24. A major theorem, proved later in this chapter, states that every closed and bounded subset of $\mathbb{R}^n$ is compact. This is the Euclidean Heine–Borel theorem.

Example 8.25. Projection need not preserve closedness without compactness of the other factor. For instance, the hyperbola

$$H = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$$

is closed in $\mathbb{R}^2$, but its projection onto the first coordinate is $\mathbb{R} \setminus \{0\}$, which is not closed. Likewise, projection of a bounded set need not be closed: the set

$$B = \left\{ \left(x, \sin \frac{1}{x} \right) : 0 < x \leq 1 \right\}$$

is bounded in $\mathbb{R}^2$, while its first-coordinate projection is $(0, 1]$, again not closed.

Theorem 8.26 (Extreme Value Theorem). *Let X be a compact topological space and let $f : X \rightarrow \mathbb{R}$ be continuous. Then f is bounded and attains both its infimum and its supremum. In other words, there exist points $c, d \in X$ such that*

$$f(c) \leq f(x) \leq f(d) \quad (x \in X).$$

Proof. The image $A := f(X)$ is compact by Theorem 8.14. To see that A is bounded above, note that otherwise the family of open intervals $(-n, n)$, $n \in \mathbb{N}$, would form an open cover of A with no finite subcover. A similar argument shows that A is bounded below. Hence $m := \inf A$ and $M := \sup A$ exist.

We show that $M \in A$. If not, then for every $n \in \mathbb{N}$ one has $A \subseteq (-\infty, M - \frac{1}{n})$, so the open rays $(-\infty, M - \frac{1}{n})$ form an open cover of A . No finite subfamily can cover A , because the largest endpoint is still strictly less than M . This contradicts compactness. Thus $M \in A$. Similarly $m \in A$. Choosing $d, c \in X$ with $f(d) = M$ and $f(c) = m$ gives the conclusion. $\square$

Definition 8.27. Let (X, d) be a metric space and let $\mathcal{A}$ be an open cover of X . A number $\delta > 0$ is called a *Lebesgue number* for $\mathcal{A}$ if every subset $E \subseteq X$ with diameter $d(E) < \delta$ is contained in some member of $\mathcal{A}$.

Theorem 8.28. (*Lebesgue Covering Lemma*) If X is a compact metric space, then every open cover of X has a Lebesgue number.

Proof. Let $\mathcal{A}$ be an open cover of X . Choose a finite subcover $A_1, \dots, A_n$. If one of the A_i equals X , then any positive number is a Lebesgue number, so assume that each A_i is proper and set $C_i := X \setminus A_i$. Define

$$\varphi(x) := \frac{1}{n} \sum_{i=1}^n d(x, C_i), \quad x \in X.$$

Each function $x \mapsto d(x, C_i)$ is continuous, so φ is continuous. Moreover $\varphi(x) > 0$ for every $x \in X$: indeed, if $x \in A_i$ and A_i is open, then some ball $B(x, \varepsilon)$ is contained in A_i , which implies $d(x, C_i) \geq \varepsilon$. By compactness, φ attains a minimum $\delta > 0$ on X .

We claim that δ is a Lebesgue number. Let $E \subseteq X$ satisfy $d(E) < \delta$, and choose $x_0 \in E$. Since

$$\delta \leq \varphi(x_0) \leq d(x_0, C_i)$$

for some index i , we obtain $B(x_0, \delta) \subseteq A_i$. Because every point of E lies within distance $< \delta$ of x_0 , the set E is contained in $B(x_0, \delta)$, hence in A_i . Thus δ is a Lebesgue number for $\mathcal{A}$. $\square$

Definition 8.29. A map $f : (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces is said to be *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

for all $x_1, x_2 \in X$.

Theorem 8.30. Every continuous function on a compact metric space is uniformly continuous.

Proof. Let $f : (X, d_X) \rightarrow (Y, d_Y)$ be continuous, with X compact. Fix $\varepsilon > 0$. The family of open sets

$$\mathcal{U} := \{f^{-1}(B(y, \varepsilon/2)) : y \in Y\}$$

covers X . By the Lebesgue Covering Lemma, this cover has a Lebesgue number $\delta > 0$. If $d_X(x_1, x_2) < \delta$, then the set $\{x_1, x_2\}$ has diameter $< \delta$, so it is contained in some $f^{-1}(B(y, \varepsilon/2))$. Therefore both $f(x_1)$ and $f(x_2)$ lie in the same ball of radius $\varepsilon/2$, and hence

$$d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Thus f is uniformly continuous. $\square$

Definition 8.31. A point $x \in X$ is called *isolated* if the singleton $\{x\}$ is open in X .

Theorem 8.32. *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. We first prove a useful claim: if $U \subseteq X$ is a nonempty open set and $x \in X$, then there exists a nonempty open set $V \subseteq U$ such that $x \notin \bar{V}$. Indeed, choose $y \in U$ with $y \neq x$; this is possible because X has no isolated points. Since X is Hausdorff, there exist disjoint open sets $W_x \ni x$ and $W_y \ni y$. Then $V := U \cap W_y$ is nonempty, open, contained in U , and satisfies $x \notin \bar{V}$ because W_x is an open neighborhood of x disjoint from V .

Now suppose, for contradiction, that X were countable. Write $X = \{x_1, x_2, \dots\}$. Starting with $U_0 := X$, construct inductively a sequence of nonempty open sets U_n such that

$$U_n \subseteq U_{n-1} \quad \text{and} \quad x_n \notin \bar{U}_n$$

for every $n \geq 1$. This is possible by applying the claim with $U = U_{n-1}$ and $x = x_n$. Then the closed sets $\bar{U}_n$ are nonempty and nested, so by compactness and Theorem 8.22,

$$\bigcap_{n=1}^{\infty} \bar{U}_n \neq \emptyset.$$

Choose x in this intersection. For each n we have $x \in \bar{U}_n$, while $x_n \notin \bar{U}_n$ by construction; hence $x \neq x_n$ for all n . This contradicts the assumption that $\{x_n\}$ enumerates X . Therefore X is uncountable. $\square$

Corollary 8.33. *Every nondegenerate interval in $\mathbb{R}$ is uncountable.*

Proof. A closed interval $[a, b]$ with $a < b$ is compact and Hausdorff, and it has no isolated points. Hence Theorem 8.32 applies. Since (a, b) contains a closed subinterval, it is also uncountable. $\square$

Limit point compactness.

Definition 8.34. A topological space X is called *limit-point compact* if every infinite subset of X has a limit point.

Theorem 8.35. *Compactness implies limit-point compactness, but the converse need not hold.*

Proof. Let X be compact and let $A \subseteq X$ be infinite. If A had no limit point, then each $a \in A$ would admit an open neighborhood U_a with

$$U_a \cap (A \setminus \{a\}) = \emptyset.$$

In particular, A would be closed. Then

$$X = (X \setminus A) \cup \bigcup_{a \in A} U_a$$

would be an open cover of X . A finite subcover would force A to be finite, a contradiction. Therefore every infinite subset of X has a limit point. $\square$

Example 8.36. The converse fails in general. On an uncountable set equipped with the cocountable topology, every infinite subset has a limit point, so the space is limit-point compact. However, as noted in Remark 8.13, the cocountable topology on $\mathbb{R}$ is not compact.

Theorem 8.37. *Let X be a metrizable space. Then the following are equivalent:*

- (i) X is compact;
- (ii) X is limit-point compact;
- (iii) X is sequentially compact.

Proof. The implication (i) $\Rightarrow$ (ii) is Theorem 8.35.

To prove (ii) $\Rightarrow$ (iii), let (x_n) be a sequence in X . If the set $\{x_n : n \in \mathbb{N}\}$ is finite, then some value occurs infinitely often, yielding a constant subsequence. If the set is infinite, limit-point compactness gives a limit point x of that set. For each $k \in \mathbb{N}$, the ball $B(x, 1/k)$ contains infinitely many terms of the sequence, so one can choose indices

$$n_1 < n_2 < \cdots$$

with $x_{n_k} \in B(x, 1/k)$ for each k . Then $x_{n_k} \rightarrow x$, proving sequential compactness.

Finally, assume (iii) and prove (i). We first show that X is totally bounded. If not, there would exist $\varepsilon > 0$ such that no finite family of ε -balls covers X . Choose $x_1 \in X$ arbitrarily. Having chosen $x_1, \dots, x_n$, choose

$$x_{n+1} \in X \setminus \bigcup_{k=1}^n B(x_k, \varepsilon).$$

Then $d(x_m, x_n) \geq \varepsilon$ for $m \neq n$, so the sequence has no convergent subsequence, contradicting sequential compactness. Thus X is totally bounded.

Next we show that every open cover of X has a Lebesgue number. Suppose $\mathcal{U}$ is an open cover with no Lebesgue number. For each $n \in \mathbb{N}$ choose a nonempty set $C_n \subseteq X$ with

$$\text{diam}(C_n) < \frac{1}{n}$$

but such that C_n is not contained in any member of $\mathcal{U}$. Pick $x_n \in C_n$. By sequential compactness, some subsequence x_{n_k} converges to a point $x \in X$. Choose $U \in \mathcal{U}$ with $x \in U$. Since U is open, there exists $r > 0$ such that $B(x, r) \subseteq U$. For k large enough we have $x_{n_k} \in B(x, r/2)$ and $\text{diam}(C_{n_k}) < r/2$, which implies $C_{n_k} \subseteq B(x, r) \subseteq U$. This contradicts the choice of C_{n_k} . Hence every open cover has a Lebesgue number.

Now let $\mathcal{U}$ be an arbitrary open cover of X , and let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. By total boundedness, there exist points $x_1, \dots, x_m \in X$ such that

$$X \subseteq \bigcup_{j=1}^m B(x_j, \delta/2).$$

Each ball $B(x_j, \delta/2)$ has diameter $< \delta$, so it is contained in some member of $\mathcal{U}$. Choosing one such member for each j gives a finite subcover of X . Therefore X is compact. $\square$

Example 8.38. Let $e_n \in [0, 1]^{\mathbb{N}}$ denote the sequence whose n th coordinate is 1 and whose other coordinates are 0. Then

$$E := \{e_n : n \in \mathbb{N}\}$$

is an infinite subset of $X = [0, 1]^{\mathbb{N}}$ with no limit point for the metric

$$d(x, y) = \sup_n |x_n - y_n|.$$

Indeed, if $m \neq n$, then $d(e_m, e_n) = 1$, so the points of E stay uniformly separated and no point of X can be a limit point of E .

Example 8.39. The subspace $[0, 1]$ is *not* compact in the lower limit topology on $\mathbb{R}$ (see Example 2.16). Indeed,

$$[0, 1] \subseteq \left(\bigcup_{n \geq 2} [0, 1 - \frac{1}{n}) \right) \cup [1, 2),$$

and the right-hand side is an open cover in the subspace topology. No finite subfamily covers points arbitrarily close to 1 from the left, so no finite subcover exists.

Definition 8.40. A space X is said to be *countably compact* if every countable open cover of X admits a finite subcover.

Example 8.41. For a T_1 -space X , countable compactness and limit-point compactness are equivalent. The forward implication is immediate, and the converse uses the fact that a countable open cover without a finite subcover yields an infinite subset with no limit point.

Example 8.42. Let (X, d) be a compact metric space, and suppose that $f : X \rightarrow X$ satisfies

$$d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in X.$$

Then f is a homeomorphism. Indeed, f is an isometric embedding, hence injective and continuous. If f were not surjective, choose $y \in X \setminus f(X)$. Since $f(X)$ is compact, there exists $\delta > 0$ such that $d(y, f(X)) \geq \delta$. For $m < n$,

$$d(f^m(y), f^n(y)) = d(y, f^{n-m}(y)) \geq \delta,$$

so the orbit $y, f(y), f^2(y), \dots$ has no convergent subsequence, contradicting compactness. Thus f is bijective; since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, the claim follows.

Remark 8.43. (i) A topological space X is compact if every open cover consisting of basis elements admits a finite subcover.

(ii) A family $\mathcal{G}$ of closed subsets of X is called a *closed basis* if

$$\{X \setminus C : C \in \mathcal{G}\}$$

is a basis for the topology of X . Similarly, one defines a *closed subbasis* by requiring the family of complements to be an open subbasis.

Theorem 8.44 (Alexander subbase theorem). *Let X be a topological space and let $\mathcal{S}$ be a subbasis for the topology of X . Then X is compact if and only if every open cover of X by members of $\mathcal{S}$ admits a finite subcover.*

Equivalently, if

$$\mathcal{C} := \{X \setminus S : S \in \mathcal{S}\}$$

is the corresponding closed subbasis, then X is compact if and only if every family $\mathcal{F} \subseteq \mathcal{C}$ with the finite intersection property satisfies $\bigcap \mathcal{F} \neq \emptyset$.

Proof. The open-cover and closed-intersection formulations are equivalent by taking complements, so we work with the closed-subbase formulation.

($\Rightarrow$) If X is compact and $\mathcal{F} \subseteq \mathcal{C}$ has the finite intersection property, then $\bigcap \mathcal{F} \neq \emptyset$ by Theorem 8.22, since every set in $\mathcal{C}$ is closed.

($\Leftarrow$) Assume that every $\mathcal{F} \subseteq \mathcal{C}$ with the finite intersection property has nonempty intersection. By Theorem 8.22, it suffices to show that *every* family of closed subsets of X with the finite intersection property has nonempty intersection.

Let $\mathcal{A}$ be a family of closed subsets of X with the finite intersection property. By Zorn's lemma, extend $\mathcal{A}$ to a maximal family $\mathcal{M}$ of closed sets having the finite intersection property (with respect to inclusion among such families). Set $\mathcal{M}_0 := \mathcal{M} \cap \mathcal{C}$. Then $\mathcal{M}_0 \subseteq \mathcal{C}$ has the finite intersection property, hence by hypothesis there exists

$$x \in \bigcap \mathcal{M}_0.$$

We claim that $x \in \bigcap \mathcal{M}$. Suppose not. Then there exists $M \in \mathcal{M}$ with $x \notin M$, so $x \in X \setminus M$, which is open. Since $\mathcal{S}$ is a subbasis, there exist $S_1, \dots, S_n \in \mathcal{S}$ such that

$$x \in S_1 \cap \dots \cap S_n \subseteq X \setminus M.$$

Let $C_i := X \setminus S_i \in \mathcal{C}$. The above inclusion implies

$$M \subseteq C_1 \cup \dots \cup C_n. \tag{8.1}$$

We now use maximality of $\mathcal{M}$ to show that at least one of the sets C_i belongs to $\mathcal{M}$. Indeed, if $C_i \notin \mathcal{M}$ for each i , then for every i the family $\mathcal{M} \cup \{C_i\}$ fails the finite intersection property, so we can choose finitely many sets $M_{i,1}, \dots, M_{i,k_i} \in \mathcal{M}$ such that

$$C_i \cap M_{i,1} \cap \dots \cap M_{i,k_i} = \emptyset.$$

Let

$$M^* := M \cap \bigcap_{i=1}^n \bigcap_{j=1}^{k_i} M_{i,j}.$$

This is an intersection of finitely many members of $\mathcal{M}$, hence $M^* \neq \emptyset$ by the finite intersection property of $\mathcal{M}$. On the other hand, by (8.1) we have $M^* \subseteq C_1 \cup \dots \cup C_n$, while the displayed emptiness conditions imply $M^* \cap C_i = \emptyset$ for each i , a contradiction. Hence $C_i \in \mathcal{M}$ for some i .

But then $C_i \in \mathcal{M}_0$, so $x \in C_i$; this contradicts $x \in S_1 \cap \dots \cap S_n$ (since $S_i = X \setminus C_i$). Therefore $x \in \bigcap \mathcal{M} \subseteq \bigcap \mathcal{A}$, and in particular $\bigcap \mathcal{A} \neq \emptyset$. This completes the proof. $\square$

Theorem 8.45 (Heine–Borel theorem on $\mathbb{R}$). *Every closed and bounded subset of $\mathbb{R}$ (with the usual topology) is compact. In particular, every closed interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. It suffices to prove that $[a, b]$ is compact, since any closed and bounded $K \subset \mathbb{R}$ is contained in some $[a, b]$ and is closed in $[a, b]$, hence compact by Theorem 8.6.

Consider the subspace $[a, b] \subset \mathbb{R}$. The collection

$$\mathcal{S} := \{[a, d] : a < d \leq b\} \cup \{(c, b] : a \leq c < b\}$$

is a subbasis for the subspace topology on $[a, b]$ (each set is the intersection of $[a, b]$ with an open ray in $\mathbb{R}$). The corresponding closed subbasis $\mathcal{C} = \{[d, b] : a \leq d \leq b\} \cup \{[a, c] : a \leq c \leq b\}$ consists of closed intervals.

Let $\mathcal{F} \subseteq \mathcal{C}$ have the finite intersection property. Define

$$A := \{c \in [a, b] : [a, c] \in \mathcal{F}\}, \quad B := \{d \in [a, b] : [d, b] \in \mathcal{F}\}.$$

Let $c_0 := \inf A$ if $A \neq \emptyset$ (and $c_0 := b$ if $A = \emptyset$), and let $d_0 := \sup B$ if $B \neq \emptyset$ (and $d_0 := a$ if $B = \emptyset$). We claim that $d_0 \leq c_0$. Otherwise $c_0 < d_0$; choose $c \in A$ and $d \in B$ with $c < d$. Then $[a, c] \cap [d, b] = \emptyset$, contradicting the finite intersection property.

Choose any $x \in [d_0, c_0] \subseteq [a, b]$. If $[a, c] \in \mathcal{F}$, then $c \geq c_0 \geq x$, so $x \in [a, c]$. If $[d, b] \in \mathcal{F}$, then $d \leq d_0 \leq x$, so $x \in [d, b]$. Hence $x \in \bigcap \mathcal{F}$, so $\bigcap \mathcal{F} \neq \emptyset$. By the Alexander subbase theorem Theorem 8.44, the interval $[a, b]$ is compact. $\square$

Theorem 8.46 (Tychonoff's theorem). *Let $\{X_i\}_{i \in I}$ be a family of nonempty compact spaces. Then the product $\prod_{i \in I} X_i$, equipped with the product topology, is compact.*

Proof. Let $X := \prod_{i \in I} X_i$ and let $\pi_i : X \rightarrow X_i$ be the coordinate projections. A subbasis for the product topology is given by sets of the form $\pi_i^{-1}(U)$, where $U \subseteq X_i$ is open. Equivalently, a closed subbasis is given by sets of the form $\pi_i^{-1}(F)$, where $F \subseteq X_i$ is closed.

By the Alexander subbase theorem Theorem 8.44, it is enough to show that any family $\mathcal{F}$ of closed subbasic sets $\pi_i^{-1}(F)$ with the finite intersection property has nonempty intersection.

Let $\mathcal{F}$ be such a family. For each $i \in I$, define

$$\mathcal{F}_i := \{ F \subseteq X_i \text{ closed} : \pi_i^{-1}(F) \in \mathcal{F} \}.$$

Then $\mathcal{F}_i$ has the finite intersection property in X_i : if $F_1, \dots, F_n \in \mathcal{F}_i$, then

$$\pi_i^{-1}(F_1 \cap \dots \cap F_n) = \pi_i^{-1}(F_1) \cap \dots \cap \pi_i^{-1}(F_n) \neq \emptyset$$

by the finite intersection property of $\mathcal{F}$, hence $F_1 \cap \dots \cap F_n \neq \emptyset$. Since each X_i is compact, Theorem 8.22 yields $\bigcap \mathcal{F}_i \neq \emptyset$. Using the axiom of choice, choose $x_i \in \bigcap \mathcal{F}_i$ (and if $\mathcal{F}_i = \emptyset$, choose $x_i \in X_i$ arbitrarily). Let $x := (x_i)_{i \in I} \in X$.

If $\pi_i^{-1}(F) \in \mathcal{F}$, then $F \in \mathcal{F}_i$ and hence $x_i \in F$, so $x \in \pi_i^{-1}(F)$. Therefore $x \in \bigcap \mathcal{F}$, proving $\bigcap \mathcal{F} \neq \emptyset$, as required. $\square$

Theorem 8.47 (Generalized Heine–Borel theorem). *Every closed and bounded subset of $\mathbb{R}^n$ (with the usual topology) is compact.*

Proof. Let $K \subset \mathbb{R}^n$ be closed and bounded. Choose real numbers $a_i < b_i$ such that

$$K \subseteq R := \prod_{i=1}^n [a_i, b_i].$$

By Theorem 8.45, each interval $[a_i, b_i]$ is compact; hence the product R , equipped with the product topology, is compact by Theorem 8.46. Moreover, the product topology on R coincides with the subspace topology inherited from $\mathbb{R}^n$, since basic open sets in the product topology are intersections of R with open rectangles in $\mathbb{R}^n$.

Finally, K is closed in R (as K is closed in $\mathbb{R}^n$ and R has the subspace topology), so K is compact by Theorem 8.6. $\square$

Definition 8.48. A topological space X is *locally compact* if every point $x \in X$ has an open neighborhood U such that $\overline{U}$ is compact.

Example 8.49. Every Euclidean space $\mathbb{R}^n$ (and likewise $\mathbb{C}^n$) is locally compact. By contrast, an infinite-dimensional Hilbert space is not locally compact: the closure of any nontrivial open ball fails to be compact.

Proposition 8.50. *Let X be a Hausdorff space. Then X is locally compact if and only if, for every $x \in X$ and every open neighborhood W of x , there exists an open set V such that*

$$x \in V \subseteq \bar{V} \subseteq W \quad \text{and} \quad \bar{V} \text{ is compact.}$$

Proof. The stated local-shrinking property immediately implies local compactness by taking $W = X$. Conversely, assume that X is locally compact and Hausdorff. Fix $x \in X$ and an open neighborhood W of x . Choose an open set $U_0 \ni x$ with $\bar{U}_0$ compact, and set $U := U_0 \cap W$. Then U is open, $x \in U \subseteq W$, and $\bar{U}$ is a closed subset of the compact Hausdorff space $\bar{U}_0$, hence compact. Thus it is enough to shrink inside the compact Hausdorff space $K := \bar{U}$.

The set $K \setminus W$ is closed in K and does not contain x . Since compact Hausdorff spaces are normal, there is an open set O in the subspace K such that

$$x \in O \subseteq \bar{O}^K \subseteq K \setminus (K \setminus W) = K \cap W.$$

Choose an open set $G \subseteq X$ with $O = G \cap K$, and put $V := G \cap U$. Then V is open in X , contains x , and

$$\bar{V}^X \subseteq \bar{O}^K \subseteq W.$$

Moreover $\bar{V}^X$ is closed in the compact space K , and is therefore compact. This gives the required set V . $\square$

8.2 Compactifications

Many spaces arising in analysis and geometry are not compact, yet they become more tractable once embedded densely in a compact space. A compactification adds ideal boundary points at infinity in a controlled way. This viewpoint packages asymptotic behavior into ordinary topological behavior and provides a bridge between local structure and global completion.

When a space X is not compact, it is often useful to embed it into a compact space that contains X as a dense subspace; such an embedding is called a *compactification* of X . A familiar example is the *two-point compactification* of $\mathbb{R}$, obtained by adjoining $-\infty$ and $+\infty$, yielding the compact ordered space $[-\infty, +\infty]$.

The simplest general procedure is to adjoin a single point. Geometrically, this is what happens when one identifies $\mathbb{R}^n$ with the sphere S^n minus one point. For the standard one-point compactification construction, however, one must assume that X is *locally compact Hausdorff*; without this hypothesis, the family of complements of compact sets need not define a topology at infinity. Accordingly, let X be a locally compact Hausdorff space and set

$$X^* := X \cup \{\infty\}.$$

We topologize X^* by declaring that the open sets are:

- (i) the open subsets of X (viewed as subsets of X^*), and
- (ii) the sets $U \subseteq X^*$ with $\infty \in U$ such that $X^* \setminus U$ is a compact subset of X .

Because X is Hausdorff, compact subsets of X are closed, and this is exactly the point that makes the topology work. If X is moreover noncompact, then X sits in X^* as a dense open subspace, and X^* is called the *one-point compactification* (or *Alexandroff compactification*) of X . If X is already compact, the same recipe simply adjoins an isolated point; the resulting space is still compact Hausdorff, but X is no longer dense, so in that case one usually does not speak of a compactification. We now prove the precise statement.

Example 8.51. Compactifications need not be unique. For instance, the interval $(-1, 1)$ sits densely inside the compact interval $[-1, 1]$, but it also sits inside the circle S^1 as $S^1 \setminus \{p\}$ for any point $p \in S^1$. Since $[-1, 1]$ is not homeomorphic to S^1 , these give genuinely different compactifications of the same space.

The next theorem records the basic properties of the one-point compactification and identifies exactly when it is Hausdorff.

Theorem 8.52 (One-point compactification theorem). *Let X be a locally compact Hausdorff space, and let $X^* = X \cup \{\infty\}$ be equipped with the topology described above. Then X^* is a compact Hausdorff space and X is an open subspace of X^* . If X is noncompact, then X is dense in X^* , so X^* is a genuine compactification of X . Conversely, if a Hausdorff space X admits a Hausdorff compactification obtained by adjoining a single point, then X is locally compact.*

Proof. Let $\mathcal{T}^*$ denote the family of subsets of X^* described above. We first verify that $\mathcal{T}^*$ is a topology. The empty set is open in X , and $X^* \in \mathcal{T}^*$ because $X^* \setminus X^* = \emptyset$ is compact.

Let $\{U_j\}_{j \in J} \subseteq \mathcal{T}^*$. If $\infty \notin \bigcup_j U_j$, then every U_j is open in X , and hence $\bigcup_j U_j$ is open in X . If $\infty \in \bigcup_j U_j$, choose j_0 with $\infty \in U_{j_0}$. Then

$$K := X^* \setminus U_{j_0}$$

is a compact subset of X . Moreover,

$$X^* \setminus \bigcup_{j \in J} U_j = K \cap \bigcap_{j \in J} (X^* \setminus U_j).$$

For each j , the set $K \cap (X^* \setminus U_j)$ is closed in K : this is clear when $\infty \in U_j$, and when $U_j \subseteq X$ is open it equals $K \setminus U_j$. Hence the displayed complement is closed in the compact space K , and is therefore compact. Thus arbitrary unions belong to $\mathcal{T}^*$.

For finite intersections it suffices to consider two sets $U, V \in \mathcal{T}^*$. If $\infty \in U \cap V$, then

$$X^* \setminus (U \cap V) = (X^* \setminus U) \cup (X^* \setminus V),$$

a finite union of compact subsets of X , hence compact. If $\infty \notin U \cap V$, then $U \cap V \subseteq X$. In this case each factor restricts to an open subset of X : this is immediate for factors already open in

X , while a neighborhood of ∞ restricts to the complement in X of a compact subset, hence to an open set because X is Hausdorff. Thus $U \cap V$ is open in X . Hence $\mathcal{T}^*$ is a topology.

By construction, the inclusion $X \hookrightarrow X^*$ identifies X with an open subspace carrying its original topology.

Compactness. Let $\{U_j\}_{j \in J}$ be an open cover of X^* . Choose j_0 with $\infty \in U_{j_0}$. Then $K := X^* \setminus U_{j_0}$ is compact in X . The family $\{U_j \cap X : j \in J\}$ is an open cover of K , so finitely many of these sets cover K . Together with U_{j_0} , they form a finite subcover of X^* . Thus X^* is compact.

Hausdorffness. If two distinct points both lie in X , they are separated by the Hausdorff property of X . Now let $x \in X$. By local compactness and Proposition 8.50, choose an open neighborhood V of x in X such that $\overline{V}^X$ is compact. Then

$$X^* \setminus \overline{V}^X$$

is an open neighborhood of ∞ , disjoint from V . Hence X^* is Hausdorff.

If X is noncompact, then every neighborhood U of ∞ satisfies $X^* \setminus U = K$ for some compact $K \subseteq X$. Since $K \neq X$, the set $U \cap X = X \setminus K$ is nonempty. Thus every neighborhood of ∞ meets X , and X is dense in X^* .

Conversely, suppose that $Y = X \cup \{\infty\}$ is a compact Hausdorff space in which X is dense and carries its given topology as a subspace. Fix $x \in X$. Since compact Hausdorff spaces are regular, choose an open neighborhood O of x in Y such that $\infty \notin \overline{O}^Y$. Then $V := O \cap X$ is an open neighborhood of x in X , and

$$\overline{V}^X \subseteq \overline{O}^Y \cap X.$$

The set $\overline{O}^Y$ is closed in the compact space Y , hence compact; since $\infty \notin \overline{O}^Y$, it is contained in X . Therefore $\overline{V}^X$ is compact. Thus X is locally compact. Since X is a subspace of the Hausdorff space Y , it is Hausdorff as well. $\square$

Remark 8.53. If X is compact, then in the one-point compactification X^* the added point ∞ is isolated: indeed $X = X^* \setminus \{\infty\}$ is compact, so $\{\infty\}$ is open by definition of the topology at ∞ . Conversely, if ∞ is isolated in X^* , then $X = X^* \setminus \{\infty\}$ is closed in the compact space X^* , hence compact.

Remark 8.54 (Stone–Čech compactification (optional preview)). Beyond the one-point construction, the Stone–Čech compactification βX provides, for suitable spaces X (e.g. completely regular T_1 spaces), a *largest* compactification characterized by a universal extension property: every continuous map $f : X \rightarrow K$ into a compact Hausdorff space K extends uniquely to a continuous map $\beta f : \beta X \rightarrow K$. We do not develop βX in detail here, but this universal perspective is the conceptual template for compactification theory.

Supplementary remarks and examples

Remark 8.55. Compactness is fundamentally an open-cover property. In metric spaces it often appears through sequences, but outside the metric setting one should resist the temptation to treat sequential compactness as the primary notion. The cover formulation is the one that behaves well under products and under continuous images.

Example 8.56. On an infinite set endowed with the cofinite topology, every open cover admits a finite subcover, so the space is compact. However, a subset is closed only when it is finite or the whole space. This shows that compactness and closedness are logically unrelated unless additional separation axioms, such as Hausdorffness, are imposed.

Problem Set

Problem guide. Approach the problems in three passes. First verify basic permanence properties of compactness and local compactness. Next practice the standard “finite-subcover extraction” arguments, especially in products. Finally use the harder problems to understand why one-point compactification and Tychonoff’s theorem are structural rather than accidental facts.

Exercise 8.57. Prove that a closed subset of a compact space is compact.

Exercise 8.58. Prove that a compact subset of a Hausdorff space is closed.

Exercise 8.59. Show that if X is compact and $f : X \rightarrow \mathbb{R}$ is continuous, then f attains its maximum and minimum values.

Exercise 8.60. Let X be locally compact Hausdorff. Construct explicitly the one-point compactification $X^* = X \cup \{\infty\}$ and show that X^* is compact Hausdorff.

Exercise 8.61. Show that a Hausdorff space X is locally compact if and only if every $x \in X$ has a neighborhood whose closure is compact.

Exercise 8.62 (Tychonoff for countable products). Let $\{X_n\}_{n \in \mathbb{N}}$ be compact spaces. Prove that $\prod_{n \in \mathbb{N}} X_n$ is compact.

Urysohn's lemma and Tietze's extension theorem

Normality is the separation hypothesis under which closed sets can be separated by continuous functions. Urysohn's lemma provides such separating functions, and Tietze's theorem upgrades this to extend continuous real-valued functions from closed subsets. These results are fundamental in analysis and in metrization theory.

Roadmap.

- Urysohn's lemma is presented as the fundamental bridge from set-theoretic separation to function-theoretic separation.
- The Tietze theorem is built from an approximation lemma and a uniformly convergent series, making the architecture of the proof as important as the final statement.
- These results prepare directly for the metrization theorem in the next chapter, where many separating functions are assembled into coordinates.

9.1 Urysohn's lemma

The abstract separation axioms become much more powerful when they can be translated into statements about continuous real-valued functions. Urysohn's lemma is the first major result of this kind: in a normal space, disjoint closed sets can be separated not merely by neighborhoods but by a continuous scalar field. This theorem is both intrinsically important and a key technical ingredient in later extension and metrization results.

Theorem 9.1 (Urysohn's lemma). *Let X be a normal space and let $F, G \subset X$ be disjoint closed sets. Then there exists a continuous function $u : X \rightarrow [0, 1]$ such that $u|_F \equiv 0$ and $u|_G \equiv 1$.*

Proof. Since X is normal, for any disjoint closed sets $A, B \subset X$ there exist disjoint open sets separating them. Let $\mathbb{D} = \left\{ \frac{k}{2^n} : n \in \mathbb{N}, 0 \leq k \leq 2^n \right\}$ be the dyadic rationals in $[0, 1]$.

We construct open sets $U_r \subset X$ for $r \in \mathbb{D}$ satisfying

(U1) $F \subset U_0$ and $U_1 = X \setminus G$.

(U2) If $r < s$, then $\overline{U_r} \subset U_s$ for all $r, s \in \mathbb{D}$.

Start with U_0 open with $F \subset U_0 \subset \overline{U_0} \subset X \setminus G$ (normality with F and G). Set $U_1 = X \setminus G$. Assume $U_{k/2^n}$ are constructed for a fixed n satisfying (U2) for adjacent dyadic rationals. For

each odd k (so $k/2^{n+1}$ lies strictly between $(k-1)/2^{n+1}$ and $(k+1)/2^{n+1}$), apply normality to the disjoint closed sets $\overline{U_{(k-1)/2^{n+1}}}$ and $X \setminus U_{(k+1)/2^{n+1}}$ to obtain an open set $U_{k/2^{n+1}}$ with

$$\overline{U_{(k-1)/2^{n+1}}} \subset U_{k/2^{n+1}} \subset \overline{U_{k/2^{n+1}}} \subset U_{(k+1)/2^{n+1}}.$$

Iterating yields the family $\{U_r\}_{r \in \mathbb{D}}$ satisfying (U1)–(U2).

Define $u : X \rightarrow [0, 1]$ by

$$u(x) = \inf\{r \in \mathbb{D} : x \in U_r\},$$

with the convention $\inf \emptyset = 1$. Then $u(x) = 0$ on F and $u(x) = 1$ on G by (U1). To see continuity, fix $a \in [0, 1]$. Using (U2), one checks that

$$\{x : u(x) < a\} = \bigcup_{\substack{r \in \mathbb{D} \\ r < a}} U_r \quad \text{and} \quad \{x : u(x) > a\} = \bigcup_{\substack{r \in \mathbb{D} \\ r > a}} (X \setminus \overline{U_r}),$$

both open; hence u is continuous. □

9.2 Tietze's extension theorem

Once continuous functions can separate closed sets, the next natural question is whether continuous data prescribed on a closed subset can be extended to the whole space. Tietze's theorem gives a striking affirmative answer for normal spaces. It is one of the clearest demonstrations that separation axioms control the existence of globally defined functions.

Lemma 9.2 (Tietze approximation step). *Let X be a normal space, let $A \subseteq X$ be closed, and let $h : A \rightarrow [-1, 1]$ be continuous. Then there exists a continuous function $g : X \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ such that*

$$|h(x) - g(x)| \leq \frac{2}{3} \quad (x \in A).$$

Proof. Set

$$F := h^{-1}([-1, -\frac{1}{3}]), \quad G := h^{-1}([\frac{1}{3}, 1]).$$

Since h is continuous and A is closed in X , the sets F and G are disjoint closed subsets of X . By Urysohn's lemma there exists a continuous map $u : X \rightarrow [0, 1]$ such that $u|_F = 0$ and $u|_G = 1$. Define

$$g(x) := \frac{2}{3}u(x) - \frac{1}{3} \quad (x \in X).$$

Then g is continuous and takes values in $[-\frac{1}{3}, \frac{1}{3}]$. Moreover, if $x \in F$, then $g(x) = -\frac{1}{3}$ and $h(x) \in [-1, -\frac{1}{3}]$, so $|h(x) - g(x)| \leq \frac{2}{3}$. If $x \in G$, then $g(x) = \frac{1}{3}$ and $h(x) \in [\frac{1}{3}, 1]$, so again $|h(x) - g(x)| \leq \frac{2}{3}$. Finally, if $x \in A \setminus (F \cup G)$, then $|h(x)| < \frac{1}{3}$ and $|g(x)| \leq \frac{1}{3}$, whence $|h(x) - g(x)| \leq \frac{2}{3}$. This proves the lemma. □

Theorem 9.3 (Tietze extension theorem). *Let X be a normal space and let $A \subset X$ be closed. Every continuous function $f : A \rightarrow [-1, 1]$ admits a continuous extension $\tilde{f} : X \rightarrow [-1, 1]$ with $\tilde{f}|_A = f$.*

Proof. Set $r_0 := f$. Applying Lemma 9.2 to r_0 , we obtain a continuous function $g_0 : X \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ such that

$$|r_0(x) - g_0(x)| \leq \frac{2}{3} \quad (x \in A).$$

Suppose inductively that for some $n \geq 0$ we have constructed continuous functions

$$g_0, g_1, \dots, g_n : X \rightarrow \mathbb{R}$$

such that, with

$$s_n := \sum_{k=0}^n g_k, \quad r_{n+1} := f - s_n|_A,$$

one has

$$\|r_{n+1}\|_{\infty, A} \leq \left(\frac{2}{3}\right)^{n+1} \quad \text{and} \quad \|g_k\|_{\infty, X} \leq \frac{1}{3} \left(\frac{2}{3}\right)^k \quad (0 \leq k \leq n).$$

Apply Lemma 9.2 to the normalized function

$$h_{n+1} := \left(\frac{3}{2}\right)^{n+1} r_{n+1} : A \rightarrow [-1, 1].$$

We obtain a continuous function $\varphi_{n+1} : X \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ such that

$$|h_{n+1}(x) - \varphi_{n+1}(x)| \leq \frac{2}{3} \quad (x \in A).$$

Now define

$$g_{n+1} := \left(\frac{2}{3}\right)^{n+1} \varphi_{n+1}.$$

Then

$$\|g_{n+1}\|_{\infty, X} \leq \frac{1}{3} \left(\frac{2}{3}\right)^{n+1},$$

and on A we have

$$\|r_{n+1} - g_{n+1}|_A\|_{\infty, A} = \left(\frac{2}{3}\right)^{n+1} \|h_{n+1} - \varphi_{n+1}|_A\|_{\infty, A} \leq \left(\frac{2}{3}\right)^{n+2}.$$

Thus the induction continues for all $n \geq 0$.

Since

$$\sum_{n=0}^{\infty} \|g_n\|_{\infty, X} \leq \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = 1,$$

the series $\sum_{n=0}^{\infty} g_n$ converges uniformly on X by the Weierstrass M-test. Let

$$\tilde{f} := \sum_{n=0}^{\infty} g_n.$$

Then $\tilde{f}$ is continuous on X , and $|\tilde{f}(x)| \leq 1$ for all $x \in X$. Finally, for $x \in A$ the remainder estimate gives

$$|f(x) - \sum_{k=0}^n g_k(x)| = |r_{n+1}(x)| \leq \left(\frac{2}{3}\right)^{n+1} \xrightarrow{n \rightarrow \infty} 0,$$

so $\tilde{f}(x) = f(x)$ on A . Hence $\tilde{f}$ is the desired extension. $\square$

Remark 9.4. The proof is constructive: one corrects the current approximation by a uniformly controlled Urysohn function at each stage, and the geometric decay of the errors guarantees uniform convergence.

Supplementary remarks and examples

Remark 9.5. Urysohn's lemma and Tietze's theorem are not merely technical results about functions into $[0, 1]$ or $\mathbb{R}$. They reveal that normal spaces admit a surprisingly rich supply of continuous functions, enough to separate closed sets and to extend data from closed subspaces.

Example 9.6. If X is discrete, every function $f : X \rightarrow [0, 1]$ is continuous. Thus Urysohn's lemma and Tietze's extension theorem become immediate in that setting. The interest of the theorems lies in the fact that normality allows one to recover a similar separation-and-extension mechanism far beyond the discrete case.

Problem Set

Problem guide. The main skill here is to keep track of what is being separated from what. In Urysohn-type arguments, monitor the nesting of open sets and the closed-set containment at every stage. In Tietze-type arguments, organize the approximation scheme before writing details. These problems are best solved slowly, with the construction written as a sequence of deliberate design choices rather than as a block of formulas.

Exercise 9.7. Let X be normal and let $F, G \subseteq X$ be disjoint closed sets. Use Urysohn's lemma to show that there exists a continuous $f : X \rightarrow [0, 1]$ with $f(F) = \{0\}$ and $f(G) = \{1\}$.

Exercise 9.8. Show that a T_1 space X is normal if and only if for every closed $F \subseteq X$ and every open set $U \supseteq F$, there exists an open V with $F \subseteq V \subseteq \overline{V} \subseteq U$.

Exercise 9.9. Assuming Tietze's extension theorem, prove that if X is normal then every continuous $f : F \rightarrow [-1, 1]$ defined on a closed $F \subseteq X$ extends to a continuous $F : X \rightarrow [-1, 1]$.

Exercise 9.10. Show that normality is inherited by closed subspaces: if X is normal and $F \subseteq X$ is closed, then F is normal.

Exercise 9.11. Let X be normal. Prove that any two disjoint closed sets can be separated by disjoint open neighborhoods.

Urysohn's metrization theorem

Metrization theorems characterize when a purely topological space arises from a metric. Urysohn's theorem states that a second countable regular space is metrizable. We present a standard approach: use Urysohn functions to embed the space into a countable product of intervals, and then pull back a metric.

Roadmap.

- The preliminary lemma shows that second countability plus regularity already implies normality, so the separation machinery from the previous chapter becomes available.
- The metrization proof then constructs a countable family of Urysohn functions that separates points from closed sets and therefore defines an embedding into $[0, 1]^{\mathbb{N}}$.
- The final metric is not guessed but transported from the Hilbert cube, which makes clear why countability is indispensable in the argument.

10.1 Statement and proof

A metrization theorem identifies abstract topological hypotheses that are strong enough to reconstruct an honest metric. Urysohn's theorem is especially important because second countability provides a countable amount of local information, while regularity supplies the separating functions needed to convert that information into coordinates. The proof shows how these ingredients combine to produce an explicit metric.

Lemma 10.1. *Every second countable regular T_1 space is normal.*

Proof. Let X be second countable, regular, and T_1 , and let $A, B \subseteq X$ be disjoint closed sets. Choose a countable base $\{B_n\}_{n \in \mathbb{N}}$ for X . For each $a \in A$, regularity yields an open set U_a such that

$$a \in U_a \subseteq \overline{U_a} \subseteq X \setminus B.$$

Choose a basic open set $B_{n(a)}$ with $a \in B_{n(a)} \subseteq U_a$. Then $\overline{B_{n(a)}} \subseteq X \setminus B$. Since the base is countable, the collection of distinct sets $B_{n(a)}$ is countable; enumerate it as $C_1, C_2, \dots$. Set

$$U := \bigcup_{k \geq 1} C_k, \quad V := X \setminus \bigcup_{k \geq 1} \overline{C_k}.$$

Then U and V are open, $A \subseteq U$, and $B \subseteq V$. Moreover $U \cap V = \emptyset$, because each $C_k \subseteq \overline{C_k}$. Thus X is normal. $\square$

Theorem 10.2 (Urysohn metrization theorem). *Let X be a T_1 regular, second countable space. Then X is metrizable; that is, there exists a metric d on X whose induced topology equals the given topology.*

Proof. Let $\{B_n\}_{n \in \mathbb{N}}$ be a countable base for X . By Lemma 10.1, the space X is normal. Consider the countable set of index pairs

$$\mathcal{P} := \{(m, n) \in \mathbb{N} \times \mathbb{N} : \overline{B_m} \subseteq B_n\}.$$

For each $(m, n) \in \mathcal{P}$, Urysohn's lemma provides a continuous function

$$f_{m,n} : X \rightarrow [0, 1]$$

such that

$$f_{m,n} = 1 \text{ on } \overline{B_m}, \quad f_{m,n} = 0 \text{ on } X \setminus B_n.$$

Enumerate this countable family as $f_1, f_2, \dots$

We first show that the sets

$$\mathcal{W} := \{f_k^{-1}((1/2, 1]) : k \in \mathbb{N}\}$$

form a basis for the topology of X . Each set in $\mathcal{W}$ is open because the corresponding f_k is continuous. Conversely, let $O \subseteq X$ be open and let $x \in O$. Choose a basic neighborhood B_n with $x \in B_n \subseteq O$. By regularity, there exists an open set U such that

$$x \in U \subseteq \overline{U} \subseteq B_n.$$

Choose a basis element B_m with $x \in B_m \subseteq U$. Then $\overline{B_m} \subseteq \overline{U} \subseteq B_n$, so $(m, n) \in \mathcal{P}$. For the corresponding function $f_{m,n}$, one has $f_{m,n}(x) = 1$ because $x \in \overline{B_m}$, while $f_{m,n}(y) = 0$ for every $y \notin B_n$. Hence

$$x \in f_{m,n}^{-1}((1/2, 1]) \subseteq B_n \subseteq O.$$

Thus every open set is a union of sets from $\mathcal{W}$, proving the claim.

Now define

$$F : X \longrightarrow [0, 1]^{\mathbb{N}}, \quad F(x) := (f_k(x))_{k \in \mathbb{N}}.$$

Since each coordinate function is continuous, F is continuous. Moreover, F is injective: if $x \neq y$, then $X \setminus \{y\}$ is an open neighborhood of x because X is T_1 . By the basis property just proved, there exists k such that

$$x \in f_k^{-1}((1/2, 1]) \subseteq X \setminus \{y\}.$$

Therefore $f_k(x) > 1/2$ while $f_k(y) \leq 1/2$, so $F(x) \neq F(y)$.

Because the sets $f_k^{-1}((1/2, 1])$ form a basis, the topology on X is exactly the initial topology generated by the family $\{f_k\}_{k \in \mathbb{N}}$. Equivalently, F is a homeomorphism of X onto the subspace $F(X) \subseteq [0, 1]^{\mathbb{N}}$; in other words, F is a topological embedding.

Finally, equip $[0, 1]^{\mathbb{N}}$ with the metric

$$D(u, v) = \sum_{k=1}^{\infty} 2^{-k} |u_k - v_k|.$$

This metric induces the product topology on $[0, 1]^{\mathbb{N}}$. Define

$$d(x, y) := D(F(x), F(y)) \quad (x, y \in X).$$

Since F is an embedding, the metric d induces exactly the original topology on X . Thus X is metrizable. $\square$

Remark 10.3. The proof shows more: X embeds homeomorphically into the Hilbert cube $[0, 1]^{\mathbb{N}}$.

Supplementary remarks and examples

Remark 10.4. Metrization theorems convert qualitative topological hypotheses into an explicit distance function. Urysohn's theorem is powerful precisely because second countability lets one produce only countably many separating functions, which can then be assembled into a single metric.

Example 10.5. The real line with its usual topology is metrizable by the Euclidean metric, but Urysohn's theorem gives a conceptual explanation. The space is regular, and rational intervals provide a countable basis. Thus the theorem recovers metrizability from purely topological input rather than from a preassigned formula for distance.

Problem Set

Problem guide. Treat this final problem set as a synthesis of the entire course. You will use bases, separation axioms, continuous functions, and product constructions together in a single argument. When solving metrization problems, do not rush to guess a metric; instead, identify the topological ingredients that make the construction possible and then check carefully that the resulting distance induces exactly the original topology.

Exercise 10.6. Let X be second countable and regular. Show that for every $x \in X$ and every open neighborhood $O \ni x$, there exists a continuous $f : X \rightarrow [0, 1]$ such that $f(x) = 1$ and $f \equiv 0$ on $X \setminus O$.

Exercise 10.7. Show that the metric

$$D(u, v) = \sum_{k=1}^{\infty} 2^{-k} |u_k - v_k|$$

induces the product topology on $[0, 1]^{\mathbb{N}}$.

Exercise 10.8. Let X be T_1 , regular, and second countable. Using the embedding into $[0, 1]^{\mathbb{N}}$ from Urysohn's metrization theorem, prove that X is Hausdorff.

Exercise 10.9. Prove that every second countable metrizable space admits a compatible complete metric if and only if it is completely metrizable (state any needed definitions precisely).

Convergence: sequences, nets, and filters

In metric spaces, many topological notions can be phrased using sequences. In general topological spaces, however, sequences may fail to detect closure and continuity. The correct replacement is the language of nets (and, equivalently, filters). We develop these notions and record the standard characterizations: closure via nets, continuity via nets, and compactness via ultrafilters. We also explain how the countability axioms recover the sequential picture.

Roadmap.

- The closure theorem for nets is the decisive reason to introduce nets: it restores in full generality the role played by sequences in first countable spaces.
- Subnets, cluster points, and the net characterization of continuity are developed as a coherent package rather than as isolated definitions.
- Ultrafilters supply a second compactness formalism, while the closing proposition explains precisely when one may safely return to sequences.

11.1 Why sequences are not enough

In metric and first countable spaces, sequences detect most of the phenomena one cares about. In arbitrary spaces, however, sequential convergence may miss genuine closure points and may fail to characterize continuity. Nets are introduced precisely to repair this defect without abandoning the intuitive idea of convergence.

In an arbitrary topological space, the set of limits of sequences from $A \subseteq X$ may be strictly smaller than the closure $\overline{A}$. This motivates the introduction of generalized sequences indexed by directed sets.

Definition 11.1 (Directed set). A *directed set* is a nonempty set D equipped with a preorder $\preceq$ such that for any $d_1, d_2 \in D$ there exists $d \in D$ with $d_1 \preceq d$ and $d_2 \preceq d$.

Definition 11.2 (Net). Let X be a set and D a directed set. A *net* in X is a function $x_\bullet : D \rightarrow X$, written $d \mapsto x_d$.

Definition 11.3 (Convergence of a net). Let X be a topological space and $x_\bullet = (x_d)_{d \in D}$ a net in X . We say $x_d \rightarrow x$ (or $x_\bullet \rightarrow x$) if for every neighborhood U of x there exists $d_0 \in D$ such that $x_d \in U$ for all $d \succeq d_0$.

Remark 11.4. Every sequence $(x_n)_{n \in \mathbb{N}}$ is a net by taking $D = \mathbb{N}$ with the usual order. Thus nets extend sequences without altering the familiar convergence notion.

11.2 Closure and cluster points via nets

One of the principal reasons for introducing nets is that they recover the familiar equivalence between belonging to a closure and arising as a limit of points from the set. This section shows that the net formalism is not an abstract luxury but the correct general replacement for sequences in arbitrary spaces.

Theorem 11.5 (Closure via nets). *Let X be a topological space, $A \subseteq X$, and $x \in X$. Then $x \in \overline{A}$ if and only if there exists a net $(a_d)_{d \in D}$ in A such that $a_d \rightarrow x$.*

Proof. If $a_d \rightarrow x$ with $a_d \in A$, then every neighborhood U of x contains some a_d , hence intersects A ; this is exactly $x \in \overline{A}$.

Conversely, assume $x \in \overline{A}$. Let D be the set of all neighborhoods of x , directed by reverse inclusion: $U \preceq V$ if and only if $U \supseteq V$. For each $U \in D$ choose $a_U \in A \cap U$ (possible since $U \cap A \neq \emptyset$). Then $(a_U)_{U \in D}$ is a net in A , and by construction $a_U \rightarrow x$. $\square$

Definition 11.6 (Cluster point of a net). Let $x_\bullet = (x_d)_{d \in D}$ be a net in X . A point $x \in X$ is a *cluster point* of $x_\bullet$ if for every neighborhood U of x and every $d_0 \in D$ there exists $d \succeq d_0$ such that $x_d \in U$.

Definition 11.7 (Subnet). Let $x_\bullet : D \rightarrow X$ be a net. A *subnet* is a net $x_{\phi(\bullet)} : E \rightarrow X$ of the form $e \mapsto x_{\phi(e)}$, where E is a directed set and $\phi : E \rightarrow D$ is cofinal: for every $d_0 \in D$ there exists $e_0 \in E$ such that $\phi(e) \succeq d_0$ for all $e \succeq e_0$.

Theorem 11.8 (Subnets and cluster points). *A point $x \in X$ is a cluster point of a net $x_\bullet$ if and only if some subnet of $x_\bullet$ converges to x .*

Proof. ($\Rightarrow$) Suppose that x is a cluster point of the net $x_\bullet = (x_d)_{d \in D}$. Let E be the set of pairs (d, U) , where $d \in D$ and U is a neighborhood of x , ordered by

$$(d_1, U_1) \preceq (d_2, U_2) \iff d_1 \preceq d_2 \text{ and } U_2 \subseteq U_1.$$

This is a directed set: given (d_1, U_1) and (d_2, U_2) , choose $d \in D$ with $d_1, d_2 \preceq d$, and use $(d, U_1 \cap U_2)$ as a common upper bound.

Since x is a cluster point, for every $(d, U) \in E$ there exists an index $\phi(d, U) \in D$ such that

$$\phi(d, U) \succeq d \quad \text{and} \quad x_{\phi(d, U)} \in U.$$

Define a net $y_{(d,U)} := x_{\phi(d,U)}$. Because $\phi(d, U) \succeq d$ for all (d, U) , the map $\phi : E \rightarrow D$ is cofinal; hence $y_\bullet$ is a subnet of $x_\bullet$. Now fix a neighborhood U of x . Choose any $d_0 \in D$. For every $(d, V) \succeq (d_0, U)$ we have $V \subseteq U$, and therefore

$$y_{(d,V)} = x_{\phi(d,V)} \in V \subseteq U.$$

Thus the subnet $y_\bullet$ is eventually in every neighborhood of x , so $y_\bullet \rightarrow x$.

($\Leftarrow$) Conversely, suppose some subnet $y_e = x_{\phi(e)}$ converges to x . Let U be a neighborhood of x , and let $d_0 \in D$. Because ϕ is cofinal, there exists $e_1 \in E$ such that $\phi(e) \succeq d_0$ for all $e \succeq e_1$. Because $y_e \rightarrow x$, there exists $e_2 \in E$ such that $y_e \in U$ for all $e \succeq e_2$. Choose $e \succeq e_1, e_2$. Then $\phi(e) \succeq d_0$ and

$$x_{\phi(e)} = y_e \in U.$$

So every neighborhood of x meets the original net arbitrarily far out, which is exactly the definition of cluster point. $\square$

11.3 Continuity in terms of nets

A good notion of generalized convergence should interact with continuous maps exactly as ordinary sequential convergence does in metric spaces. The net formulation of continuity confirms that this is indeed the case. It also provides a conceptually economical way to transport many sequential arguments to general topological spaces.

Theorem 11.9 (Continuity preserves limits of nets). *Let $f : X \rightarrow Y$ be a map between topological spaces. Then f is continuous if and only if for every net $x_\bullet$ in X and every $x \in X$,*

$$x_\bullet \rightarrow x \quad \Rightarrow \quad f(x_\bullet) \rightarrow f(x).$$

Proof. If f is continuous and $x_\bullet \rightarrow x$, let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x , so eventually $x_d \in f^{-1}(V)$, hence eventually $f(x_d) \in V$.

Conversely, assume the net condition. To prove continuity, let $U \subseteq Y$ be open and take $x \in f^{-1}(U)$. If $f^{-1}(U)$ were not a neighborhood of x , then $x \in \overline{X \setminus f^{-1}(U)}$. By Theorem 11.5 there exists a net $x_\bullet$ in $X \setminus f^{-1}(U)$ with $x_\bullet \rightarrow x$. Then $f(x_\bullet) \rightarrow f(x) \in U$, so eventually $f(x_\bullet) \in U$, contradicting $x_\bullet \subseteq X \setminus f^{-1}(U)$. Thus $f^{-1}(U)$ is a neighborhood of each of its points, hence open. $\square$

11.4 Filters and ultrafilters (optional but standard)

Filters encode convergence through families of large sets rather than through indexed points. This language is often more flexible for compactness arguments and is particularly well suited to maximality principles. Ultrafilters, in particular, provide an elegant reformulation of compactness that complements the viewpoint developed earlier through open covers.

Definition 11.10 (Filter). Let X be a set. A *filter* on X is a nonempty family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:

- (i) $\emptyset \notin \mathcal{F}$;
- (ii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
- (iii) if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

A filter $\mathcal{U}$ is an *ultrafilter* if it is maximal among filters with respect to inclusion.

Definition 11.11 (Filter convergence). Let X be a topological space, $x \in X$, and $\mathcal{F}$ a filter on X . We write $\mathcal{F} \rightarrow x$ if every neighborhood of x belongs to $\mathcal{F}$.

Theorem 11.12 (Compactness via ultrafilters). *A topological space X is compact if and only if every ultrafilter on X converges to some point of X .*

Proof. ($\Rightarrow$) Assume that X is compact, and let $\mathcal{U}$ be an ultrafilter on X . Suppose, for contradiction, that $\mathcal{U}$ does not converge to any point of X . Then for each $x \in X$ there exists an open neighborhood $U_x \ni x$ such that $U_x \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, this implies $X \setminus U_x \in \mathcal{U}$. The family $\{U_x : x \in X\}$ is an open cover of X , so compactness yields a finite subcover

$$X = U_{x_1} \cup \cdots \cup U_{x_n}.$$

Taking complements, we obtain

$$\bigcap_{k=1}^n (X \setminus U_{x_k}) = \emptyset.$$

But each set $X \setminus U_{x_k}$ lies in $\mathcal{U}$, and filters are closed under finite intersections, so $\emptyset \in \mathcal{U}$, a contradiction. Hence $\mathcal{U}$ converges to some point of X .

($\Leftarrow$) Conversely, assume that every ultrafilter on X converges, and suppose that X is not compact. Choose an open cover $\{U_i\}_{i \in I}$ of X with no finite subcover. Let

$$\mathcal{A} := \left\{ X \setminus \bigcup_{j \in F} U_j : F \subseteq I \text{ finite} \right\}.$$

Because the cover has no finite subcover, every member of $\mathcal{A}$ is nonempty, and $\mathcal{A}$ has the finite intersection property. By the ultrafilter lemma (equivalently, by Zorn's lemma), there exists an ultrafilter $\mathcal{U}$ on X containing $\mathcal{A}$. By hypothesis, $\mathcal{U} \rightarrow x$ for some $x \in X$. Since $\{U_i\}_{i \in I}$ covers X , choose $i_0 \in I$ with $x \in U_{i_0}$. Because $\mathcal{U} \rightarrow x$, the neighborhood U_{i_0} belongs to $\mathcal{U}$. On the other hand, taking $F = \{i_0\}$ in the definition of $\mathcal{A}$, we have $X \setminus U_{i_0} \in \mathcal{A} \subseteq \mathcal{U}$. Thus both U_{i_0} and $X \setminus U_{i_0}$ lie in $\mathcal{U}$, so their intersection $\emptyset$ lies in $\mathcal{U}$, again impossible. Therefore X must be compact. $\square$

11.5 Countability and the sequential viewpoint

After seeing why sequences fail in general, it is natural to ask when they suffice after all. First countability gives a precise answer: a countable local base allows one to replace nets by sequences in many fundamental arguments. This section makes explicit the bridge between abstract topology and the more familiar sequential language.

The first countability axiom guarantees that, in many arguments, nets may be replaced by sequences.

Proposition 11.13. *Let X be first countable, $A \subseteq X$, and $x \in X$. Then $x \in \overline{A}$ if and only if there exists a sequence (a_n) in A with $a_n \rightarrow x$.*

Proof. Assume first that $x \in \overline{A}$, and let $(U_n)_{n \in \mathbb{N}}$ be a countable local base at x . Replace it by the decreasing local base

$$V_n := U_1 \cap U_2 \cap \cdots \cap U_n \quad (n \in \mathbb{N}).$$

Since $x \in \overline{A}$, each V_n meets A ; choose $a_n \in A \cap V_n$. Let U be any neighborhood of x . Because (U_n) is a local base, there exists N such that $U_N \subseteq U$. Then for every $n \geq N$ we have $a_n \in V_n \subseteq V_N \subseteq U_N \subseteq U$. Hence $a_n \rightarrow x$.

The converse holds in every topological space: if $a_n \rightarrow x$ with $a_n \in A$, then every neighborhood of x contains some term of the sequence, and therefore meets A . Thus $x \in \overline{A}$. $\square$

Supplementary remarks and examples

Remark 11.14. Sequences detect topology perfectly in first countable spaces, but arbitrary spaces require a more flexible convergence language. Nets and filters are not optional technicalities: they are the correct replacement for sequences when countability is absent.

Example 11.15. Let X be an uncountable set equipped with the cocountable topology. If a sequence (x_n) converges to $x \in X$, then it is eventually constant with value x . Indeed, the set of values taken by the sequence away from x is countable, so one may remove all of them and still obtain an open neighborhood of x . This shows that sequences alone can severely under-report the true convergence structure of a space.

Problem Set

Problem guide. Work these problems in two layers. First master the definitions of directed set, net, subnet, filter, and ultrafilter. Then use them as tools: closure should be translated into existence of convergent nets, continuity into preservation of convergence, and compactness into the existence of cluster points or convergent subnets. Whenever a sequential argument fails, ask what the corresponding net argument should be.

Exercise 11.16. Give an example of a topological space X and a subset $A \subseteq X$ for which $\overline{A}$ is strictly larger than the set of sequential limits of points of A . (One classical example uses the first uncountable ordinal with the order topology.)

Exercise 11.17. Let $f : X \rightarrow Y$ be a map. Prove Theorem 11.9 directly from the open-set definition of continuity.

Exercise 11.18. Show that a space X is compact if and only if every net in X has a convergent subnet.

Baire category

The Baire category theorem is a fundamental tool that converts “countably many bad sets” into a small exceptional set. It underlies many existence and genericity arguments in analysis and dynamical systems, and it is one of the standard bridges between topology and functional analysis.

Roadmap.

- The main theorem is presented in its dense- G_δ formulation, which is the version most often used in applications.
- The proof is organized around a nested-ball construction in a complete metric space, illustrating how completeness converts approximate solvability into an actual point.
- The closing corollary packages the theorem in the language of comeagre sets, which is the form most useful for genericity arguments in analysis.

12.1 The Baire category theorem

The Baire category theorem is one of the main principles through which topology influences analysis. Its content is qualitative rather than measure-theoretic: a complete metric space cannot be exhausted by countably many topologically negligible sets. The theorem underlies many genericity arguments and explains why large sets often arise as countable intersections of dense open sets.

Definition 12.1 (Nowhere dense and meagre sets). Let X be a topological space. A set $A \subseteq X$ is *nowhere dense* if $\text{int}(\overline{A}) = \emptyset$. A set is *meagre* (or of *first category*) if it is a countable union of nowhere dense sets. A set is *comeagre* if its complement is meagre.

Theorem 12.2 (Baire category theorem). *Every complete metric space is of second category in itself. Equivalently, if X is a complete metric space, then the intersection of countably many dense open subsets of X is dense.*

Proof. We prove the dense-intersection formulation. Let $G_1, G_2, \dots$ be dense open subsets of the complete metric space (X, d) , and fix a nonempty open set $W \subseteq X$. Choose $x_0 \in W$ and $r_0 > 0$ such that $\overline{B}(x_0, r_0) \subseteq W$, replacing r_0 by a smaller radius if necessary.

Since G_1 is dense and open, the nonempty open set $B(x_0, r_0) \cap G_1$ contains an open ball $B(x_1, r_1)$ whose closed ball satisfies

$$\overline{B}(x_1, r_1) \subseteq B(x_0, r_0) \cap G_1, \quad 0 < r_1 < 2^{-1}.$$

Inductively, once $\overline{B}(x_{n-1}, r_{n-1})$ has been chosen, the open set $B(x_{n-1}, r_{n-1}) \cap G_n$ is nonempty; choose x_n and $r_n > 0$ so that

$$\overline{B}(x_n, r_n) \subseteq B(x_{n-1}, r_{n-1}) \cap G_n, \quad 0 < r_n < 2^{-n}.$$

Thus the closed balls are nested and $\overline{B}(x_n, r_n) \subseteq G_n$ for every $n \geq 1$.

If $m \geq n$, then $x_m \in \overline{B}(x_n, r_n)$, and therefore

$$d(x_m, x_n) \leq r_n < 2^{-n}.$$

Hence (x_n) is Cauchy. By completeness, $x_n \rightarrow x$ for some $x \in X$. For each fixed n , all terms x_m with $m \geq n$ lie in the closed ball $\overline{B}(x_n, r_n)$, so $x \in \overline{B}(x_n, r_n)$. Consequently

$$x \in W \cap \bigcap_{n=1}^{\infty} G_n.$$

Since every nonempty open set W meets $\bigcap_n G_n$, that intersection is dense in X . □

Corollary 12.3. *In a complete metric space, every comeagre set contains a dense G_δ set, and every dense G_δ set is comeagre.*

Proof. If A is comeagre, then $X \setminus A = \bigcup_{n=1}^{\infty} N_n$, where each N_n is nowhere dense. The set

$$G := \bigcap_{n=1}^{\infty} (X \setminus \overline{N_n})$$

is a dense G_δ set by the Baire category theorem, and $G \subseteq A$. Conversely, if $G = \bigcap_n U_n$ is dense and each U_n is open, then each U_n is dense. Hence

$$X \setminus G = \bigcup_n (X \setminus U_n)$$

is a countable union of closed sets with empty interior, so G is comeagre. □

Remark 12.4. Many spaces occurring in analysis are Baire spaces even without an explicit metric: for instance, every locally compact Hausdorff space is a Baire space. This principle frequently turns “countably many bad sets” into a “small” exceptional set.

Notation and Glossary

A quick reference for recurring notation and terminology used throughout the notes.

Notation

X, Y, Z	topological spaces (unless otherwise specified).
$\mathcal{T}$	a topology on X ; the pair $(X, \mathcal{T})$ denotes a topological space.
$\mathcal{P}(X)$	the power set of X .
$\overline{A}$	closure of $A \subseteq X$.
$\text{int}(A)$	interior of $A \subseteq X$.
∂A	boundary of $A \subseteq X$.
A'	derived set (set of limit points) of $A \subseteq X$.
$\partial(A)$	boundary operator used in parts of the notes, typically $\partial(A) = \overline{A} \cap \overline{(X \setminus A)}$.
$\mathcal{T}_Y$	subspace topology on $Y \subseteq X$.
$\prod_{\lambda \in I} X_\lambda$	product space with index set I , equipped with the product topology unless stated otherwise.
π_λ	the λ -th coordinate projection in a product.
$q : X \rightarrow Y$	a quotient map; Y carries the quotient topology induced by q .
T_0, T_1, T_2	separation axioms (with T_2 meaning Hausdorff).

First/second countable

countability axioms: existence of a countable neighborhood base at each point / existence of a countable base for the topology.

Glossary of key terms

Topology	A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ containing $\emptyset$ and X , closed under arbitrary unions and finite intersections.
Basis / subbasis	A basis $\mathcal{B}$ generates $\mathcal{T}$ by unions; a subbasis $\mathcal{S}$ generates $\mathcal{T}$ by finite intersections and unions.

Continuous map	$f : X \rightarrow Y$ is continuous if $f^{-1}(U)$ is open in X whenever U is open in Y .
Homeomorphism	A bijection $f : X \rightarrow Y$ that is continuous with continuous inverse; equivalently, a continuous bijection that is open (or closed).
Connected	X is connected if it is not the union of two disjoint nonempty open sets.
Path connected	Any two points can be joined by a continuous map $[0, 1] \rightarrow X$.
Component	A maximal connected subset of X ; components partition X .
Compact	Every open cover admits a finite subcover.
Locally compact	Each point has a neighborhood with compact closure (in Hausdorff spaces, this is equivalent to the usual definition).
One-point compactification	For locally compact Hausdorff X , the space $X^* = X \cup \{\infty\}$ obtained by adjoining one point so that complements of compact sets form neighborhoods of ∞ .
Normal	Disjoint closed sets can be separated by disjoint open neighborhoods.
Urysohn function	In a normal space, a continuous $f : X \rightarrow [0, 1]$ separating two disjoint closed sets.
Metrizable	The topology arises from a metric.

Problem Sets

A consolidated collection of problem sets corresponding to the main topics of the course. The problems are arranged roughly in the order in which the material is developed in the text; starred problems are intended to be more challenging or supplementary.

Problem Set 1: Topological spaces and bases

1. List all possible topologies on the set $\{a, b\}$ and also on the set $\{a, b, c\}$.
2. Let X be a set with exactly 4 elements. Does there exist a topology τ on X such that there are precisely 14 open sets in the topological space (X, τ) ? Justify.
3. Examine whether τ is a topology on X , where
 - (a) $X = \mathbb{R}$ and $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.
 - (b) $X = \mathbb{R}$ and $\tau = \{[a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.
 - (c) $X = \mathbb{R}$ and $\tau = \{(a, \infty) : a \in \mathbb{Q}\} \cup \{\emptyset, \mathbb{R}\}$.
 - (d) $X = \mathbb{R}$ and $\tau = \{[-a, a) : a \in \mathbb{R}, a > 0\} \cup \{\emptyset, \mathbb{R}\}$.
 - (e) $X = \mathbb{N}$ and $\tau = \{\{n, n+1, n+2, \dots\} : n \in \mathbb{N}\} \cup \{\emptyset\}$.
 - (f) $X = \mathbb{R}$ and $\tau = \{G \subset \mathbb{R} : \mathbb{Q} \not\subset G\} \cup \{\mathbb{R}\}$.
 - (g) $X = \mathbb{R}$ and $\tau = \{G \subset \mathbb{R} : G \subset \mathbb{Q} \text{ or } \mathbb{Q} \subset G\}$.
 - (h) $X = [0, 1]$ and $\tau = \{G \subset [0, 1] : GG \subset G\}$.

Here, for $A \subset [0, 1]$, the notation AA means $AA = \{xy : x, y \in A\}$.
 - (i) X is an infinite set and $\tau = \{\emptyset\} \cup \{G \subset X : X \setminus G \text{ is infinite}\}$.
 - (j) X is a nonempty set, $x_0 \in X$ and $\tau = \{G \subset X : X \setminus G \text{ is finite or } x_0 \in X \setminus G\}$.
 - (k) X is an uncountable set and $\tau = \{G \subset X : G \text{ is countable or } X \setminus G \text{ is countable}\}$.

4. Let $\tau = \{G_k : k \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}^2\}$, where for each $k \in \mathbb{R}$, $G_k = \{(x, y) \in \mathbb{R}^2 : x > y + k\}$. Prove that τ is a topology on $\mathbb{R}^2$.

Is τ a topology on $\mathbb{R}^2$ if ‘ $k \in \mathbb{R}$ ’ is replaced by (a) ‘ $k \in \mathbb{N}$ ’? (b) ‘ $k \in \mathbb{Q}$ ’? Justify.

5. Let X, Y be nonempty sets and let $f : X \rightarrow Y$.
 - (a) If $\mathfrak{J}'$ is a topology on Y , then show that $\mathfrak{J} = \{f^{-1}(H) : H \in \mathfrak{J}'\}$ is a topology on X .
 - (b) If τ is a topology on X and if f is one-to-one and onto, then show that $\tau' = \{f(G) : G \in \tau\}$ is a topology on Y .

Show also that, in general, both conditions that f be one-to-one and onto are necessary.

6. Let $\mathcal{X}$ be the class of all (proper) prime ideals of a commutative ring R with unity and for each $E \subset R$, let $\mathcal{V}(E) = \{P \in \mathcal{X} : E \subset P\}$. Prove that $\tau = \{\mathcal{X} \setminus \mathcal{V}(E) : E \subset R\}$ is a topology

on $\mathcal{X}$.

(τ is called the Zariski topology on $\mathcal{X}$ and the topological space $(\mathcal{X}, \tau)$ is called the prime spectrum of R , denoted by $\text{Spec}(R)$.)

7. Let τ_f and τ_c denote respectively the cofinite and the cocountable topologies on a nonempty set X . Find a necessary and sufficient condition on X such that $\tau_f = \tau_c$.
8. Let (X, τ) be a topological space in which every singleton subset of X is open. Prove that τ is the discrete topology on X .
9. Let τ be a topology on an infinite set X such that every infinite subset of X is open in (X, τ) . Show that τ is the discrete topology on X .
10. Let τ be a topology on an infinite set X such that every infinite subset of X is closed in (X, τ) . Prove that τ is the discrete topology on X .
11. Let τ and τ' be topologies on $\mathbb{R}$ such that every countable subset of $\mathbb{R}$ is open in $(\mathbb{R}, \tau)$ and every uncountable subset of $\mathbb{R}$ is open in $(\mathbb{R}, \tau')$. Determine (with justification) which of the following is true.
 - (a) τ is strictly finer than τ' .
 - (b) τ' is strictly finer than τ .
 - (c) τ and τ' are not comparable.
 - (d) $\tau = \tau'$.
12. Prove that on a finite set with exactly n elements there are at most $2^{(2^n - 2)}$ distinct topologies. (Finding a tractable closed formula for the exact number of topologies on an arbitrary finite set is a difficult enumeration problem. Exact values are known in many small cases. For example, if $k(n)$ denotes the number of distinct topologies on a set with exactly n elements, then it is already known that $k(4) = 355$, $k(5) = 6942$, $k(6) = 209527$, $k(7) = 9535241$.)
13. Prove that uncountably many distinct topologies can be defined on any infinite set.
14. Let X be a nonempty set and let $\lambda > 0$. Consider the metric d on X defined by

$$d(x, y) = \begin{cases} \lambda & \text{if } x \neq y, \\ 0 & \text{if } x = y \ (x, y \in X). \end{cases}$$
 Find the topology on X induced by d .
15. Let d be a metric on a nonempty set X and let $\lambda > 0$. Consider the metrics d_1 , d_2 and d_3 on X defined by $d_1(x, y) = \lambda d(x, y)$, $d_2(x, y) = \min\{1, d(x, y)\}$ and $d_3(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ for all $x, y \in X$. Prove that d , d_1 , d_2 and d_3 induce the same topology on X .
16. Let d be a metric on a nonempty set X and let τ_d be the topology on X induced by d . Prove that every finite subset of X is closed in (X, τ_d) .
Without using this fact, show also that every finite subset of $\mathbb{R}$ is closed in $\mathbb{R}$ with the usual topology.
17. Let X be a topological space. If A is an open subset of X and B is a closed subset of X , then prove that $A \setminus B$ is an open subset of X and $B \setminus A$ is a closed subset of X .

18. Let $\mathcal{X}$ be the class of all (proper) prime ideals of a commutative ring R with unity. Consider the Zariski topology $\tau = \{\mathcal{X} \setminus \mathcal{V}(E) : E \subset R\}$ on $\mathcal{X}$, where $\mathcal{V}(E) = \{P \in \mathcal{X} : E \subset P\}$ for each $E \subset R$. If $P \in \mathcal{X}$, then show that $\{P\}$ is a closed set in the topological space $(\mathcal{X}, \tau)$ if and only if P is a maximal ideal of R .
19. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a class of closed subsets of a topological space X . Assume that for each $x \in X$, there exists an open subset G_x of X containing x such that $\{\alpha \in \Lambda : G_x \cap A_\alpha \neq \emptyset\}$ is a finite set. Show that $\bigcup_{\alpha \in \Lambda} A_\alpha$ is closed in X .
20. Let G be a nonempty open subset of $\mathbb{R}$ (with the usual topology) such that $x - y \in G$ for all $x, y \in G$. Show that $G = \mathbb{R}$.
(Thus $\mathbb{R}$ is the only open subgroup of the additive group $\mathbb{R}$ with the usual topology.)
21. Let F be a nonempty closed subset of $\mathbb{R}$ (with the usual topology) such that $x - y \in F$ for all $x, y \in F$. Show that either $F = \mathbb{R}$ or $F = \alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.
(This provides the class of all closed subgroups of the additive group $\mathbb{R}$ with the usual topology.)
22. Prove that every subgroup of the group $(\mathbb{R}, +)$ is either cyclic (and hence closed in $(\mathbb{R}, \tau_u)$) or dense in $(\mathbb{R}, \tau_u)$.
23. Consider the topology $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is finite or } 2 \notin G\}$ on $\mathbb{R}$. If τ_u is the usual topology on $\mathbb{R}$, examine whether $\tau \cup \tau_u$ is a topology on $\mathbb{R}$.
24. State TRUE or FALSE with justification for each of the following statements.
 - (a) There exists a topological space with exactly 100 (distinct) open sets.
 - (b) If $\{\tau_n\}_{n=1}^\infty$ is an ascending chain of topologies on a nonempty set X , then $\bigcup_{n=1}^\infty \tau_n$ must be a topology on X .
25. Let τ_u be the usual topology on $\mathbb{R}$. Show that $\tau_u|_{\mathbb{N}}$ is the discrete topology on $\mathbb{N}$ but that $\tau_u|_{\mathbb{Q}}$ is not the discrete topology on $\mathbb{Q}$.
26. Consider $\mathbb{R}$ with the usual topology. Find the relative topology on $\{\frac{1}{n} : n \in \mathbb{N}\}$.
27. Is it possible to define a topology τ on $\mathbb{R}$ such that τ is not the discrete topology on $\mathbb{R}$ and the relative topology on $[0, 1]$ induced by τ is the discrete topology on $[0, 1]$?
28. Let τ_l denote the lower limit topology on $\mathbb{R}$. Examine whether the set $\{x \in \mathbb{Q} : x^2 > 5\}$ is closed in the topological space $(\mathbb{Q}, \tau_l|_{\mathbb{Q}})$.
29. Let (X, τ) be a topological space and let $A (\neq \emptyset) \subset X$. Prove that $\tau' = \{G \cup (H \cap A) : G, H \in \tau\}$ is a topology on X such that A is open in the topological space (X, τ') and $\tau|_A = \tau'|_A$.
30. Let (X, τ) be a topological space such that for each nonempty finite subset A of X (or, for each subset A of X containing exactly two elements), $\tau|_A$ is the indiscrete topology on A . Is it necessary that τ is the indiscrete topology on X ?
31. Let F be a closed subspace of a topological space X and let G be an open set in F . If H is an open set in X containing G , then prove that $G \cup (H \setminus F)$ is open in X .

32. Let Y and Z be subspaces of a topological space X and let $A \subset Y \cap Z$. If A is open in both Y and Z , then show that A is open in $Y \cup Z$.
33. Let Y be a nonempty subset of a topological space X and let Z be a nonempty open subset of $X \setminus Y$. If $A \subset Z$ such that A is open in $Y \cup Z$, then show that A is open in X .
34. Let X be a topological space and $A \subset X$. Show that A is closed in X if and only if for every $x \in \overline{A}$, there exists an open set G in X containing x such that $G \cap A$ is closed in G .
35. Let X be a topological space and $A \subset X$. If for each $a \in A$, there exists an open set G in X containing a such that $G \cap A$ is closed in G , then show that there exist an open set V in X and a closed set F in X such that $A = V \cap F$.
36. Consider the subspace $(X, \tau_u|_X)$ of the topological space $(\mathbb{R}, \tau_u)$, where $X = \{m + n\pi : m, n \in \mathbb{Z}\}$ and τ_u is the usual topology on $\mathbb{R}$. Examine whether $\{0\}$ is an open set in $(X, \tau_u|_X)$.
37. Let (X, d) be a metric space and let $Y (\neq \emptyset) \subset X$. If ρ is the metric on Y induced by d (i.e. $\rho = d|_{Y \times Y}$), τ_d is the topology on X induced by d and τ_ρ is the topology on Y induced by ρ , then show that $\tau_\rho = \tau_d|_Y$.
38. Consider $\mathbb{R}$ with the usual topology and let $x \in \mathbb{R}$. Find the closure of the set $\{x + r : r \in \mathbb{Q}\}$.
39. Consider $\mathbb{R}$ with the lower limit topology. Does there exist a subset A of $\mathbb{R}$ such that $\overline{A} = [0, 1) \cup (1, 2)$? Justify.
40. Let X be a topological space and let $A, B \subset X$. Prove that $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$, and also that the inclusion can be strict.
41. Let τ, τ' be topologies on a nonempty set X . Prove that $\tau \subset \tau'$ if and only if $\overline{A}^\tau \supset \overline{A}^{\tau'}$ for all $A \subset X$.
(Hence $\tau = \tau'$ if and only if $\overline{A}^\tau = \overline{A}^{\tau'}$ for all $A \subset X$. However, even if $\tau \subsetneq \tau'$, it can happen that $\overline{A}^\tau = \overline{A}^{\tau'}$ for some $A \subset X$.)
42. If G is an open set in a topological space X , then show that $\overline{G} = \overline{X \setminus \overline{G}}$.
43. Let A be a subset of a topological space X . Show that $\overline{A} \setminus A$ is closed in X if and only if there exist a closed set F in X and an open set G in X such that $A = F \cap G$.
44. Let G be an open set in a topological space X and let $A \subset X$. Prove that $G \cap A = \emptyset$ if and only if $G \cap \overline{A} = \emptyset$.
Hence, or otherwise, show that if U and V are disjoint open sets in X , then $(\overline{U})^0 \cap (\overline{V})^0 = \emptyset$.
45. Prove that a subset G of a topological space X is open in X if and only if $\overline{G \cap A} = \overline{G} \cap \overline{A}$ for all $A \subset X$.
46. Let Y be a closed subspace of a topological space X . Let $A \subset X$ and let H be an open set in Y such that $A \cap Y \subset H$. Prove that $A \cap \overline{Y \setminus H}^X = \emptyset$.
47. Let A, B be subsets of a topological space. Show that $(A \setminus B)^0 \subset A^0 \setminus B^0$ and also show that the equality need not occur in the above inclusion.

48. Prove that a subset F of a topological space X is closed in X if and only if $(F \cup A^0)^0 = (F \cup A)^0$ for all $A \subset X$.
49. Let Y be a subspace of a topological space X and let $A \subset Y$. Show that $\text{Int}_Y(A) \supset Y \cap \text{Int}_X(A)$ and $\text{Int}_X(A) = \text{Int}_Y(A) \cap \text{Int}_X(Y)$, although in general, $\text{Int}_Y(A) \neq Y \cap \text{Int}_X(A)$.
50. For every subset A of a topological space X , show that $(X \setminus A)^0 = X \setminus \overline{A}$.
(Other similar ‘commutative’ relations are true for closure, interior and complement. For example, $X \setminus A^0 = \overline{X \setminus A}$.)
51. Let A, B be subsets of a topological space. Prove that $\overline{A} \cap B^0 = \overline{A \cap B} \cap B^0$.
52. Let G be an open subset of a topological space. Prove that $G \subset (\overline{G})^0$, and that the inclusion can be strict. Prove, however, that $\overline{G} = \overline{(\overline{G})^0}$.
53. If F is a closed set in a topological space, then show that $(\overline{F^0})^0 = F^0$.
54. If A is a subset of a topological space, then show that $\overline{A^0} = \overline{(\overline{A^0})^0}$.
55. Let A be an open set in a topological space X and let $B \subset X$. Show that $\overline{A \cap B}^0 = (\overline{A})^0 \cap (\overline{B})^0$.
56. If $\tau = \{\{n, n+1, \dots\} : n \in \mathbb{N}\} \cup \{\emptyset\}$, then in the topological space $(\mathbb{N}, \tau)$, find $\{4, 13, 28, 37\}'$ and also find all the subsets A of $\mathbb{N}$ for which $A' = \mathbb{N}$.
57. If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then in the topological space $(\mathbb{R}, \tau)$, determine $[4, 10]'$ and $\mathbb{Z}'$.
58. If $\tau = \{G \subset \mathbb{R} : 2 \in G\} \cup \{\emptyset\}$, then determine (with justification) $(2, 4)'$ and $[2, 4]'$ in the topological space $(\mathbb{R}, \tau)$.
59. Let A be a subset of a topological space X such that each subset of A is closed in X . Show that A has no limit point in X .
60. If $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is finite or } 0 \in \mathbb{R} \setminus G\}$, then determine (with justification) $\mathbb{Q}^0$, $\overline{\mathbb{R} \setminus \mathbb{Q}}$, $[0, 1]'$ and $\partial \mathbb{Q}$ in the topological space $(\mathbb{R}, \tau)$.
61. Let A be a subset of a topological space. Prove that $\partial A^0 \subset \partial A$ and $\partial \overline{A} \subset \partial A$ and that both these inclusions can be strict.
62. Let A be a subset of a topological space. Show that $(A \cap \partial A)^0 = \emptyset$. Is it necessary that $(\partial A)^0 = \emptyset$?
63. Consider $\mathbb{R}$ with the usual topology. Does there exist a set $A \subset \mathbb{R}$ such that $\partial A = [0, 1]$? Justify.
64. Let A, B be subsets of a topological space X . Prove that $\partial(A \cup B) \subset \partial A \cup \partial B$, and that this inclusion can be strict. Show, however, that if $\overline{A} \cap \overline{B} = \emptyset$, then $\partial(A \cup B) = \partial A \cup \partial B$.
65. Let A, B be subsets of a topological space such that $\partial A \cap \partial B = \emptyset$. Prove that $(A \cup B)^0 = A^0 \cup B^0$.
66. State TRUE or FALSE with justification for each of the following statements.
 - (a) Every uncountable subset of $\mathbb{R}$ (with the usual topology) has a limit point in $\mathbb{R}$.

- (b) If every singleton subset of a topological space X is dense in X , then X must be an indiscrete space.
 - (c) If no proper subset of a topological space X is dense in X , then X must be a discrete space.
 - (d) A nonempty open set in a topological space X cannot be nowhere dense in X .
67. Let A be a nonempty subset of a topological space X . Show that A is dense in the subspace $\overline{A}^X$.
68. Let Y be a subspace of a topological space X . If $D \subset Y$ is dense in Y , then prove that D is dense in the subspace $\overline{Y}^X$.
(Hence, if Y is a dense subspace of a topological space X and if $D \subset Y$ is dense in Y , then D is dense in X .)
69. Let D be a dense set in a topological space X and let $Y (\neq \emptyset) \subset X$. Show that $D \cap Y$ need not be dense in Y , but if moreover Y is open in X , then $\overline{D \cap Y}^X = \overline{Y}^X$ and hence deduce that $D \cap Y$ is dense in Y .
70. Let D_1 and D_2 be dense sets in a topological space X . Show that $D_1 \cap D_2$ need not be dense in X , but if moreover D_1 or D_2 is open in X , then $D_1 \cap D_2$ is dense in X .
71. Let F be a closed subset of a topological space X . Prove that F is nowhere dense in X if and only if $X \setminus F$ is dense in X . Is this result true for an arbitrary subset F of X ? Justify.
72. Prove that a subset A of a topological space X is nowhere dense in X if and only if $A \subset \overline{X \setminus \overline{A}}$.
73. Let A be a subset of a topological space X . Prove that the following statements are equivalent.
- (a) A is nowhere dense in X .
 - (b) $(X \setminus A)^0$ is dense in X .
 - (c) $A \subset \partial \overline{A}$.
74. Let Y be a nonempty open set in a topological space X and let $A \subset Y$. If A is nowhere dense in the subspace Y , then show that A is nowhere dense in X .
75. Prove that for each $1 \leq m < n$, $A = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{m+1} = x_{m+2} = \dots = x_n = 0\}$ is nowhere dense in $\mathbb{R}^n$ with the usual topology.
76. Show that for a set A in a topological space X , ∂A need not be nowhere dense in X , but that ∂A is nowhere dense in X if A is either open or closed in X .
77. Prove that in a topological space X , every closed nowhere dense set is the boundary of some open set in X .
78. Prove that every finite union of nowhere dense sets in a topological space X is nowhere dense in X .
Give examples to show that an infinite union of nowhere dense sets in a topological space X can/need not be nowhere dense in X .

79. Let X be a topological space and let $x \in X$. If x has a finite local basis in X , then show that x has a local basis in X consisting of precisely one member.
80. Let $\mathcal{B}$ be a class of subsets of a nonempty set X . Show that $\mathcal{B}$ is a basis for the discrete topology on X iff $\mathcal{B} \supset \{\{x\} : x \in X\}$.
81. Show that $\{(a, b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for the usual topology on $\mathbb{R}$.
82. Show that $\{[a, b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for a topology τ on $\mathbb{R}$ which is strictly coarser than the lower limit topology on $\mathbb{R}$ but strictly finer than the usual topology on $\mathbb{R}$. Determine the closures of $(0, \sqrt{2})$ and $(\sqrt{2}, 3)$ in the topological space $(\mathbb{R}, \tau)$.
83. Examine whether each of the following classes is a basis for some topology on $\mathbb{R}$.
- (a) $\{[a, b] : a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}, a < b\}$
 - (b) $\{[a, b] : a, b \in \mathbb{R}, a < b\}$
 - (c) $\{[a, b] : a, b \in \mathbb{Q}, a < b\}$
84. Let τ_u and τ_c denote respectively the usual topology and the cocountable topology on $\mathbb{R}$. Examine whether $\tau_u \cup \tau_c$ is a basis for some topology on $\mathbb{R}$.
85. Let X be any nonempty set. Prove that $\mathcal{S} = \{X \setminus \{x\} : x \in X\}$ is a subbasis for the cofinite topology τ_f on X . Is $\mathcal{S}$ a basis for τ_f on X ? Is $\mathcal{S}$ a basis for some topology on X ? Justify.
86. For each $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 \neq 0$, let $E(a, b, c) = \{(x, y) \in \mathbb{R}^2 : ax + by + c = 0\}$ and let τ be the topology on $\mathbb{R}^2$ generated by the class $\{E(a, b, c) : a, b, c \in \mathbb{R}, a^2 + b^2 \neq 0\}$. Examine whether $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is an open set in the topological space $(\mathbb{R}^2, \tau)$.
87. Let τ be the topology on $\mathbb{R}$ having $\mathcal{S} = \{\mathbb{R} \setminus \{x\} : x \in \mathbb{R}\} \cup \{\{x\} : x(\neq 2) \in \mathbb{R}\}$ as a subbasis. If A is an infinite subset of $\mathbb{R}$, then show that $A' \neq \emptyset$ in the topological space $(\mathbb{R}, \tau)$.
88. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ and let $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$. Show that $\mathcal{B}$ is a basis for some topology on $\mathbb{R}$. (This topology is called the K -topology on $\mathbb{R}$.)
Prove that the K -topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$ but that it is not comparable with the lower limit topology on $\mathbb{R}$.
89. Let $\mathcal{B}$ be a basis for a topology τ on a nonempty set X . Prove that a set $D \subset X$ is dense in X if and only if $D \cap B \neq \emptyset$ for every $B(\neq \emptyset) \in \mathcal{B}$.
Does this remain true if $\mathcal{B}$ is merely a subbasis for τ on X ? Justify.
90. Let τ be the topology on $\mathbb{R}$ having $\{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b] : a, b \in \mathbb{R}, a < b\}$ as a subbasis. Examine whether $\mathbb{Q}$ is a dense set in the topological space $(\mathbb{R}, \tau)$.
91. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on a nonempty set X . Show that there is a unique smallest topology on X containing all τ_α ($\alpha \in \Lambda$) and a unique largest topology on X contained in all τ_α ($\alpha \in \Lambda$).

92. Consider the topologies $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$ on the set $X = \{a, b, c\}$. Find the smallest topology on X containing both τ_1 and τ_2 and the largest topology on X contained in both τ_1 and τ_2 .
93. State TRUE or FALSE with justification for each of the following statements.
- (a) If τ is the topology on $\mathbb{R}$ having $\{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$ as a basis, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$, and if τ_u is the usual topology on $\mathbb{R}$, then the relative topology on $A = \{x \in \mathbb{R} : x > 0\}$ induced by τ is strictly finer than the relative topology on A induced by τ_u .
 - (b) If a topology τ on $\mathbb{R}$ is strictly finer than both the lower limit topology and the cocountable topology on $\mathbb{R}$, then τ must be the discrete topology on $\mathbb{R}$.
 - (c) If $\mathcal{B}$ and $\mathcal{B}'$ are bases for topologies τ and τ' respectively on a nonempty set X , then $\{B \cap B' : B \in \mathcal{B}, B' \in \mathcal{B}'\}$ must be a basis for some topology on X .
 - (d) If τ and τ' are topologies on a nonempty set X , then $\tau \cup \tau'$ must be a basis for some topology on X .
94. It is known that if $\mathbb{R}$ is equipped with either the indiscrete topology or the cofinite topology, then the sequence $(\frac{1}{n})_{n=1}^{\infty}$ converges to 1. Mention (with justification) another topology on $\mathbb{R}$ with respect to which the sequence $(\frac{1}{n})_{n=1}^{\infty}$ converges to 1.
95. Consider $X = (0, 1]$ with the relative topology τ induced by the usual topology on $\mathbb{R}$. Show that the sequence $(\frac{1}{n})_{n=1}^{\infty}$ in X does not converge in the topological space (X, τ) .
96. If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then in the topological space $(\mathbb{R}, \tau)$, prove that the sequence
- (a) $(n)_{n=1}^{\infty}$ converges to every point of $\mathbb{R}$.
 - (b) $(-n)_{n=1}^{\infty}$ does not converge.
 - (c) $((-1)^n)_{n=1}^{\infty}$ converges to x if and only if $x \in (-\infty, -1]$.
97. If $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$, then determine (with justification) all the convergent sequences and all their limits in the topological space $(\mathbb{R}, \tau)$.
98. State TRUE or FALSE with justification for each of the following statements.
- (a) The sequence $(\frac{(-1)^n}{n})_{n=1}^{\infty}$ converges in $\mathbb{R}$ with the lower limit topology.
 - (b) There exists a topology τ on $\mathbb{R}$ such that the sequence $(1, 2, 3, \dots)$ converges to the unique limit 0 in $(\mathbb{R}, \tau)$.
 - (c) If (X, τ) is a topological space such that every sequence in X converges to every point of X in (X, τ) , then τ must be the indiscrete topology on X .

Problem Set 2: Continuous maps and homeomorphisms

1. Let X, Y be topological spaces and let $x_0 \in X$. If $f : X \rightarrow Y$ is continuous at x_0 and G is an open set in Y containing $f(x_0)$, then is it necessary that $f^{-1}(G)$ is an open set in X ? Justify.

2. Let A be a nonempty subset of a topological space X and let $f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$
Prove that $f : X \rightarrow (\mathbb{R}, \tau_u)$ is continuous if and only if $\partial A = \emptyset$, where τ_u is the usual topology on $\mathbb{R}$.
3. Let τ_f and τ_c denote respectively the cofinite topology and the cocountable topology on $\mathbb{R}$. If $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$
then determine all the points of $\mathbb{R}$ at which $g : (\mathbb{R}, \tau_c) \rightarrow (\mathbb{R}, \tau_f)$ is continuous.
4. Let τ_l denote the lower limit topology on $\mathbb{R}$ and let $f(x) = -x$ for all $x \in \mathbb{R}$. Show that $f : (\mathbb{R}, \tau_l) \rightarrow (\mathbb{R}, \tau_l)$ is discontinuous at each point of $\mathbb{R}$.
5. Let $f : X \rightarrow \mathbb{R}$ be continuous at a point $x_0 \in X$, where X is a topological space and $\mathbb{R}$ is equipped with the usual topology. If $f(x_0) > 0$, then show that there exists an open set G in X containing x_0 such that $f(x) > 0$ for all $x \in G$.
6. Let (X, τ) and (Y, τ') be topological spaces such that every map from (X, τ) to (Y, τ') is continuous. Prove that τ is the discrete topology on X or τ' is the indiscrete topology on Y .
7. Let X, Y be nonempty sets and let $x_0 \in X, y_0 \in Y$. If $\tau = \{G \subset X : x_0 \in G\} \cup \{\emptyset\}$ and $\tau' = \{H \subset Y : y_0 \in H\} \cup \{\emptyset\}$, then show that a map $f : (X, \tau) \rightarrow (Y, \tau')$ is continuous if and only if $f(x_0) = y_0$ or f is constant.
8. Let τ_f denote the cofinite topology on $\mathbb{R}$. Examine whether $\varphi : (\mathbb{R}, \tau_f) \rightarrow (\mathbb{R}, \tau_f)$ is continuous, if φ is defined as
 - (a) $\varphi(x) = x^3 - 4x - 1$ for all $x \in \mathbb{R}$.
 - (b) $\varphi(x) = \sin x$ for all $x \in \mathbb{R}$.
 - (c) $\varphi(x) = x + \sin x$ for all $x \in \mathbb{R}$.
 - (d) $\varphi(x) = e^x$ for all $x \in \mathbb{R}$.
9. Show that $\tau = \{G \subset \mathbb{N} : n \in G \text{ and } m|n \Rightarrow m \in G\}$ is a topology on $\mathbb{N}$ which is different from the discrete topology on $\mathbb{N}$.
Show also that a map $f : (\mathbb{N}, \tau) \rightarrow (\mathbb{N}, \tau)$ is continuous if and only if for all $m, n \in \mathbb{N}$, $m|n \Rightarrow f(m)|f(n)$.
10. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on a nonempty set X , let (Y, τ) be a topological space and let $f : X \rightarrow Y$. Prove that $f : \left(X, \bigcap_{\alpha \in \Lambda} \tau_\alpha\right) \rightarrow (Y, \tau)$ is continuous if and only if for each $\alpha \in \Lambda$, $f : (X, \tau_\alpha) \rightarrow (Y, \tau)$ is continuous.
11. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on a nonempty set Y , let (X, τ) be a topological space and let $f : X \rightarrow Y$. If τ' is the unique smallest topology on Y containing $\bigcup_{\alpha \in \Lambda} \tau_\alpha$, then show that $f : (X, \tau) \rightarrow (Y, \tau')$ is continuous if and only if for each $\alpha \in \Lambda$, $f : (X, \tau) \rightarrow (Y, \tau_\alpha)$ is continuous.

12. Let τ_f and τ_u denote respectively the cofinite topology and the usual topology on $\mathbb{R}$. If $\varphi : (\mathbb{R}, \tau_f) \rightarrow (\mathbb{R}, \tau_u)$ is continuous, then prove that φ is a constant map.
13. Let X, Y be topological spaces and let $f : X \rightarrow Y$. Prove that f is continuous iff $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.
14. Let X, Y be topological spaces and let $f : X \rightarrow Y$. Prove that f is continuous iff $\partial(f^{-1}(B)) \subset f^{-1}(\partial B)$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.
15. Let X, Y be topological spaces and let $f : X \rightarrow Y$. Prove that f is continuous iff $f^{-1}(B^0) \subset (f^{-1}(B))^0$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.
16. Let X, Y be topological spaces and let $f : X \rightarrow Y$. Prove that f is continuous if and only if $f(A') \subset \overline{f(A)}$ for all $A \subset X$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $A \subset X$.
17. Let $f : X \rightarrow Y$ be continuous, where X, Y are topological spaces. If $x \in X$ is a limit point of a subset A of X , then is it necessary that $f(x)$ is a limit point of $f(A)$? Justify.
18. Let X, Y be topological spaces and let $f : X \rightarrow Y$ be continuous and onto. If A is a dense set in X , then prove that $f(A)$ is dense in Y and that the onto condition on f is, in general, necessary for this.
Show also that $f(A)$ can be dense in Y even when $A \subset X$ is not dense in X and $f : X \rightarrow Y$ is neither continuous nor onto.
19. Consider the topology $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is finite or } 0 \in \mathbb{R} \setminus G\}$ on $\mathbb{R}$. Let τ_u be the usual topology on $\mathbb{R}$ and let $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau_u)$ be continuous. Show that there exists a countable subset A of $\mathbb{R}$ such that $f(x) = f(0)$ for all $x \in \mathbb{R} \setminus A$.
20. Consider the topology τ on $\mathbb{R}$ having $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \cap \mathbb{Q} : a, b \in \mathbb{R}, a < b\}$ as a basis. Also, let τ_u be the usual topology on $\mathbb{R}$.
(a) Examine whether $\mathbb{R} \setminus \mathbb{Q}$ is closed in the topological space $(\mathbb{R}, \tau)$.
(b) If $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau_u)$ is a continuous map such that $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$, then show that $f(x) = 0$ for all $x \in \mathbb{R}$.
21. Let X, Y be topological spaces. Prove that a map $f : X \rightarrow Y$ is an open map iff $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$ for all $B \subset Y$.

22. Let X, Y be topological spaces. Show that $f : X \rightarrow Y$ is a closed map if and only if for each $y \in Y$ and for each open set G in X with $f^{-1}(\{y\}) \subset G$, there exists an open set H in Y containing y such that $f^{-1}(H) \subset G$.
23. Let X, Y, Z be topological spaces and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous maps such that $g \circ f : X \rightarrow Z$ is a homeomorphism. If g is one-to-one, then show that both f and g are homeomorphisms.
24. Examine whether the following pairs of topological spaces are homeomorphic.
 - (a) $(\mathbb{R}, \tau_u)$ and $(\mathbb{R}, \tau_c)$, where τ_u and τ_c are respectively the usual topology and the cocountable topology on $\mathbb{R}$.
 - (b) $(\mathbb{R}, \tau_f)$ and $(\mathbb{R}, \tau_c)$, where τ_f and τ_c are respectively the cofinite topology and the cocountable topology on $\mathbb{R}$.
 - (c) $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \tau')$, where $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$ and $\tau' = \{G \subset \mathbb{R} : 0 \notin G\} \cup \{\mathbb{R}\}$.
25. Consider $\mathbb{R}^2$ with the usual topology. Show that the subspaces $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $\{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ and $\{(x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 < 1\}$ of $\mathbb{R}^2$ are homeomorphic.
26. State TRUE or FALSE with justification for each of the following statements.
 - (a) If X is an indiscrete topological space, then there cannot exist any non-constant continuous map $f : X \rightarrow (\mathbb{R}, \tau_u)$.
 - (b) If τ_u is the usual topology on $\mathbb{R}$, then every one-to-one, onto and continuous map $f : (\mathbb{R}, \tau_u) \rightarrow (\mathbb{R}, \tau_u)$ is a homeomorphism.
 - (c) Every continuous map $f : (\mathbb{R}, \tau_u) \rightarrow (\mathbb{R}, \tau_u)$ is open or closed.
 - (d) If τ is the cofinite topology on $\mathbb{R}$, then every bijection $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)$ is a homeomorphism.
 - (e) If X, Y are topological spaces such that X is homeomorphic to a subspace of Y and Y is homeomorphic to a subspace of X , then X and Y must be homeomorphic.
27. Let X, Y be topological spaces and let $f : X \rightarrow Y$. Show that the map φ of X to the subspace $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $X \times Y$, defined by $\varphi(x) = (x, f(x))$ for all $x \in X$, is a homeomorphism if and only if f is continuous.
Deduce that every topological space X is homeomorphic to the diagonal $\Delta = \{(x, x) : x \in X\}$ of $X \times X$.
28. Let τ_l be the lower limit topology on $\mathbb{R}$. Examine whether the set $\{(x, y) \in \mathbb{R}^2 : x + y < 0\}$ is closed in the product space $(\mathbb{R}, \tau_l) \times (\mathbb{R}, \tau_l)$.
29. If $X = (\mathbb{R}, \tau_c)$, where τ_c is the cocountable topology on $\mathbb{R}$, then determine the interior of the set $(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}) \times \cdots$ in the countably infinite product $X^{\mathbb{N}}$ with the product topology.
30. If τ_c is the cocountable topology on $\mathbb{R}$, then show that the product topology of $(\mathbb{R}, \tau_c)$ and $(\mathbb{R}, \tau_c)$ is strictly finer than the cocountable topology on $\mathbb{R} \times \mathbb{R}$.

31. Let X, Y be topological spaces and let $A \subset X, B \subset Y$. Prove that $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$.
32. Let A, B, C, D be topological spaces and let $f : A \rightarrow B$ and $g : C \rightarrow D$ be continuous. If $\varphi(a, c) = (f(a), g(c))$ for all $(a, c) \in A \times C$, then show that $\varphi : A \times C \rightarrow B \times D$ is continuous.
33. Let Λ be an infinite set and let for each $\alpha \in \Lambda$, X_α be a discrete space. Prove that $\prod_{\alpha \in \Lambda} X_\alpha$ is a discrete space in the box topology, but if for each $\alpha \in \Lambda$, X_α has more than one point, then $\prod_{\alpha \in \Lambda} X_\alpha$ is not a discrete space in the product topology.
34. Consider the usual topology on $\mathbb{R}$ and let $A = \{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite}\}$. Show that A is dense in $\mathbb{R}^{\mathbb{N}}$ with the product topology.
35. Determine the closure of $\{(x_n) \in \mathbb{R}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite}\}$ in $\mathbb{R}^{\mathbb{N}}$ with the box topology.

Problem Set 3: Quotient spaces and compact spaces

1. State TRUE or FALSE with justification for each of the following statements.
 - (a) If $A = \{x \in \mathbb{R} : x^8 - x^7 \leq 200\}$ and $B = \{x^2 - 2x : x \in (0, \infty)\}$, then $A \cap B$ is a compact set in $\mathbb{R}$ with the usual topology.
 - (b) If every proper closed subset of a topological space X is compact, then X must be compact.
 - (c) If every subset of a Hausdorff space X is compact, then X must be a discrete space.
 - (d) A topological space in which every compact subset is closed must be Hausdorff.
 - (e) If X is a topological space such that every continuous map $f : X \rightarrow \mathbb{R}$ (with the usual metric on $\mathbb{R}$) is bounded, then X must be compact.
 - (f) Every topological space which is a continuous image of a topological space with the Bolzano-Weierstrass property must have the Bolzano-Weierstrass property.
 - (g) If τ_c denotes the cocountable topology on $\mathbb{R}$, then the topological space $(\mathbb{R}, \tau_c)$ is locally compact.
 - (h) The one-point compactification of $\mathbb{N}$ (with the usual topology) is homeomorphic to $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ (with the usual topology).
2. Let $p : X \rightarrow Y$ be continuous and suppose that there is a continuous map $f : Y \rightarrow X$ such that $p \circ f = \text{id}_Y$. Show that p is a quotient map.
3. Let A be a subspace of a topological space X , and let $r : X \rightarrow A$ be a continuous retraction; that is, $r(a) = a$ for every $a \in A$. Show that r is a quotient map.
4. Let $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $p(x, y) = x$, and let $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ or } y = 0\}$. Let $q = p|_A : A \rightarrow p(A)$. Show that q is a quotient map that is neither open nor closed.

5. Let $p : X \rightarrow Y$ be an open map and let A be open in X . Show that the restriction $q = p|_A : A \rightarrow p(A)$ is an open map.
6. Define equivalence relations on $X = \mathbb{R}^2$ by
 - (a) $(x_0, y_0) \sim (x_1, y_1)$ if $x_0 + y_0^2 = x_1 + y_1^2$;
 - (b) $(x_0, y_0) \sim (x_1, y_1)$ if $x_0^2 + y_0^2 = x_1^2 + y_1^2$.
 Let X^* denote the corresponding quotient space. In each case, describe a familiar space homeomorphic to X^* .
7. Let Y be the quotient space obtained from the K -topology $\mathbb{R}_K$ by collapsing the set K to a point. Let $p : \mathbb{R}_K \rightarrow Y$ be a quotient map.
 - (a) Show that Y satisfies T_1 axiom but not Hausdorff.
 - (b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \rightarrow Y \times Y$ is not a quotient map.
8. Show that $[0, 1]$ is not limit-point compact in the lower-limit topological space $\mathbb{R}_\ell$.
9. If X is compact Hausdorff under each of two topologies τ and τ' , show that either $\tau = \tau'$ or the two topologies are not comparable.
10. Let X be any nonempty set and let $x_0 \in X$. If $\tau = \{G \subset X : x_0 \notin G\} \cup \{X\}$, then find all the compact subsets of the topological space (X, τ) .
11. Consider the topology τ on $\mathbb{R}$ having

$$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$$
 as a basis, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Examine whether $[0, 1]$ is a compact set in the topological space $(\mathbb{R}, \tau)$.
12. Let τ_c be the cocountable topology on a nonempty set X . Find all the (a) compact (b) sequentially compact subsets of the topological space (X, τ_c) .
13. If $a, b \in \mathbb{R}$ such that $a < b$, then show that $[a, b]$ is not a compact subset of $\mathbb{R}$ with the lower limit topology.
14. Let (x_n) be a sequence in a topological space X and let $x_n \rightarrow x \in X$. Show that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a compact subset of X .
15. Prove that the union of finitely many compact sets in a topological space is compact. Also, show that the union of infinitely many compact sets in a topological space need not be compact.
16. Let A and B be compact subsets of a topological space X . Show that $A \cap B$ need not be compact but if X is Hausdorff, then $A \cap B$ must be compact.
17. Let G be topological group.
 - (a) If A and B are compact in G , then show that $A \cdot B$ is compact.
 - (b) Let H be subgroup of G and $p : \rightarrow G/H$ be the quotient map. If H is compact, show that p is closed.
 - (c) Let H be a compact subgroup of G . Show that if G/H is compact, then G is compact.

18. Let A and B be compact subspaces of the topological spaces X and Y respectively. If W is an open set in $X \times Y$ containing $A \times B$, then prove that there exist open sets G and H in X and Y respectively such that $A \times B \subset G \times H \subset W$.
19. For each $n \in \mathbb{N}$, let $X_n = [0, 1]$ with the usual topology, and $X = \prod_{n=1}^{\infty} X_n$.
 - (a) Show that X is not limit-point compact with respect to uniform topology.
 - (b) Show that $\prod_{n=1}^{\infty} X_n$ with the box topology is not compact.
20. Let X, Y be topological spaces with Y compact. Prove that the projection map $p_X : X \times Y \rightarrow X$ is a closed map.
21. Let X, Y be topological spaces with Y compact. If $f : X \rightarrow Y$ is such that $\{(x, f(x)) : x \in X\}$ (i.e. the graph of f) is closed in the product space $X \times Y$, then prove that f is continuous.
22. Show that the closure of a compact subset of a topological space need not be compact. Show, however, that the closure of a compact subset of a regular space is compact.
23. Let X be a topological space. If for each $x \in X$, there exists an open set G in X containing x such that $\overline{G}$ is a compact Hausdorff space, then show that X is Hausdorff.
24. Show that a compact topological space need not be separable but that every compact metrizable space must be separable.
25. If X is a Lindelöf space and Y is a compact space, then show that the product space $X \times Y$ is a Lindelöf space.
26. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of closed nowhere dense subsets of a compact Hausdorff space X . Show that

$$\text{int}\left(\bigcup_{n=1}^{\infty} A_n\right) = \emptyset.$$
27. Let X and Y be topological spaces such that Y is compact. Let $p : X \rightarrow Y$ be continuous, closed and onto such that $p^{-1}(\{y\})$ is a compact set in X for each $y \in Y$. Show that X is compact.
28. Let X and Y be topological spaces such that X is second countable. Let $f : X \rightarrow Y$ be continuous, closed and onto such that $f^{-1}(\{y\})$ is a compact set in X for each $y \in Y$. Show that Y is second countable.
29. Let f be a continuous map from a compact space X onto a Hausdorff space Y and let g be a map from Y to a topological space Z . If $g \circ f$ is continuous, then prove that g is continuous.
30. Let X be a metrizable topological space. Prove that the following statements are equivalent.
 - (a) X is compact.
 - (b) X is bounded with respect to every metric on X that induces the topology of X .
 - (c) Every continuous map $f : X \rightarrow \mathbb{R}$ (with the usual metric on $\mathbb{R}$) is bounded.

31. Let (X, d) be a compact metric space and let $f : X \rightarrow X$ be such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that f is onto and so f is a homeomorphism. Show also that the compactness of (X, d) is, in general, necessary.
32. Let X be a compact Hausdorff space and let $f : X \rightarrow X$ be continuous. Show that there exists a nonempty compact set K in X such that $f(K) = K$.
33. Show that a T_1 -space X has the Bolzano-Weierstrass property if and only if every countable open cover of X has a finite subcover.
34. Let X be a topological space with the Bolzano-Weierstrass property and let Y be a first countable space. Show that the projection map $p_Y : X \times Y \rightarrow Y$ is closed.
35. Let Y be a nonempty dense subset of a Hausdorff space X . If the subspace Y is locally compact, then show that Y is open in X .
36. Let X be a compact Hausdorff space and let $\mathbb{R}$ be equipped with the usual topology. Consider the real vector space $C(X, \mathbb{R})$ of all continuous maps from X to $\mathbb{R}$, where the vector space operations are defined pointwise. Prove that $C(X, \mathbb{R})$ is finite dimensional if and only if X is finite.
37. Let K be a nonempty compact subset and let C be a nonempty closed subset of a locally compact Hausdorff space X such that $K \cap C = \emptyset$. Prove that there exists a continuous map $f : X \rightarrow [0, 1]$ (with the usual topology) such that $f(K) = \{0\}$ and $f(C) = \{1\}$.
38. Let (f_n) be a monotone increasing sequence of real-valued continuous functions on a compact topological space X . If f_n converges pointwise to a continuous function f , show that f_n converges to f uniformly.
39. Show that a compact Hausdorff space is metrizable if and only if it is second countable.

Problem Set 4: Connected spaces

1. State TRUE or FALSE with justification for each of the following statements.
 - (a) It is possible that $\mathbb{R}^2$ can be written as a countable union of path-connected subsets.
 - (b) The set of all polynomials on $\mathbb{R}$ whose zero-set complements are connected is countable.
 - (c) There exists a non-empty open and connected set $A \subset \mathbb{R}^n$ such that every real-valued function on A is continuous.
 - (d) If a metric space X is path connected, then there exists a continuous function $f : [0, 1] \rightarrow X$ which is onto.
 - (e) There exists a discontinuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the graph G_f is connected in $\mathbb{R}^2$ but $\text{int}(\overline{G_f})$ is non-empty in $\mathbb{R}^2$.
 - (f) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^3 \in \mathbb{R} \setminus \mathbb{Q}\}$ is a disconnected subset of $\mathbb{R}^2$ (with the usual topology).
 - (g) If a topological space X has only finitely many components, then each component of X must be open in X .

- (h) If τ_l denotes the lower limit topology on $\mathbb{R}$, then the topological space $(\mathbb{R}, \tau_l)$ is totally disconnected.
 - (i) $\mathbb{R}$ with the cofinite topology is a locally connected space.
2. Show that a topological space X is connected if and only if for every $x, y \in X$ with $x \neq y$, there exists a connected subspace of X containing both x and y .
 3. Prove that a topological space X is connected if and only if every nonempty proper subset of X has a nonempty boundary.
 4. Let C be a connected subspace of a topological space X and let $A \subset X$ such that $C \cap A \neq \emptyset$ and $C \cap (X \setminus A) \neq \emptyset$. Prove that $C \cap \partial A \neq \emptyset$.
 5. Consider the topology τ on $\mathbb{R}$ having $\{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \cap \mathbb{Q} : a, b \in \mathbb{R}, a < b\}$ as a basis. Show that the topological space $(\mathbb{R}, \tau)$ is connected.
 6. Show that $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is a compact set in } (\mathbb{R}, \tau_u)\} \cup \{\emptyset\}$ is a topology on $\mathbb{R}$, where τ_u is the usual topology on $\mathbb{R}$. Examine whether the topological space $(\mathbb{R}, \tau)$ is (a) compact (b) connected.
 7. Let A and B be nonempty closed (or, open) subsets of a topological space such that both $A \cup B$ and $A \cap B$ are connected. Show that A and B are also connected.
 8. If A and B are connected subsets of a topological space such that $A \cap \overline{B} \neq \emptyset$, then prove that $A \cup B$ is connected.
 9. If X, Y are connected spaces and if $A \subsetneq X, B \subsetneq Y$, then show that $(X \times Y) \setminus (A \times B)$ is connected.
 10. Let $\{A_n\}_{n=1}^\infty$ be a sequence of connected subspaces of a topological space X such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Show that $\bigcup_{n=1}^\infty A_n$ is connected.
 11. If $\{A\} \cup \{A_\alpha : \alpha \in \Lambda\}$ is a class of connected subsets of a topological space such that $A \cap A_\alpha \neq \emptyset$ for all $\alpha \in \Lambda$, then show that $A \cup \left(\bigcup_{\alpha \in \Lambda} A_\alpha\right)$ is connected.
 12. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a class of connected subsets of a topological space such that $A_\alpha \cap A_\beta \neq \emptyset$ for all $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$. Prove that $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected.
 13. Let $\{A_n\}_{n=1}^\infty$ be a decreasing sequence of nonempty compact connected sets in a Hausdorff space. Prove that $\bigcap_{n=1}^\infty A_n$ is nonempty, compact and connected.
 14. Let X be a connected space and let there exist a non-constant continuous map from X to $\mathbb{R}$ (with the usual topology). Show that X is uncountable.
 15. Prove that every connected T_3 -space containing at least two points must be uncountable.
 16. Let X be a connected space and let τ_l be the lower limit topology on $\mathbb{R}$. Prove that every continuous map from X to $(\mathbb{R}, \tau_l)$ is a constant map.
(Hence, in particular, every continuous map from $(\mathbb{R}, \tau_u)$ to $(\mathbb{R}, \tau_l)$ is a constant map, where τ_u denotes the usual topology on $\mathbb{R}$.)

17. For each $m, n \in \mathbb{N}$, let $B_{m,n} = \{mk + n : k \in \mathbb{Z}\} \cap \mathbb{N}$. Show that
 - (a) $\{B_{m,n} : m, n \in \mathbb{N}, \text{g.c.d.}(m, n) = 1\}$ is a basis for some topology τ on $\mathbb{N}$.
 - (b) for each prime $p \in \mathbb{N}$, $\{pk : k \in \mathbb{N}\}$ is a closed set in $(\mathbb{N}, \tau)$ and hence deduce that there exist infinitely many prime numbers.
 - (c) $(\mathbb{N}, \tau)$ is connected, Hausdorff but not compact.
 - (d) if P is the set of all primes, then $P^0 = \emptyset$.
18. Consider $\mathbb{R}^2$ with the usual topology. If A is a countable subset of $\mathbb{R}^2$, then show that $\mathbb{R}^2 \setminus A$ is path connected.
19. Prove that every open connected subspace of $\mathbb{R}^2$ (with the usual topology) is path connected.

Problem Set 5: Countability and separation axioms

1. State TRUE or FALSE with justification for each of the following statements.
 - (a) If A and B are nonempty subsets of topological spaces X and Y respectively such that $A \times B$ is closed in the product space $X \times Y$, then A must be closed in X and B must be closed in Y .
 - (b) If τ_l denotes the lower limit topology on $\mathbb{R}$, then $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$ is an open set in the product space $(\mathbb{R}, \tau_l) \times (\mathbb{R}, \tau_l)$.
 - (c) There cannot exist topologies τ, τ' on an infinite set X such that the product topology for (X, τ) and (X, τ') coincides with the cofinite topology on $X \times X$.
 - (d) If τ_u and τ_l denote respectively the usual topology and the lower limit topology on $\mathbb{R}$, then the product space $(\mathbb{R}, \tau_u) \times (\mathbb{R}, \tau_l)$ is not metrizable.
2. Prove that every closed (respectively, open) subset of a metrizable space is a G_δ (respectively, an F_σ) set.
Also, show that the metrizability condition is, in general, necessary.
3. Let (X, τ) be a metrizable topological space and let τ_u be the usual topology on $\mathbb{R}$. Show that τ is the weakest topology on X with respect to which every continuous map from (X, τ) to $(\mathbb{R}, \tau_u)$ remains continuous.
4. Let X be a first countable space and let $A \subset X$. Prove that
 - (a) A is closed in X if and only if for every sequence $(a_n) \subset A$ and for every $x \in X$, $a_n \rightarrow x \Rightarrow x \in A$.
 - (b) A is open in X if and only if for every sequence $(x_n) \subset X$ and for every $a \in A$, $x_n \rightarrow a \Rightarrow x_n \in A$ eventually.
5. If $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$, then show that the topological space $(\mathbb{R}, \tau)$ is first countable but not second countable.

6. Let X be a first countable space and let $G \subset X$. If for every nonempty countable set A in X , $G \cap A$ is open in the subspace A , then show that G is open in X .
7. Let $X = \mathbb{Z}_+ \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, and let

$$\begin{aligned} \tau = \mathcal{P}(X \setminus \{(0, 0)\}) \\ \cup \left\{ G \subset X : (0, 0) \in G, \{m \in \mathbb{Z}_+ : \{n \in \mathbb{Z}_+ : (m, n) \notin G\} \text{ is infinite}\} \right. \\ \left. \text{is finite} \right\}. \end{aligned}$$

Prove that τ is a topology on X which is different from the discrete topology on X . Determine all convergent sequences in (X, τ) and hence conclude that (X, τ) is not first countable. This space is called the Arens–Fort space; it shows that a countable topological space need not be first countable.

8. Prove that a topological space is second countable if and only if it has a countable subbasis.
9. Let A be an uncountable subset of a second countable space. Show that uncountably many points of A are limit points of A .
10. Let X be a separable space. Prove that every class of pairwise disjoint open sets in X is countable.
Hence deduce that the set of all isolated points of X is countable.
11. Prove that every topological space can be considered as a subspace of a separable space.
(From this it follows immediately that a subspace of a separable space need not be separable.)
12. If Y is a nonempty open subset of a separable space (X, τ) , then show that the subspace $(Y, \tau|_Y)$ is separable.
13. Let Y be a dense subspace of a first countable separable space X . Show that Y is separable.
14. Let τ_l denote the lower limit topology on $\mathbb{R}$. Prove that every subspace of the topological space $(\mathbb{R}, \tau_l)$ is separable.
15. Prove or disprove: $\mathbb{R}$ with the cocountable topology is a Lindelöf space.
16. Prove that every metrizable Lindelöf space is second countable.
17. Let X be any set with at least two elements. Show that there exists a topology τ on X such that (X, τ) is a T_0 -space but not a T_1 -space.
18. Prove that a topological space X is a T_0 -space if and only if for all $x, y \in X$ with $x \neq y$, $\overline{\{x\}} \neq \overline{\{y\}}$.
19. For a topological space (X, τ) , prove that the following statements are equivalent.
 - (a) (X, τ) is a T_1 -space.
 - (b) For each $x \in X$, $\{x\} = \bigcap \{G \in \tau : x \in G\}$.
 - (c) τ is finer than the cofinite topology on X .
20. Let (X, τ) be a T_1 -space and $A \subset X$. Show that $A = \bigcap \{G \in \tau : A \subset G\}$.

21. Show that for every convergent sequence in a topological space X to have a unique limit in X , it is necessary but not sufficient that X is a T_1 -space.
22. Let X be a T_1 -space and let $A \subset X$, $x \in X$. Show that
 - (a) $x \in A'$ if and only if every open set in X containing x contains infinitely many points of A .
 - (b) A' is closed in X .
23. Let X be a first countable T_1 -space and let $A \subset X$, $x \in X$. Prove that $x \in A'$ if and only if there exists a sequence of distinct points in A converging to x in X .
Also, show that both T_1 and first countability conditions are, in general, necessary.
24. Show that in a first countable T_1 -space, every singleton set is a G_δ set.
Show that both first countability and T_1 conditions are, in general, necessary.
Also, give an example of a topological space which is not first countable but in which every singleton set is a G_δ set.
25. Prove that a topological space (X, τ) is a Hausdorff space if and only if for each $x \in X$, $\{x\} = \bigcap \{\overline{G} : G \in \tau, x \in G\}$.
26. If X is a Hausdorff space, then show that
 - (a) for each $x \in X$, $\bigcap \{F \subset X : F \text{ is closed in } X, x \in F\} = \{x\}$.
 - (b) for each $x \in X$, $\bigcap \{G \subset X : G \text{ is open in } X, x \in G\} = \{x\}$.
 Also, show that a topological space X satisfying (a) and (b) need not be Hausdorff.
27. Show that a topological space X is a Hausdorff space if and only if $\{(x, x) : x \in X\}$ is a closed subset of the product space $X \times X$.
28. Show that a topological space X is a discrete space if and only if $\{(x, x) : x \in X\}$ is an open subset of the product space $X \times X$.
29. Let X be a Hausdorff space and let $f : X \rightarrow X$ be a continuous map such that $f \circ f = f$. Prove that $f(X)$ is a closed subset of X .
30. Let X, Y be topological spaces and let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be continuous such that $g(f(x)) = x$ for all $x \in X$. If Y is a Hausdorff space, then show that X is a Hausdorff space and $f(X)$ is closed in Y .
31. Let D be a dense subset of a topological space X and let Y be a Hausdorff space. If $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are continuous such that $f(x) = g(x)$ for all $x \in D$, then show that $f(x) = g(x)$ for all $x \in X$.
32. Let X_0 be a dense subspace of a topological space X and let Y be a Hausdorff space. Prove that every continuous map $f_0 : X_0 \rightarrow Y$ can have at most one continuous extension $f : X \rightarrow Y$. Show also that a continuous map $f_0 : X_0 \rightarrow Y$ need not have any continuous extension $f : X \rightarrow Y$.

33. Let X and Y be Hausdorff spaces and let $f : X \rightarrow Y$ be onto. Show that $f : X \rightarrow Y$ is a homeomorphism if and only if $\overline{A} = f^{-1}(\overline{f(A)})$ for all $A \subset X$.
34. Let $x_1, \dots, x_n$ be distinct points of a Hausdorff space X . Prove that there exist pairwise disjoint open sets $G_1, \dots, G_n$ in X such that $x_i \in G_i$ for $i = 1, \dots, n$.
35. Prove that every infinite Hausdorff space contains an infinite set A such that each point of A is an isolated point of A .
(Hence every infinite Hausdorff space contains an infinite discrete subspace).
36. Prove that a first countable space X is a Hausdorff space if and only if every convergent sequence in X has a unique limit in X .
37. Let X, Y be topological spaces with Y Hausdorff. If $f : X \rightarrow Y$ is continuous, then prove that $G_f = \{(x, f(x)) : x \in X\}$ (the graph of f) is closed in the product space $X \times Y$.
Show that the Hausdorff condition on Y is, in general, necessary.
38. Prove that a topological space Y is Hausdorff if and only if for every topological space X and for any continuous maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$, the set $\{x \in X : f(x) = g(x)\}$ is closed in X . Hence show that the set of all fixed points of a continuous map from a Hausdorff space Y to itself is closed in Y .
39. State TRUE or FALSE with justification for each of the following statements.
 - (a) If τ and τ' are topologies on a nonempty set X such that both (X, τ) and (X, τ') are T_1 -spaces, then $(X, \tau \cap \tau')$ must be a T_1 -space.
 - (b) There exists a Hausdorff space with exactly 100 (distinct) open sets.
 - (c) If $x_1, x_2, \dots$ are distinct points in a Hausdorff space X , then there must exist pairwise disjoint open sets $G_1, G_2, \dots$ in X such that $x_n \in G_n$ for all $n \in \mathbb{N}$.
 - (d) Every second countable Hausdorff space is metrizable.
 - (e) If X and Y are topological spaces such that the product space $X \times Y$ is Hausdorff, then both X and Y must be Hausdorff.
 - (f) If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then the topological space $(\mathbb{R}, \tau)$ is normal but not regular.
40. Prove that every regular T_0 -space is a T_3 -space but that a normal T_0 -space need not be a T_4 -space.
41. Let X be a T_3 -space and let $x, y \in X$ with $x \neq y$. Prove that there exist open sets G and H in X containing x and y respectively such that $\overline{G} \cap \overline{H} = \emptyset$.
42. Let A and B be disjoint closed subsets of a normal space X . Prove that there exist open subsets G and H of X containing A and B respectively such that $\overline{G} \cap \overline{H} = \emptyset$.
43. Let X be a T_4 -space and let Y be a topological space. If $f : X \rightarrow Y$ is continuous, closed and onto, then show that Y is a T_4 -space.
44. Let X be a normal space. Show that X is regular if and only if X is completely regular.

45. Let (X, τ) be a completely regular space. Show that the weak topology on X induced by the family of all continuous maps $f : (X, \tau) \rightarrow (\mathbb{R}, \tau_u)$ is the same as τ .
46. Let K be a compact set and C be a closed set in a regular space X such that $K \cap C = \emptyset$. Prove that there exist disjoint open sets G and H in X such that $K \subset G$ and $C \subset H$.
47. Show that a compact Hausdorff space is metrizable if and only if it is second countable.

Problem Set 6: Nets, filters, and general convergence

1. Give an example of a topological space X and a subset $A \subset X$ such that there exists a point $x \in \overline{A}$ which is *not* the limit of any sequence from A .
2. Let $(x_i)_{i \in I}$ be a net in a topological space X and let $x \in X$. Prove that the following are equivalent:
 - (a) $x_i \rightarrow x$;
 - (b) for every neighborhood U of x there exists $i_0 \in I$ such that $i \geq i_0$ implies $x_i \in U$;
 - (c) for every open set U with $x \in U$, the set $\{i \in I : x_i \in U\}$ is cofinal in I .
3. Prove that $x \in \overline{A}$ if and only if there exists a net (a_i) in A converging to x .
4. Prove that a function $f : X \rightarrow Y$ between topological spaces is continuous if and only if it preserves limits of nets: whenever $x_i \rightarrow x$ in X , we have $f(x_i) \rightarrow f(x)$ in Y .
5. Let $\mathcal{F}$ be a filter on X and let $x \in X$. Prove that $\mathcal{F}$ converges to x if and only if every neighborhood of x belongs to $\mathcal{F}$, equivalently, $\mathcal{N}(x) \subset \mathcal{F}$. Also formulate the corresponding cluster-point condition for filters.
6. Show that if (X, τ) is first countable and $x \in \overline{A}$, then there exists a sequence in A converging to x . Conclude that in first countable spaces, closure can be tested using sequences.
7. $[\star]$ Let (x_i) be a net in X . Prove that x is a cluster point of (x_i) (every neighborhood of x meets $\{x_i : i \geq i_0\}$ for all i_0) if and only if some subnet of (x_i) converges to x .
8. Let X be compact. Prove that every net in X has a convergent subnet. (This is the net version of sequential compactness, and it holds without countability assumptions.)
9. Let $(X_j)_{j \in J}$ be a family of spaces and $X = \prod_{j \in J} X_j$ with the product topology. Prove that a net $x_i = (x_i^{(j)})$ converges to $x = (x^{(j)})$ in X if and only if for every $j \in J$ we have $x_i^{(j)} \rightarrow x^{(j)}$ in X_j .
10. $[\star]$ Let $\mathcal{U}$ be an ultrafilter on X . Show that X is compact if and only if every ultrafilter on X converges.

Problem Set 7: Compactness, products, and compactifications

1. Prove that a continuous image of a compact space is compact. Give an example showing that the preimage of a compact set under a continuous map need not be compact.

2. Let X be Hausdorff. Prove that every compact subset of X is closed.
3. Show that if X is compact and Y is Hausdorff, then every continuous bijection $f : X \rightarrow Y$ is a homeomorphism.
4. Let X be a space. Prove that X is compact if and only if every family of closed sets with the finite intersection property has nonempty intersection.
5. Let X be compact and $A \subset X$. Show that A is compact if and only if it is closed in X .
6. Let (X, d) be a metric space. Prove that compactness is equivalent to completeness plus total boundedness.
7. Prove that a product of finitely many compact spaces is compact. (*Hint*: use the finite-intersection characterization.)
8. $[\star]$ (Tychonoff in a special case.) Let $X = \{0, 1\}^J$ with the product topology. Prove that X is compact. (*Hint*: identify X with the set of ultrafilters on $\mathcal{P}(J)$, or use the finite intersection property directly.)
9. Let X be locally compact Hausdorff. Construct the one-point compactification $X^* = X \cup \{\infty\}$ and prove:
 - (a) X^* is compact and Hausdorff;
 - (b) X is open and dense in X^* ;
 - (c) X^* is Hausdorff if and only if X is locally compact Hausdorff.
10. $[\star]$ Let X be non-compact locally compact Hausdorff. Show that X^* is connected if and only if X is connected and not compact.

Problem Set 8: Baire category and applications

1. A set $A \subset X$ is *nowhere dense* if $\text{int}(\overline{A}) = \emptyset$, and *meagre* if it is a countable union of nowhere dense sets. Show that:
 - (a) every nowhere dense set has empty interior;
 - (b) the closure of a nowhere dense set is nowhere dense;
 - (c) a countable union of meagre sets is meagre.
2. Prove the Baire Category Theorem for complete metric spaces: if (X, d) is complete and (U_n) is a sequence of open dense sets, then $\bigcap_{n=1}^{\infty} U_n$ is dense.
3. Conclude that $\mathbb{R}$ is not a countable union of nowhere dense sets (in particular, not countable).
4. Let X be a Baire space. Prove that every nonempty open subset of X is not meagre in X .
5. $[\star]$ Let $C([0, 1])$ be the space of real-valued continuous functions on $[0, 1]$ with the sup norm. Show that the set of nowhere differentiable functions is dense (and in fact comeagre) in $C([0, 1])$. (*You may quote a standard theorem, for example, the Banach–Mazur theorem, if known.*)

6. Use Baire category to show that if (X, d) is complete and $f_n : X \rightarrow \mathbb{R}$ are continuous with $f_n(x) \rightarrow f(x)$ pointwise, then the set of points where f is continuous is a G_δ set. (*Hint: oscillation of f .*)
7. Let X be compact Hausdorff and $C(X)$ the Banach space of continuous real functions with the sup norm. Show that if (f_n) is Cauchy in $\|\cdot\|_\infty$, then (f_n) converges uniformly to some $f \in C(X)$.
8. $[\star]$ Show that every locally compact Hausdorff space is a Baire space.

Problem Set 9: Urysohn–Tietze, metrization, and separation

1. Let X be normal and $A, B \subset X$ be disjoint closed sets. Prove Urysohn's Lemma: there exists $f : X \rightarrow [0, 1]$ continuous with $f|_A = 0$ and $f|_B = 1$.
2. Use Urysohn's Lemma to show that every normal T_1 space is completely regular.
3. State and prove the Tietze Extension Theorem for normal spaces: if X is normal, $A \subset X$ closed, and $f : A \rightarrow [-1, 1]$ continuous, then f extends to a continuous $F : X \rightarrow [-1, 1]$.
4. Let X be a space and $\mathcal{F} \subset C(X, [0, 1])$ a family of continuous functions separating points from closed sets. Show that the evaluation map

$$e : X \rightarrow [0, 1]^{\mathcal{F}}, \quad e(x) = (f(x))_{f \in \mathcal{F}}$$

is an embedding.

5. Prove that every completely regular T_1 space embeds in a cube $[0, 1]^I$ for some index set I .
6. (Urysohn metrization.) Prove that every regular second countable space is metrizable. (*You may assume the existence of a countable base and construct a metric via Urysohn functions.*)
7. Show that a compact Hausdorff space is metrizable if and only if it is second countable.
8. $[\star]$ (Nagata–Smirnov, special case.) Prove that every metrizable space admits a σ -locally finite base.
9. Let X be Hausdorff and second countable. Prove that X is separable. Is the converse true? Give an example.
10. $[\star]$ Give an example of a normal space X for which $X \times X$ is not normal. (*Hint: the Sorgenfrey line.*)

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