Fourier transform:

Fourier analysis is basically study of functions via exploiting the under-lined symmetries. We have seen while discussing the Fourier Series that, if the function is periodic on $\mathbb{R}$, then it is enough to study the function on a period. Each period essentially corresponds only one Fourier coeff. and hence, a periodic function can be completely determined by countably many complex numbers. However, if f is not periodic, then it needs slitly different treatment, though the essence of both cases is same. That is, how the function on $\mathbb{R}^n$ (or $\mathbb{T}^n$) transform under translations? Suppose the function f transforms under the translation by a multiplication of absolute value 1. That is,

$$f(x+y) = \varphi(x)f(y), \text{ where } |\varphi(x)| = 1.$$

Then $f(x) = \varphi(x)f(0)$. That is, f is completely determine by φ. Moreover,

$$\varphi(x)\varphi(y)f(0) = \varphi(x)f(y) = f(x+y) = \varphi(x+y)f(0).$$

$$\Rightarrow \quad \varphi(x+y) = \varphi(x)\varphi(y), \text{ if } f \not\equiv 0.$$

Hence, to determine all such f's that transform as above, it is enough to find all those φ's s.t (2)

$$\varphi(x+y) = \varphi(x)\varphi(y).$$

Theorem: If φ is a measurable function on $\mathbb{R}^n$ with $\varphi(x+y) = \varphi(x)\varphi(y)$ & $|\varphi(x)| = 1$. Then $\exists\ \xi \in \mathbb{R}^n$ s.t $\varphi(x) = e^{2\pi i x\cdot\xi}$.

Proof: First, we consider φ on $\mathbb{R}$. Let $a \in \mathbb{R}$ be s.t. $A^{-1} = \int_0^a \varphi(t)\,dt \neq 0$.

(such a exists, otherwise by FTC, $\varphi = 0$ a.e.

Then $\varphi(x) = A\int_0^a \varphi(x)\varphi(t)\,dt = A\int_0^a \varphi(x+t)\,dt$

& $\varphi(x) = \int_x^{x+a} \varphi(t)\,dt$

This implies, φ is continuous, being integral of $\varphi \in L^1_{loc}(\mathbb{R})$. Further, φ is integral of the continuous function φ, hence $\varphi \in C^1(\mathbb{R})$.

This gives, $\varphi'(x) = A[\varphi(x+a) - \varphi(x)] = B\varphi(x)$,

where $B = A[\varphi(a) - 1]$.

$$\Rightarrow \frac{d}{dx}\left[e^{-Bx}\varphi(x)\right] = 0 \Rightarrow e^{-Bx}\varphi(x) = \text{const}$$

Since, $\varphi(0) = 1$, $\varphi(x) = e^{Bx}$, Since $|\varphi(x)| = 1$, it follows that B must be purely imaginary. i.e. $B = 2\pi i \xi$, for some $\xi \in \mathbb{R}$.

For $\varphi : \mathbb{R}^n \to \mathbb{C}$, define $\varphi_j(t) = \varphi(t e_j)$, where $\{e_1, e_2, \ldots, e_n\}$ is S.B. Then

(3)

$$\varphi_j(t+s) = \varphi_j(t)\varphi_j(s).$$

$$\Rightarrow \varphi_j(x) = e^{2\pi i \xi_j x}, \quad x \in \mathbb{R}.$$

$$\Rightarrow \varphi(y) = \varphi\left(\sum y_j e_j\right) = \prod_{j=1}^{n} \varphi(y_j) = e^{2\pi i y \cdot \xi}$$

where $\xi = (\xi_1, \ldots, \xi_n)$.

__Cor__: If $\varphi : \mathbb{T} \xrightarrow{\text{m'ble}} \mathbb{C}$ and $\varphi(x+y) = \varphi(x)\varphi(y)$, with $|\varphi(x)| = 1$. Then $\varphi(x) = e^{2\pi i n x}$.

Proof: Notice that φ is periodic with period 1 iff $\varphi(0) = \varphi(1)$ iff $e^{2\pi i x \xi} = 1$ iff $\xi \in \mathbb{Z}$. That is, $\varphi(x) = e^{2\pi i n x}$.

__Exercise__: If $\varphi : \mathbb{T}^n \xrightarrow{\text{m'ble}} \mathbb{C}$ and $|\varphi(x)| = 1$,

$\varphi(s+t) = \varphi(s)\varphi(t)$. Then show that

$$\varphi(t) = e^{2\pi i t \cdot \alpha}, \quad \alpha \in \mathbb{Z}^n.$$

Thus, we conclude that those functions which transform as above satisfying

$$f(x+y) = e^{2\pi i x \cdot \xi} f(y). \text{ for some}$$

$\xi \in \mathbb{R}^n$ (or $\mathbb{Z}^n$).

For the time being we consider (4)

$$f(x+y) = e^{ix\cdot\xi} f(y).$$

Let $f \in L^1(\mathbb{R})$ (or $L^1(\mathbb{R}^n)$). Then we define its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\cdot\xi} f(x)\,dx.$$

Lemma: Let $f \in L^1(\mathbb{R}^n)$. Then

(i) $(\tau_y f)^\wedge(\xi) = e^{-i\xi\cdot y}\hat{f}(\xi)$, where

$$\tau_y f(x) = f(x-y).$$

(ii) If $g(x) = e^{i\alpha\cdot x} f(x)$, then

$$\hat{g}(\xi) = \hat{f}(\xi-\alpha) = (\tau_\alpha \hat{f})(\xi).$$

(iii) If $g(x) = \overline{f(-x)}$, then $\hat{g}(\xi) = \overline{\hat{f}(\xi)}$.

(iv) If $g(x) = f(\frac{x}{\lambda})$, $\lambda > 0$, then $\hat{g}(\xi) = \lambda \hat{f}(\lambda\xi)$.

(v) $|\hat{f}(\xi)| \le \|f\|_1$ (uniformly bounded).

(vi) If $f, g \in L^1(\mathbb{R}^n)$, then

$$(f*g)^\wedge(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

(Hint: use Fubini's theorem, and change of variable etc.)

Lemma: Let $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$.

Proof: Let $x_n, y_n \in \mathbb{R}^n$, be such that

$$|x_n - y_n| \to 0.$$

Then $\left|\hat{f}(x_n) - \hat{f}(y_n)\right| = \left|\int f(\xi)\left(e^{-i x_n \cdot \xi} - e^{-i y_n \cdot \xi}\right) d\xi\right|$

$$\leq \int |f(\xi)| \left| e^{-i(x_n - y_n)\cdot\xi} - 1 \right| d\xi.$$

for each fixed ξ, $e^{-ix\cdot\xi}$ is uniformly cont.

it follows by DCT that

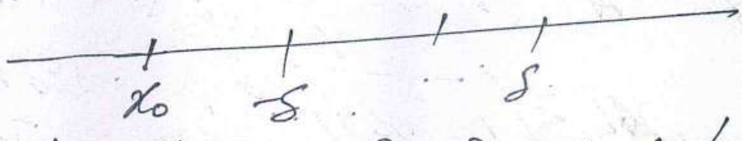
$$\left|\hat{f}(x_n) - \hat{f}(y_n)\right| \to 0 \quad \text{as } n \to \infty.$$

Hence $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$.

Lemma: Let $f \in L^1(\mathbb{R})$ & f is uniformly cont, then $\lim_{|x| \to \infty} f(x) = 0$.

Proof: If $\lim_{|x| \to \infty} f(x) \neq 0$, then for some $\epsilon_0 > 0$,

$\exists\, x_0 \in \mathbb{R}$ s.t. $|f(x_0)| > \epsilon_0$, $|x_0| > \delta$, $\forall \delta > 0$.



By Continuity at x_0, $\exists\ \delta_0 > 0$ s.t.

$|x - x_0| < \delta_0 \Rightarrow |f(x) - f(x_0)| < \epsilon_0/2 \Rightarrow |f(x)| > \epsilon_0/2$.

By uniform continuity, $|f(x)| > \epsilon_0/2$ ~~for~~ on each interval of length $2\delta_0$ since

$y \in (x_0 - 2\delta_0, x_0 - \delta) \Rightarrow |x_0 - y| < \delta_0 \Rightarrow |f(y)| > \epsilon_0/2$.

$$\therefore \int_{|y| > \delta} |f(y)|\, dy = \sum \int_{x_0 - n\delta_0}^{x_0 - (n+1)\delta_0} |f(y)|\, dy = \sum_{n \in \mathbb{Z}} \delta\, \epsilon_0/2 = \infty.$$

We use this fact to prove the following result.

Theorem: Let $f \in L^1(\mathbb{R})$ and $x f(x) \in L^1(\mathbb{R})$. Then $\hat{f}$ is differentiable and

(6)

$$\frac{d}{d\xi}\hat{f}(\xi) = -(ixf)^{\wedge}(\xi)$$

Proof: $$\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} = \int f(x) e^{-i\xi x}\left(\frac{e^{-ihx}-1}{h}\right) dx$$

notice that $\left|\frac{e^{-ihx}-1}{h}\right| \leq |x|$ & $\frac{e^{-ihx}-1}{h} \to -ix$

Hence, the integrand on the RHS is bounded by $|x f(x)| \in L^1(\mathbb{R})$. By DCT, it follows that $$\frac{d}{d\xi}\hat{f}(\xi) = \int f(x) e^{-i\xi x}(-ix)\, dx$$
$$= (-ixf)^{\wedge}(\xi).$$

Theorem: Let $f \in L^1(\mathbb{R})$, and $F(x) = \int_{-\infty}^{x} f(y)\, dy$. If $F \in L^1(\mathbb{R})$, then $\hat{F}(\xi) = \frac{1}{i\xi}\hat{f}(\xi)$, $\xi \neq 0$.

(Equivalently, if $f, f' \in L^1(\mathbb{R})$, then
$$\hat{f'}(\xi) = i\xi \hat{f}(\xi).$$

Proof: By FTC, it follows that
$$F' = f \text{ a.e. on } \mathbb{R}.$$

Since, $f \in L^1(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty} F(x)\, e^{-ix \cdot y}\, dx = \frac{F(x)\, e^{-ixy}}{-iy}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{f(x)\, e^{-ixy}}{-iy}\, dx$$

Since $F(x)e^{-ixy} \in L^1(\mathbb{R})$ & unif cont, by the previous lemma,

$$\hat{F}(y) = \frac{1}{iy}\hat{f}(y), \text{ if } y \neq 0.$$ (7)

or $\hat{f'}(y) = iy\,\hat{f}(y)$, if $f, f' \in L^1(\mathbb{R})$.

Lemma 9: Let $C_c^{\infty}(\mathbb{R})$ be the space of all infinitely differentiable functions on $\mathbb{R}$ having compact support. Then

$$\overline{C_c^{\infty}(\mathbb{R})} = L^1(\mathbb{R}).$$

Proof: Since $\overline{C_c(\mathbb{R})} = L^1(\mathbb{R})$, for $\epsilon > 0$, $\exists\, g \in C_c(\mathbb{R})$ s.t. $\|g - f\|_1 < \epsilon$

Now, consider $\varphi \in C_c^{\infty}(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi = 1$.

For $t > 0$, let $\varphi_t(x) = t^{-1}\varphi(\frac{x}{t})$. Then $\int \varphi_t = 1$.

Hence, $g * \varphi_t \in C_c^{\infty}(\mathbb{R})$ (Exercise)

Now,

$$g * \varphi_t(x) - g(x) = \int \big(g(x-y) - g(x)\big)\varphi_t(y)\, dy$$

$$= \int \big(g(x - tz) - g(x)\big)\varphi(z)\, dz \quad \text{— (1)}$$

$$\Rightarrow \|g * \varphi_t - g\|_1 \leq \int \|\tau_{tz} g - g\|_1 |\varphi(z)|\, dz \quad \text{— (*)}$$

for small t, $\|\tau_{tz} g - g\|_1 < \epsilon$. By DCT, it follows that $\| g * \varphi_t - g \|_1 < \epsilon$ for $|t| < \delta$.

i.e. $\| g * \varphi_t - f \|_1 < 2\epsilon$ for $|t| < \delta$.

<u>Exercise</u>: For $1 \le p < \infty$. Show that

$$\overline{C_c^\infty(\mathbb{R})} = L^p(\mathbb{R})$$

and $\overline{C_c^\infty(\mathbb{R})} = C_0(\mathbb{R})$

(Hint: use Minkowski integral inequality in (1)).

<u>Riemann - Lebesgue Lemma</u>:

If $f \in L^1(\mathbb{R})$, then $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$.

<u>Proof</u>: Since, $f \in L^1(\mathbb{R})$, for $\epsilon > 0$, $\exists\, g \in C_c^\infty(\mathbb{R})$

s.t. $\| g - f \|_1 < \epsilon$.

Given g is differentiable, $\hat{g'}(x) = (ix)\hat{g}(x)$

i.e. $|ix\, \hat{g}(x)| \le \| g' \|_1 < \infty$.

$\Rightarrow \quad |\hat{g}(x)| \to 0$ as $|x| \to \infty$.

Now, $|\hat{f}(x) - \hat{g}(x)| \le \| f - g \|_1 < \epsilon$

Letting $|x| \to \infty$, $\lim_{|x| \to \infty} |\hat{f}(x)| \le \epsilon$, $\forall\, \epsilon > 0$.

$\Rightarrow \quad \lim_{|x| \to \infty} \hat{f}(x) = 0$.

Notice that $(L^1(\mathbb{R}))^{\wedge} \underset{\neq}{\subset} C_0(\mathbb{R})$. In fact, the inclusion is injective but <u>not</u> surjective. 9
That is, every continuous function vanishing at ∞, need not be F.T. of an L^1-func.
This is based on the fact that F.T. of an L^1-function not too far from being an L^1-function.

Suppose $g \in C_0(\mathbb{R})$ is an odd function s.t. $g = \hat{f}$, for some $f \in L^1(\mathbb{R})$.

Then $\left| \int_1^b \frac{\hat{f}(x)}{x} dx \right| \le A < \infty$, where A is independent of b. This follows by the fact that $\left| \int_\alpha^\beta \frac{\sin t}{t} dt \right| \le B < \infty$, where B is free of choice of $\alpha, \beta \in \mathbb{R}$.

Since $\hat{f}$ is odd (as g is odd).

$$\hat{f}(x) = -i \int_{\mathbb{R}} f(t) \sin tx \, dt.$$

Consider
$$\left| \int_{-n}^{n} f(t) \left(\int_1^b \frac{\sin tx}{x} dx \right) dt \right| \quad \text{--- } (*)$$
$$\le \int_{-n}^{n} |f(t)| \left| \int_1^b \frac{\sin tx}{x} dx \right| dt$$
$$\le \int_{-n}^{n} |f(t)| B \le \|f\|_1 B.$$

Notice that by Fubini's theorem, we can

interchange the integrals in (*).

$$\therefore \left| \int_1^b \frac{\hat{f}(x)}{x} dx \right| \le \|f\|_1 B < \infty.$$

But $g(x) = \begin{cases} \frac{1}{\log x} & x > 0 \\ -\frac{1}{\log x} & x < 0 \\ 0 & x = 0. \end{cases}$

Then $g \in C_0(\mathbb{R})$ & g is odd. However,

$$\left| \int_1^b \frac{1}{x \log x} dx \right| = \infty.$$

Ex. let $f(x) = e^{-\pi x^2}$, the Gaussian. Then

$$F(\xi) = \hat{f}(\xi) = \int e^{-2\pi i x \xi} f(x) dx = f(\xi).$$

We know that $\int e^{-\pi x^2} dx = 1$ (Exercise)

$$F'(\xi) = \int (-2\pi i \xi) f(x) e^{-2\pi i x \xi} d\xi$$

$= (-2\pi i x f)^{\wedge}(\xi)$ $(\because f, xf \in L^1(\mathbb{R}))$

$= i (f')^{\wedge}(\xi)$ $(\because f'(x) = -2\pi x e^{-\pi x^2})$

$= i (2\pi i \xi) \hat{f}(\xi) = -2\pi \xi F(\xi).$

i.e. $F'(\xi) = -2\pi \xi F(\xi)$

$\Rightarrow \frac{d}{d\xi}\left(F(\xi) e^{\pi \xi^2}\right) = 0$

$\Rightarrow F(\xi) e^{\pi \xi^2} = $ const.

Since $F(0) = 1$, $\Rightarrow F(\xi) = e^{-\pi \xi^2}$.

Remark: For $\delta > 0$, let $f_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}$

Then $\hat{f_\delta}(x) = e^{-\pi \delta x^2} \to 0$ as $\delta \to 0$, however, $f_\delta(x) \to 1$ as $\delta \to 0$. Hence, we cannot see both f_δ & $\hat{f_\delta}$ exist together. That is, f_δ & $\hat{f_\delta}$ both cannot be localized together. (This is known as Heisenberg uncertainty principle, we elaborate later.) (11)

Ex. If $f(x) = e^{-\pi x^2}$, then show that
$$|f(x)| \le \frac{M}{1+x^2}.$$

Lemma: Let $f, h \in L^1(\mathbb{R})$ and
$$h(x) = \int_{\mathbb{R}} H(\xi)\, e^{ix\cdot\xi} d\xi$$
for some $H \in L^1(\mathbb{R})$. Then
$$(h * f)(x) = \int H(\xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

Proof:
$$h * f(x) = \int h(x-y) f(y)\, dy$$
$$= \iint H(\xi)\, e^{i(x-y)\xi} f(y)\, dy\, d\xi$$
$$= \int H(\xi) \left(\int e^{-iy\cdot\xi} f(y)\, dy\right) e^{ix\xi} d\xi$$
$$= \int H(\xi) \hat{f}(\xi) e^{ix\xi} d\xi.$$

Next, we shall consider seqn of good kernel on $\mathbb{R}$. Some time it is known as summability kernel (or approximation of identity

Good Kernels on $\mathbb{R}$:

A seq^n of functions $\{K_\lambda\} \subset L^1(\mathbb{R})$ is said to be "good kernels" if

(12)

(i) $\int K_\lambda(x)\,dx = 1$

(ii) $\int |K_\lambda(x)|\,dx \leq M$ as $\lambda \to \infty$.

(iii) $\lim\limits_{\lambda\to\infty} \int\limits_{|x|>\delta} |K_\lambda(x)|\,dx = 0, \ \forall \delta > 0.$

We can easily construct a seq^n of good kernel in the following way. Let $f \in L^1(\mathbb{R})$ be such that $\int_{\mathbb{R}} f(x)\,dx = 1$.

Write $K_\lambda(x) = \lambda f(\lambda x)$, $\lambda > 0$. Then

(i) $\int K_\lambda(x)\,dx = \int f(y)\,dy = 1 \quad (\because y = \lambda x)$

(ii) $\|K_\lambda\|_1 = \|f\|_1 < \infty \quad \forall \lambda > 0$.

(iii) $\int\limits_{|x|>\delta} |K_\lambda(x)|\,dx = \int\limits_{|y|>\lambda\delta} |f(y)|\,dy = \int\limits_{\mathbb{R}} \left(f - \chi_{\{|y|\leq\delta\lambda\}} f\right) dy$

since $f(x) - \chi_{\{|y|\leq\delta\lambda\}}(x) \xrightarrow{p.w} 0$ as $\lambda \to \infty$, and $|f - \chi_{\{|y|\leq\delta\lambda\}} f| \leq 2|f| \in L^1$. By DCT,

$\int\limits_{|x|>\delta} |K_\lambda(x)|\,dx \to 0$ as $\lambda \to \infty$.

Hence, $\{K_\lambda\}_{\lambda>0}$ is a family of good kernels.

Theorem: Let $f \in L^1(\mathbb{R})$ (or $f \in L^p(\mathbb{R})$, $1 \le p < \infty$). Then $\lim_{\lambda \to \infty} \| f - K_\lambda * f \|_p = 0$. If $f \in L^\infty(\mathbb{R})$ and f is ~~not~~ continuous at x. Then (13)

$$\lim_{\lambda \to \infty} f * K_\lambda(x) = f(x).$$

Proof: $|K_\lambda * f(x) - f(x)| \le \int_{\mathbb{R}} |K_\lambda(y)(f(x-y) - f(x))| \, dy$ —(1)

By Minkowski integral inequality (if $p > 1$)

$$\| K_\lambda * f - f \|_p \le \int_{\mathbb{R}} |K_\lambda(y)| \, \| \tau_y f - f \|_p \, dy.$$

For small $|y| < \delta$, $\| \tau_y f - f \|_p < \epsilon$. Hence,

$$\| K_\lambda * f - f \|_p \le \int_{|y| < \delta} |K_\lambda(y)| \cdot \epsilon + \int_{|y| > \delta} |K_\lambda(y)| \, \| \tau_y f - f \|_p \, dy$$

$$\le \epsilon M + \int_{|y| > \delta} |K_\lambda(y)| \, 2\|f\|_p \, dy$$

$$\le \epsilon M + 2\|f\|_p \, \epsilon, \ \delta > 0.$$

If $f \in L^\infty(\mathbb{R})$, f cont at x, then from (1),

$$|K_\lambda * f(x) - f(x)| \le \int_{\mathbb{R}} |K_\lambda(y)| \, |f(x-y) - f(x)| \, dy$$

For small $|y| < \delta$, $|f(x-y) - f(x)| < \epsilon$. Hence

$$|K_\lambda * f(x) - f(x)| \le \epsilon M + 2\|f\|_\infty \, \epsilon, \text{ for } \delta > 0.$$

Hence, $K_\lambda * f(x) \longrightarrow f(x)$ as $\lambda \to \infty$.

The Fejer Kernel on $\mathbb{R}$ is given by (14)

$$K_\lambda(x) = \lambda K(\lambda x), \text{ where}$$

$$K(x) = \frac{1}{2\pi}\left(\frac{\sin x/2}{x/2}\right)^2 = \int_{-1}^{1}(1-|\xi|)e^{ix\xi}\,d\xi.$$

(It can be seen by evaluating the integral)

$$\therefore K_\lambda(x) = \frac{1}{2\pi}\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)e^{i\xi x}\,d\xi.$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\left(1-\frac{|\xi|}{\lambda}\right)\chi_{[-\lambda,\lambda]}(\xi)\, e^{i\xi x}\,d\xi.$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} G_\lambda(\xi)\, e^{i\xi x}\,d\xi, \text{ where}$$

$G_\lambda(\xi) = \left(1-\frac{|\xi|}{\lambda}\right)\chi_{[-\lambda,\lambda]}(\xi)$ is compactly supported.

To show K_λ is a good Kernel, we all need to show that $\int_{\mathbb{R}} K(x)\,dx = 1$.

For this, we use the fact that Fejer Kernel for circle $F_n(x) = \frac{1}{(n+1)}\left(\frac{\sin (n+1)\, x/2}{\sin x/2}\right)^2$.

satisfies $\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\delta}^{\delta} F_n(x)\,dx = 1.$

We know that $\lim_{x\to 0}\left(\frac{\sin x/2}{x/2}\right)^2 = 1.$ For $\epsilon = 1-\left(\frac{\sin \delta}{\delta}\right)$

for some small $\epsilon > 0$, $\exists\ \delta' > 0$ s.t.

$$\left|\left(\frac{\sin x/2}{x/2}\right)^2 - 1\right| < 1-\left(\frac{\sin \delta}{\delta}\right)^2$$

i.e $\left(\frac{\sin\delta}{\delta}\right)^2 \le \left(\frac{\sin x/2}{x/2}\right)^2$

for $|x|<\delta'$ (small). Hence,

$$\frac{1}{2\pi(n+1)}\left(\frac{\sin\delta}{\delta}\right)^2\left(\frac{\sin(n+1)x/2}{\sin x/2}\right)^2.$$

$$\le \frac{1}{2\pi(n+1)}\left(\frac{\sin x/2}{x/2}\cdot\frac{\sin(n+1)x/2}{\sin x/2}\right)^2$$

$$\le \frac{1}{2\pi(n+1)}\left(\frac{\sin(n+1)x/2}{\sin x/2}\right)^2$$

let $K_n(x) = \frac{1}{2\pi(n+1)}\left(\frac{\sin(n+1)x/2}{x/2}\right)^2$. then

$$\frac{1}{2\pi}\left(\frac{\sin\delta}{\delta}\right)^2\int_{-\delta}^{\delta} F_n(x)\,dx \le \int_{-\delta}^{\delta} K_n(x)\,dx \le \frac{1}{2\pi}\int_{-\delta}^{\delta} F_n(x)\,dx.$$

Since $\lim_{n\to\infty}\int_{\delta}^{\delta} K_n(x)\,dx = \int_{-\infty}^{\infty} K(x)\,dx$, it follows

that $\left(\frac{\sin\delta}{\delta}\right)^2\cdot 1 \le \|K\|_1 \le 1$, $\forall\ \delta>0$ (small).

$\Rightarrow \|K\|_1 = 1$.

Hence, $\{K_\lambda\}_{\lambda>0}$ is a family of good kernels

let $f\in L^1(\mathbb{R})$, then by the fact that

$$f * K_\lambda(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\left(1-\frac{|\xi|}{\lambda}\right)\chi_{[-\lambda,\lambda]}(\xi)\,\hat{f}(\xi)\,e^{ix\cdot\xi}\,d\xi,$$

it follows that

$$(*)\quad f = \lim_{\lambda\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\left(1-\frac{|\xi|}{\lambda}\right)\chi_{[-\lambda,\lambda]}\,\hat{f}(\xi)\,e^{ix\cdot\xi}\,d\xi$$

in the L^1-norm.

Thus, if $\hat{f}(\xi)=0$, $\forall\ \xi \in \mathbb{R}$. Then by *

$\|f\|_1 = 0 \Rightarrow f = 0$ a.e. (16)

(Fourier uniqueness theorem).

<u>Fourier Inversion</u>:

<u>Theorem</u>: Let $f, \hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\, e^{i\xi \cdot x}\, d\xi \qquad -(1)$$

holds for almost all $x \in \mathbb{R}$.

We know that

$$(**) \quad f(x) = \lim_{\lambda \to \infty} \int_{\mathbb{R}} \chi_{[-\lambda,\lambda]}(\xi)\left(1-\frac{|\xi|}{\lambda}\right)\hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi$$

holds in L^1-norm. Hence, it follows that $\exists$ a subsequence such that $(**)$ holds. Therefore, w.l.g. we can assume $(**)$ holds a.e. Since $\left|\chi_{[-\lambda,\lambda]}(\xi)\left(1-\frac{|\xi|}{\lambda}\right)\hat{f}(\xi)\right| \le 2|\hat{f}(\xi)| \in L^1(\mathbb{R})$

and $\chi_{[-\lambda,\lambda]}(\xi)\left(1-\frac{|\xi|}{\lambda}\right)\hat{f}(\xi) \to \hat{f}(\xi)$ as $\lambda \to \infty$,

By DCT, we get

$$f(x) = \lim_{\lambda\to\infty} \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(x)\, e^{ix\cdot\xi} \quad \text{a.e.}$$

That is, if $f, \hat{f} \in L^1(\mathbb{R})$, then

$$f = (\hat{f})^{\vee} \quad \text{a.e.}$$

Notice that Fejer kernel $K_\lambda \in L^1(\mathbb{R})$ (as $\int K_\lambda(x)dx = \int K(x)dx = 1$,) and

$$K_\lambda(x) = \int_{\mathbb{R}} G_\lambda(\xi) e^{ix\xi} d\xi, \quad \text{——(2)}$$

where $G_\lambda(\xi) = \chi_{[-\lambda,\lambda]}(\xi)\left(1 - \frac{|\xi|}{\lambda}\right) \in L^1(\mathbb{R})$, In fac $\check{G}_\lambda \in L^1(\mathbb{R})$. Therefore, by inversion formula,

$$G_\lambda = (\check{G}_\lambda)^\wedge = \hat{K}_\lambda(x) \quad (\text{from } (2))$$

That is, $\hat{K}_\lambda(x) = \chi_{[-\lambda,\lambda]}(x)\left(1 - \frac{|x|}{\lambda}\right)$.

Plancherel theorem:

We know that if $f \in L^1(\mathbb{R})$, then $\hat{f} = \mathcal{F}(f)$ is a uniformly continuous function on $\mathbb{R}$. However, for $f \in L^2(\mathbb{R})$, $\hat{f}$ exists uniquely as a function in $L^2(\mathbb{R})$ and satisfies the isometry

$$\|\hat{f}\|_2 = \|f\|_2.$$

This can be seen using the fact that $\mathcal{F}$ is a conti linear function on dense set $L^1 \cap L^2$ to L^2.

Further, using Riesz-Thorin interpolation theorem, for $f \in L^p(\mathbb{R})$, $1 \le p \le 2$, $\hat{f}$ exists as function in $L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$. (This we see later). Finally, for $p > 2$, we shall see that $\hat{f}$ exists as "distribution".

That is, $\hat{f}$ defined by the relation

$$\langle \hat{f}, \varphi \rangle = \int f(x)\, \hat{\varphi}(x)\, dx, \quad \varphi \in C_c^{\infty}(\mathbb{R}).$$

Now, for $f \in L^1 \cap L^2(\mathbb{R})$, we define (18)

$$\hat{f}(\xi) = \int e^{-2\pi i x \xi} f(x)\, dx.$$

Then $f * K_\lambda(x) = \int_{\mathbb{R}} G_\lambda(\xi) \hat{f}(\xi) e^{2\pi i x \xi} d\xi$,

where $G_\lambda(\xi) = \left(1 - \frac{|\xi|}{\lambda}\right) \chi_{[-\lambda, \lambda]}(\xi)$.

Let $\check{f}(x) = \overline{f(-x)}$, and $g = f * \check{f}$. Then

$g \in L^1(\mathbb{R})$ and $\hat{g}(x) = \hat{f}(x)\overline{\hat{f}(x)} = |\hat{f}(x)|^2$.

Further, $g(x) = \int f(x-y)\, \overline{f(-y)}\, dy$

$$= \int f(x+y)\, \overline{f(y)}\, dy$$

$$= \langle f_{-x}, f \rangle.$$

As $x \mapsto f_x$ is cont on $\mathbb{R} \to L^2(\mathbb{R})$

and $\langle \cdot, \cdot \rangle$ is cont, it follows that

g is continuous & $|g(x)| \le \|f_x\|_2 \|f\|_2$

ie $|g(x)| \le \|f\|_2^2$.

Notice that $g \in L^\infty$ & g is cont.

$$g * K_\lambda(0) = \int_{\mathbb{R}} G_\lambda(\xi)\, \hat{g}(\xi)\, d\xi \to g(0), \text{ as } \lambda \to \infty$$

That is, $\lim_{\lambda \to \infty} \int G_\lambda(\xi)\, \hat{g}(\xi)\, d\xi = \|f\|_2^2 = g(0)$.

$$\lim_{\lambda\to\infty} \int G_\lambda(\xi)\,|\hat{f}(\xi)|^2 d\xi = \|f\|_2^2.$$

Since $G_\lambda(\xi) \uparrow 1$. by monotone conv. theorem it follows that

$$\int |\hat{f}(\xi)|^2 d\xi = \|f\|_2^2$$

i.e $\|\hat{f}\|_2 = \|f\|_2$, if $f \in L^1 \cap L^2$.

Let $Y = \{\hat{f} : f \in L^1 \cap L^2\}$. Then

$$\mathcal{F} : L^1 \cap L^2(\mathbb{R}) \xrightarrow{\text{onto}} Y \text{ isometry.}$$

We claim that $\overline{Y} = L^2(\mathbb{R})$. By H-B theorem, it is enough to show that $Y^\perp = \{0\}$. If $h \in Y^\perp \subset L^2$, then the fact that $G_\lambda e_x$ (when $e_x(\xi) = e^{2\pi i x\xi}$) belongs to $L^1 \cap L^2$,

$$(G_\lambda e_x)^\wedge = (G_\lambda e_x)^\vee = \tau_x G_\lambda = \tau_x K_\lambda \in Y,$$

for each $x \in \mathbb{R}$. Hence, we get

$$\langle \tau_x K_\lambda, h\rangle = 0$$

$$\Rightarrow K_\lambda * \bar{h}(x) = 0.$$

But $\|K_\lambda * \bar{h} - \bar{h}\|_2 \to 0$ as $\lambda \to \infty$

$$\Rightarrow \|\bar{h}\|_2 = 0 \Rightarrow Y^\perp = \{0\}.$$

Hence, $\mathcal{F}$ can be extended on L^2 onto L^2 with $\|\mathcal{F}(f)\|_2 = \|f\|_2$.

For this, $\mathcal{F}: L^1 \cap L^2 \subset L^2 \to Y \subset L^2$.

Let $g \in L^2(\mathbb{R})$, then $\exists\ \mathcal{F}(g_n) \in Y$ with $g_n \in L^1 \cap L^2$ s.t. $\mathcal{F}(g_n) \xrightarrow{L^2} g$ and (20)

$$\|\mathcal{F}(g_n)\|_2 = \|g_n\|_2.$$

It implies, g_n is a. C.C. in $L^1 \cap L^2$.

Hence, $\exists\ f \in L^2$ s.t. $g_n \xrightarrow{L^2} f$, & it implies $\mathcal{F}(g_n) \xrightarrow{L^2} \mathcal{F}(f)$. Then

$$\|\mathcal{F}(f)\|_2 = \|f\|_2.$$

Remark: Let $f \in L^2(\mathbb{R})$, then $\chi_{[-n,n]} f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

If we write $\varphi_n(x) = \int_{-n}^{n} e^{-2\pi i x \xi} f(\xi)\, d\xi$,

then $\|\varphi_n - \hat{f}\|_2 = \|(\chi_{[-n,n]} f)^\wedge - \hat{f}\|_2$

$$= \|\chi_{[-n,n]} f - f\|_2 \to 0.$$

Thus, $\hat{f}(\xi) = \lim \int_{-a}^{a} f(x)\, e^{-2\pi i x \xi}\, dx$ exists in the L^2-norm.

Ex. Let $H(t) = e^{-|x|}$. Show that

$$\hat{H}(x) = \int_{\mathbb{R}} H(t)\, e^{itx}\, dt = \frac{2}{1+x^2}.$$

Note that if $f \in L^2(\mathbb{R})$, then $\|\hat{f}\|_2 = \|f\|_2$.

By polarization identity, $\int f \bar{g} = \int \hat{f}\, \bar{\hat{g}}$, for $f, g \in L^2(\mathbb{R})$.

More on Convolution:

Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $f*g$ is an uniformly continuous and bounded function on $\mathbb{R}$, with $\|f*g\|_u \le \|f\|_p \|g\|_q$. In particular, if $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, then $f*g \in C_0(\mathbb{R})$.

Proof: By Holder inequality, we get

$$|f*g(x)| \le \int |f(x-y)||g(y)|dy$$
$$\le \|\tau_x f\|_p \|g\|_q$$
$$= \|f\|_p \|g\|_q$$

$\Rightarrow f*g$ is bounded. Further,

$$|\tau_x (f*g)(y) - (f*g)(y)| \le \int |\tau_x f(y-\xi) - f(y-\xi)| |g(\xi)| d\xi$$
$$\le \|\tau_x f - f\|_p \|g\|_q.$$

$$\Rightarrow \|\tau_x (f*g) - f*g\|_u \le \|\tau_x f - f\|_p \|g\|_q.$$

Since $x \to \tau_x f$ is uniformly cont on $\mathbb{R}$ to $L^1(\mathbb{R})$, it follows that $f*g$ is unif. cont on $\mathbb{R}$.

Let $1<p<\infty$, then $1<q<\infty$, since $\frac{1}{p}+\frac{1}{q}=1$. For given $\epsilon>0$, $\exists$ $f_n, g_n \in C_c^\infty(\mathbb{R})$ s.t. $\|f_n - f\|_p < \epsilon$ & $\|g_n - g\|_q < \epsilon$, (since $\overline{C_c(\mathbb{R})} = L^r(\mathbb{R})$ if $1 \le r < \infty$).

Hence,

$$\|f_n * g_n - f*g\|_\infty \le \|f_n - f\|_p \|g_n\|_q + \|f\|_p \|g_n - g\|_q.$$

Since $g_n \xrightarrow{L^q} g$, $\exists\ M_q > 0$ s.t $\|g_n\| \le M_q$.

$\therefore \|f_n * g_n - f*g\|_\infty \le \epsilon M_q + \|f\|_p \epsilon$

Thus, $f_n * g_n \longrightarrow f*g$ uniformly, but $C_0(\mathbb{R})$ is a complete space, hence $f*g \in C_0(\mathbb{R})$.

<u>Riesz Thorin interpolation theorem:</u>

Let (X, S, μ) and (Y, T, ν) be two σ-finite measure spaces. Let $p_i, q_i \in [1, \infty]$, $i = 0, 1$, and define $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$ & $\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$, where $0 \le t \le 1$. If T is a linear map from $L^{p_0}(\mu) + L^{p_1}(\mu) \to L^{q_0}(\nu) + L^{q_1}(\nu)$ such that

$$\|Tf\|_{q_i} \le M_i \|f\|_{p_i};\ i = 0, 1.$$

Then $\|Tf\|_{q_t} \le M_0^{1-t} M_1^{t} \|f\|_{p_t}$.

(For a prove, we refer to Real Analysis by G.B. Folland)

Using R-T thm we see that F.T of a function $f \in L^p(\mathbb{R})$, $1 \le p \le 2$, exists as a function in L^q, $\frac{1}{p} + \frac{1}{q} = 1$.

Hausdorff Young inequality:
let $1 \le p \le 2$. Then for $f \in L^p(\mathbb{R})$, $\hat{f} \in L^q(\mathbb{R})$,
with $\|\hat{f}\|_q \le \|f\|_p$, where $\frac{1}{p} + \frac{1}{q} = 1$.
note that if $1 \le p \le 2$, then $q \in [2, \infty]$.
Similarly, if $f \in L^p(S^1)$, $1 \le p \le 2$, then $\hat{f} \in l^q(\mathbb{Z})$,
$\frac{1}{p} + \frac{1}{q} = 1$ and $\|\hat{f}\|_{l^q} \le \|f\|_p$.

Proof: We know that $F: L^1(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$ satisfies
$$\|F(f)\|_\infty \le \|f\|_1 \quad \text{and} \quad F: L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$
with $\|F(f)\|_2 = \|f\|_2$.
$$\text{Let } \frac{1}{p_t} = \frac{1-t}{1} + \frac{t}{2} \text{ and } \frac{1}{q_t} = \frac{1-t}{\infty} + \frac{t}{2}.$$
Note that $\frac{1}{p_t} + \frac{1}{q_t} = 1$, $\frac{1}{p} + \frac{1}{q} = 1$. So we can choose
$t \in (0,1)$ s.t. $\frac{1}{q} = \frac{t}{2}$ & $\frac{1}{p} = \frac{1-t}{1} + \frac{t}{2}$.
Hence by R-T inequality, we set
$$\|F(f)\|_q \le \|f\|_p.$$
Thus, F.T is a bounded linear function from L^p to L^q.

Young's inequality: let $1 \le p, q, r \le \infty$ and
$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If $f \in L^p$ and $g \in L^q$, then
$f * g \in L^r$ and $\|f * g\|_r \le \|f\|_p \|g\|_q$.

Proof: Case I If $p=1$, $q=r$, then

$\|f*g\|_r = \|f*g\|_q \le \|f\|_1 \|g\|_q$

(by Minkowski integral inequality)

Case II If $p=\frac{q}{q-1}$, $r=\infty$ $\left(\frac{1}{p}+\frac{1}{q}=1,\ 1<p,q<\infty\right)$,

then $\|f*g\|_r = \|f*g\|_\infty \le \|f\|_p \|g\|_q$.

$(\because\ f*g \in C_0(\mathbb{R}))$.

Case III $1\le q\le\infty$, fix $g\in L^q$ and write

$T_g(f) = f*g$. Then

(i) $T_g : L^1 \longrightarrow L^q$, satisfies $\|T_g(f)\|_q \le \|g\|_q \|f\|_1$,

(ii) $T_g : L^{q'} \longrightarrow L^\infty$ satisfies

$\|T_g(f)\|_\infty \le \|f\|_{q'} \|g\|_q$, where $\frac{1}{q}+\frac{1}{q'}=1$.

For Riesz-Thorin interpolation theorem, let

$p_0=1$, $q_0=q$, $p_1=q'$, $q_1=\infty$, $M_0=\|g\|_q$

and $M_1=\|g\|_q$. Then

$\|T_g(f)\|_{q_t} \le M_0^{1-t} M_1^{t} \|f\|_{p_t}$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0}+\frac{t}{p_1} = 1-t+\frac{t}{q'} \text{ and}$$

$$\frac{1}{q_t} = \frac{1-t}{q_0}+\frac{t}{q_1} = \frac{1-t}{q}.$$

If we want $q_t=r$, then $\frac{1}{r}=\frac{1-t}{q}$.

Hence $\frac{q}{r}=1-t$, $t=1-\frac{q}{r}$. Thus $\frac{1}{p_t}=\frac{1}{p}$,

Since $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ & $\frac{1}{q}+\frac{1}{q'}=1$.

Hence $\|T_g(f)\|_r \le \|f\|_p \|g\|_q$.

notice that by Hausdorff-Young inequality, if $1 \le p \le 2$, then for $f \in L^p(\mathbb{R})$, $\hat{f} \in L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Hence by continuity we can define

$$\hat{f}(\xi) \overset{L^2}{=} \lim_{n\to\infty} \int_{-n}^{n} e^{-ix\xi} f(x)\, dx.$$

However, if $1<p<2$, we donot know how the $\hat{f}$ looks like. For example if $f \in L^1(\mathbb{R})$, then $\lim_{\lambda\to\infty} \|f * K_\lambda - f\|_1 = 0$ and

$$(*) \quad f(x) = \lim_{\lambda\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} G_\lambda(\xi) \hat{f}(\xi)\, e^{i\xi x} d\xi$$

holds in $L^1(\mathbb{R})$.

For $1<p<2$, we can generalized $(*)$. For this, we need to verify the following:

If $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, $1<p<2$, then $f * g \in L^p$ and $(f*g)^\wedge = \hat{f}\hat{g}$.

Since, $\overline{C_c^\infty(\mathbb{R})} = L^p(\mathbb{R})$, for $\epsilon > 0$, $\exists$ $g_n \in C_c^\infty(\mathbb{R})$ s.t. $\|g - g_n\|_p < \epsilon$.

note that $\hat{g}_n \in L^1(\mathbb{R})$ and

$$F(g_n * f) = F(g_n) F(f). \quad \text{--- } (**)$$

As $F: L^p \longrightarrow L^2$, is cont. linear map.

From (**), it follows that:

$$F(g*f) = F(g)F(f).$$

Now, consider $f = K_\lambda$ (Fejer kernel on $\mathbb{R}$), then $(K_\lambda * g)^\wedge = \hat{K}_\lambda \hat{g} = G_\lambda \hat{g}$, where

$$G_\lambda(\xi) = \left(1 - \frac{|\xi|}{\lambda}\right) \chi_{[-\lambda,\lambda]}(\xi).$$

Since $\hat{g} \in L^{q}(\mathbb{R})$, $q > 2$, it is easy to see that $G_\lambda \hat{g} \in L^2(\mathbb{R})$. By inversion formula,

$$K_\lambda * g(x) = \frac{1}{2\pi} \int G_\lambda(\xi) \hat{g}(\xi) e^{ix\xi} d\xi, \text{ and}$$

$K_\lambda * g \in L^2(\mathbb{R})$. Since K_λ is a good kernel and $K_\lambda * g \to g$ in $L^p(\mathbb{R})$, we can write the following result.

<u>Theorem</u>: Let $1 \le p \le 2$ and $g \in L^p(\mathbb{R})$. Then

$$g(x) = \lim_{\lambda \to \infty} \frac{1}{2\pi} \int_{\mathbb{R}} G_\lambda(\xi) \hat{f}(\xi) e^{ix\xi} d\xi$$

in $L^p(\mathbb{R})$.

<u>Cor</u>: $\{ f \in L^p : 1 \le p \le 2, \text{ supp } \hat{f} \text{ is compact} \}$ is dense in $L^p(\mathbb{R})$.

Notice that if $f, g \in L^1(\mathbb{R})$, then $F(f*g) = F(f)F(g)$, where F is the Fourier transform.

Question: Does F is unique that satisfies

$$F(f*g) = F(f)F(g)\ ?$$ (27)

Note that if we write
$$F(f) = \int f(x) e^{-it_0 x}\,dx = \hat{f}(t_0),$$
then F is a cont. linear functional on $L^1(\mathbb{R})$. We ~~that~~ shall see that such any continuous linear functional is only F.T.

<u>Riesz Theorem</u>: Let $1 \le p < \infty$ and (X, S, μ) be a σ-finite measure space. Then for every cont. linear functional T on $L^p(\mu)$, $\exists !$ $g \in L^q(\mu)$, where $\frac{1}{p}+\frac{1}{q}=1$ such that

$$T(f) = \int fg$$

Now, suppose φ is a cont linear functional on $L^1(\mathbb{R})$ with $\|\varphi\| \le 1$ and $\varphi(f*g) = \varphi(f)\varphi(g)$, $\forall f, g \in L^1(\mathbb{R})$. Then by Riesz theorem, $\exists !$ $\beta \in L^\infty(\mathbb{R})$ such that

$$\varphi(f) = \int f(x)\beta(x)\,dx.$$

Then
$$\varphi(f*g) = \int\left(\int f(x-y)\, g(y)\, dy\right)\beta(x)\,dx$$
$$= \int g(y)\,\varphi(f_y)\,dy,$$

where $f_y(x) = f(x-y)$. On the other hand,

$$\varphi(f * g) = \varphi(f)\varphi(g) = \varphi(f)\int g(y)\beta(y)\,dy.$$

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Hence $\int (\varphi(f_y) - \varphi(f)\beta(y))g(y)\,dy = 0$,

$\forall g \in L^1(\mathbb{R})$. By uniqueness in the first theorem, it follows that

(1) — $\varphi(f)\beta(y) = \varphi(f_y)$ a.e. y.

Since $y \mapsto f_y$ is cont on $\mathbb{R}$ to $L^1(\mathbb{R})$ and φ is cont on $L^1(\mathbb{R})$ to $\mathbb{C}$, it follows that RHS of (1) is continuous. Hence, we can assume $\beta(y)$ is continuous, except on a set of measure zero.

By replacing $y \mapsto x+y$, we get.

$$\varphi(f)\beta(x+y) = \varphi(f_{x+y}) = \varphi((f_x)_y)$$
$$= \varphi(f_x)\beta(y)$$
$$= \varphi(f)\beta(x)\beta(y)$$

Since φ is non-zero, we can find $f \in L^1(\mathbb{R})$ such that $\varphi(f) \neq 0$. Hence

$$\beta(x+y) = \beta(x)\beta(y)$$

Hence, by a similar argument as on page 2,

$\exists\, t_0 \in \mathbb{R}$, s.t. $\beta(x) = e^{-it_0 x}$

Hence $\varphi(f) = \int f(x) e^{-it_0 x}\,dx = \hat{f}(t_0)$.

Notice that for every φ (except $\varphi = 0$), $\exists!$ $t \in \mathbb{R}$ s.t $\varphi(f) = \hat{f}(t)$, because of $s \neq t$, then $\exists$ $f \in L^1(\mathbb{R})$ s.t $\hat{f}(t) \neq \hat{f}(s)$.

Poisson Summation formula:

For $f \in L^1(\mathbb{R})$, write $\varphi(t) = 2\pi \sum_{j=-\infty}^{\infty} f(t+2\pi j)$.

Then φ is a 2π-periodic function on $\mathbb{R}$.

Let $\varphi_n(t) = 2\pi \sum_{j=-n}^{n} f(t+2\pi j)$. Then $\varphi_n \to \varphi$ p.w.

and $\left|2\pi \sum_{j=-n}^{n} f(t+2\pi j)\right| \leq 2\pi \sum_{j=-n}^{n} |f(t+2\pi j)|$

$\leq 2\pi \sum_{j=-\infty}^{\infty} |f(t+2\pi j)| = g(t)$.

Since $f \in L^1(\mathbb{R})$, $g \in L^1(S^1)$. By DCT, we get

$\lim \int_{S^1} |\varphi_n(t)| = \int_{S^1} |\varphi| dt$. But $\frac{1}{2\pi}\int_{S^1} |\varphi_n(t)| \leq \int_{\mathbb{R}} |f|$,

hence $\frac{1}{2\pi}\int_{S^1} |\varphi| dt \leq \int_{\mathbb{R}} |f| dt$

$\Rightarrow \|\varphi\|_{L^1(S^1)} \leq \|f\|_{L^1(\mathbb{R})}$

Let $f \in L^1(\mathbb{R})$, then

(*) $\sum_{j=-\infty}^{\infty} f(t+2\pi j) = \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijt}$, for a.e. $t \in \mathbb{R}$.

Proof: To prove this identity, it is enough to prove that fourier coeff. of LHS $= \hat{f}(j)$.

Now, $\frac{1}{2\pi}\int_0^{2\pi}\sum_{j=-\infty}^{\infty} f(t+2\pi j)e^{-int}\,dt = \sum_{j=-\infty}^{\infty}\int_0^{2\pi} f(t+2\pi j)e^{-int}\,dt$ (30)

(by Beppo-Levi theorem)

$$= \int_{\mathbb{R}} f(t)e^{-int}\,dt = \hat{f}(n).$$

Hence, by uniqueness of the Fourier series, we get the required identity.

Ex. Prove that $\sum \frac{1}{(n+x)^2} = \frac{\pi^2}{(\sin \pi x)^2}$

(Hint: Take $g(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{o.w.} \end{cases}$ in the Poisson summation formula (*)).

<u>L^p-derivative of a function on $\mathbb{R}$:</u>

For $h \in \mathbb{R}$ & f a function on $\mathbb{R}$, define

$$\Delta_h f(x) = \frac{f(x+h)-f(x)}{h}.$$

Def^n: A function $f \in L^p(\mathbb{R})$ is said to be differentiable in L^p-sense if $\exists\, g \in L^p(\mathbb{R})$ s.t. $\lim_{|h|\to 0} \|\Delta_h f - g\|_p = 0$.

<u>Lemma</u>: Let $1 \le p, q \le \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Suppose $f \in L^p$ has derivative f' in L^p-sense then $(f*g)'$ exists in the ordinary sense when $g \in L^q$ and

$$(f*g)' = f'*g$$

Proof: We know that $f*g$ is continuous and $f' \in L^p$. therefore $f'*g$ is also continuous.

Thus;

$$|\Delta_h(f*g)(x) - f'*g(x)| = |(\Delta_h f - f')*g(x)|$$
$$\leq \|\Delta_h f - f'\|_p \|g\|_q$$
$$\to 0 \text{ as } |h| \to 0.$$

Hence $(f*g)' = f'*g$.

Theorem: Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Then f has derivative in L^p-sense iff f is absolutely continuous on each bounded interval $[a,b]$ (except on a set of measure zero) and its pointwise derivative $f' \in L^p(\mathbb{R})$.

To prove this we need a fact that $AC[a,b]$ is a complete space under the norm: $\|f\|_{AC} = |f(a)| + \int_a^b |f'(t)|dt$.

We know that $f \in AC[a,b]$ iff f' exists a.e, $f' \in L'[a,b]$ and $f(x) = f(a) + \int_a^x f'(t)dt$.

Hence $\|f\|_{AC} < \infty$ and $\|f\|_{AC} = 0$

$\Rightarrow f(a) = 0$, $f'(t) = 0$ a.e. $\Rightarrow f(t) = f(a) = C$ $\because$ $f' = 0$ a.e. $\Rightarrow f$ is const, a non-trivial result (Referred to Royden book).

Hence, $(AC[a,b], \|\cdot\|_{AC})$ is a n.l.s. Pf

f_n is a Cauchy seq^n, then $f_n(a)$ &

f_n' are Cauchy sequences in $\mathbb{C}$ & $L'[a,b]$.

respectively. Let $f_n(a) \to f_a$ & $f_n' \to g$ in L'. Write

$$f(x) = f_a + \int_a^x g(t)\,dt$$

Then f is abs. cont and

$$\|f_n - f\|_{AC} \leq |f_n(a) - f_a| + \int_a^b |g(t) - f_n'(t)|\,dt$$

Hence, $f_n \to f \in AC[a,b]$.

<u>Proof of the theorem:</u>

For simplicity, consider $p=1$ & $r=\infty$.

Suppose f has L'-derivative (a derivative in L'-sense). Then $\exists\ g \in L'(\mathbb{R})$ such that $\lim_{|h|\to 0} \|\Delta_h f - g\|_1 = 0$. By the previous lemma, $(f * K_\lambda)'$ exists ordinarily and satisfies

$$(f * K_\lambda)' = g * K_\lambda.$$

Note that $(f*K_\lambda)'$ and $f*K_\lambda$ are bounded continuous functions on $\mathbb{R}$. Hence by MVT,

$$f*K_\lambda \in AC[a,b],\ \forall\ a,b \in \mathbb{R}.$$

That is, (1) $f * K_\lambda(x) = f * K_\lambda(x_0) + \int_{x_0}^{x} (f * K_\lambda)'(t)\,dt,$

for some $x_0 \in [a,b]$. Since $f * K_\lambda \xrightarrow{L^1} f$, it follows that $f * K_\lambda(x) \longrightarrow f(x)$ a.e.

(as a subsequence of $f * K_\lambda$). Hence, we can choose $x_0 \in [a,b]$.

As $(f * K_\lambda)' = g * K_\lambda \longrightarrow g$ in L^1, we can take limit in (1) and hence

$$f(x) = f(x_0) + \int_{x_0}^{x} g(t)\,dt \quad \text{a.e. } x \in \mathbb{R}.$$

This implies, $f' = g$ a.e. on $\mathbb{R}$, and $f' = g \in L^1(\mathbb{R})$.

Conversely, suppose $f \in AC[a,b]$, $\forall\, a,b \in \mathbb{R}$ and pointwise derivative f' exists a.e. belongs to $L^1(\mathbb{R})$. Then

$$\frac{f(x+h) - f(x)}{h} - f'(x) = \frac{1}{h} \int_0^h \left(f'(x+t) - f'(x) \right) dt$$

(Since $f \in AC[a,b]$ etc)

Since $f' \in L^1(\mathbb{R})$, by Minkowski integral inequality, it follows that

$$\|\Delta_h f - f'\|_1 \le \frac{1}{|h|}\int_0^{|h|} \|\tau_t f' - f'\|_1\, dt \qquad (34)$$

$$< \|\tau_t f' - f'\|_1 < \epsilon$$

while $|h| < \delta$, as $|t| < |h| < \delta$.

Thus, f' is the L^1-derivative of f.

If $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $L^p(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$.

Hence, all the above calculations make sense, and same conclusion is followed by Minkowski integral inequality.

C^∞-form of Urysohn lemma:

Let K be a cpt set that contained in an open set $O \subset \mathbb{R}$. Then $\exists\, f \in C_c^\infty(\mathbb{R})$ s.t. $0 \le f \le 1$, $f|_K = 1$ and supp $f \subset O$.

Proof: Let $\delta = d(K, O^c)$. Then $\delta > 0$, and let $V = \{x : d(x, K) < \delta/3\}$. Suppose $\varphi \in C_c^\infty(\mathbb{R})$ s.t. $\int \varphi = 1$, $\varphi(x) = 0$ if $|x| > \delta/3$.

Write $f = \chi_V * \varphi$. Then $f|_K = 1$, $0 \le f \le 1$, and $\operatorname{supp}(f) \subset \{x : d(x, K) < 2\delta/3\} \subset O$ and $f \in C_c^\infty(\mathbb{R})$. (Note that φ can be constructed by taking $\varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \ge 1. \end{cases}$