# Residue theorem and its applications

# Residue

**Question:** Let $\gamma$ be a simple closed contour in a simply connected domain $D$, and suppose $z_0$ doesn't lie on $\gamma$. If $f$ has singularity only at $z_0$, what could be the value for $\int_\gamma f(z)dz$?

- **Recall: Laurent's Theorem:** Let $0 \le r < R$, and $f$ be analytic in the annulus $\mathrm{Ann}(z_0, r, R)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where the convergence is absolute and uniform in $\overline{\mathrm{Ann}(z_0, r_1, R_1)}$ if $r < r_1 < R_1 < R$. The coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

for any $r < s < R$.

Moreover, this series is unique.

- Put $n = -1$. The **answer to the above question** is $2\pi i\, a_{-1}$.

# Residue

**Definition:** Let $z = z_0$ be an isolated singularity of $f$, and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be the Laurent series expansion of $f$ about $z_0$, then the residue of $f$ at $z_0$ is the coefficient $a_{-1}$.

- We denote $\text{Res}(f, z_0) = a_{-1}$.
- If $f$ has a removable singularity at $z = z_0$, then $\text{Res}(f, z_0) = 0$.
- If $f(z) = \dfrac{\sin z}{z}$, then $\text{Res}(f, 0) = 0$.
- Let $f(z) = e^{\frac{2}{z}}$ and $g(z) = e^{\frac{1}{z^2}}$. Then $\text{Res}(f, 0) = 2$, and $\text{Res}(g, 0) = 0$.

# Residue at poles

- If $f$ has a simple pole (pole of order one) at $z = z_0$, then

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

In this case, we have $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$.

- If $f$ has a pole of order $m \, (m > 1)$ at $z = z_0$, then $f(z) = \frac{g(z)}{(z-z_0)^m}$, where $g(z_0) \neq 0$. Since $g$ is analytic at $z_0$, we can write

$$g(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots .$$

Hence,

$$f(z) = \frac{g(z)}{(z - z_0)^m} = \frac{b_0}{(z - z_0)^m} + \frac{b_1}{(z - z_0)^{m-1}} + \cdots + \frac{b_{m-1}}{(z - z_0)} + \sum_{k=0}^{\infty} b_{m+k} (z - z_0)^k.$$

This implies that $\text{Res}(f, z_0) = b_{m-1} = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(z)$.

# Residue at poles

- Thus, we summarize that if $f$ has a pole of order $m\,(>1)$ at $z = z_0$, then

$$\text{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}[f(z)(z-z_0)^m].$$

- Let $f(z) = \dfrac{z}{(z-1)(z+1)^2}$, then

$$\text{Res}(f, 1) = \lim_{z\to 1} f(z)(z-1) = \frac{1}{4}$$

and

$$\text{Res}(f, -1) = \lim_{z\to -1} \frac{d}{dz}[f(z)(z+1)^2] = -\frac{1}{4}.$$

# Cauchy's residue theorem

**Cauchy residue theorem:** Let $f$ be analytic in the interior enclosed by a simple closed contour $\gamma$ (positively orientated) except for finitely many isolated singularities $a_1, a_2, \ldots, a_n$ in the interior. If the points $a_1, a_2, \ldots, a_n$ do not lie on $\gamma$, then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, a_k).$$

Proof. Apply **Cauchy's theorem for multiply connected domain**.

- $\int_\gamma f(z)dz = (2\pi i)\times$ sum of the residues of $f$ at singular points that are in the interior enclosed by $\gamma$.
- $\displaystyle\int_{|z|=1} \frac{1}{z(z-2)}dz = 2\pi i \times \operatorname{Res}(f, 0)$. (The point $z = 2$ does not lie in the interior enclosed by the unit circle.)

# Argument Principele

**Definition:** A function $f$ is said to be **meromorphic** in a domain $D$ if $f$ is analytic throughout on $D$, except possibly on the poles of $f$ in $D$.

- Suppose $f$ is meromorphic in the interior enclosed by a closed contour $\gamma$, and analytic on $\gamma$, without zero on $\gamma$. Let $\Gamma = f(\gamma)$, then $\Gamma$ is a closed contour (not necessarily be simple).
- As $z$ traverses $\gamma$ in the positive direction, its image by $w = f(z)$ traverses $\Gamma$ in a particular direction (may be different than $\gamma$) that determines the orientation of $\Gamma$.
- Fix $f(z_0) = w_0 \in \Gamma$. Let $\phi_0 = \arg w_0$. Take $w \in \Gamma$ and run $\arg w$ continuously, starting with the value $\phi_0$.
- When $w$ returns to the point $w_0$ (in this case, $z$ traverses from $z_0$ to $z_0$), $\arg w$ assumes a particular value, say $\phi_1$.
- The change in $\arg w$ (independent of the point $w_0$) is $\phi_1 - \phi_0$, which is an integral multiple of $2\pi$.
- The integer $\frac{1}{2\pi}(\phi_1 - \phi_0)$ represents orientation, and the number of times the point $w$ rotates around the origin is called the **winding number**.

# Argument principle

**Question:** Can we determine the winding number by counting the zeros and poles of $f$ lying in the interior enclosed by a closed contour $\gamma$?

The argument principle gives an answer.

**Argument principle:** Suppose a function $f(z)$ is meromorphic in the interior enclosed by a positively oriented simple closed contour $\gamma$ such that

- $f(z)$ is analytic on $\gamma$ and $f(z) \neq 0$ on $\gamma$,
- $Z =$ Number of **zeros** of $f$ counted according to multiplicity, and
- $P =$ Number of **poles** of $f$ counted according to multiplicity.

Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = Z - P.$$

# Argument principle

The contour integral $\int_\gamma \frac{f'(z)}{f(z)}\,dz$ can be interpreted in the following (informal) ways:

- as the **total change in the argument** of $f(z)$ as $z$ travels on $\gamma$, exhibiting the name of the theorem. Suppose (!)

$$\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)}$$

  holds on $\gamma$, then the integration of $\frac{f'(z)}{f(z)}$ over $\gamma$ gives

$$\log f|_\gamma = [\overset{0}{\cancel{\ln|f(z)|}} + i\arg f(z)]|_\gamma .$$

- as $2\pi i$ times the **winding number** of the path $f(\gamma)$ around the origin. By substituting $w = f(z)$, we get

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_{f(\gamma)} \frac{1}{w}\,dw.$$

# Argument principle

**Proof.** If $f$ is analytic and has a zero of order $m\,(m > 1)$ at $z = a$, then $f(z) = (z-a)^m g(z)$, where $g(a) \neq 0$. Hence, $\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$. By residue theorem, we get

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\, m.$$

If $f$ has a pole of order $n\,(> 1)$ at $z = a$, then $f(z) = (z-a)^{-n} g(z)$, where $g(a) \neq 0$. Hence, $\frac{f'(z)}{f(z)} = \frac{-n}{z-a} + \frac{g'(z)}{g(z)}$. Again by residue theorem, we get

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(-n).$$

Combining the above two results, we obtain

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(Z - P).$$

- Evaluate: $\displaystyle\int_{|z-\frac{\pi}{2}|=1} \tan z\, dz$.
- Evaluate: $\displaystyle\int_{|z|=1} \frac{dz}{\sin z}$. [Hint. $f(z) = \tan(z/2)$]

# Rouché's theorem

**Theorem: (Rouché's theorem)** Let $f$ and $g$ be two analytic functions in the interior enclosed by a simple closed contour $\gamma$ and on $\gamma$ such that $|g(z)| < |f(z)|$ holds for each point $z$ on $\gamma$. Then $f(z)$ and $f(z) + g(z)$ have same number of zeros, counted according to their multiplicity in the interior enclosed by $\gamma$.

**Proof:** Since $|g(z)| < |f(z)|$ holds on $\gamma$, if $f(z) = 0$ for some $z$ on $\gamma$, then $|g(z)| < 0$, which is impossible. Hence $f$ will never vanish on $\gamma$. Now, $f + g = f\left(1 + \frac{g}{f}\right)$ on $\gamma$. Therefore, $\arg(f + g) = \arg(f) + \arg\left(1 + \frac{g}{f}\right)$. Notice that $\left|\frac{g}{f}\right| < 1$, we conclude that $1 + \frac{g}{f}$ lies in the RHP. Also, change in the argument of $1 + \frac{g}{f}$ around $\gamma$ is zero. Thus, $\arg(f + g) = \arg(f)$. By argument principle, $f(z)$ and $f(z) + g(z)$ have same number of zeros, counted according to their multiplicity.

**Example:** Determine the number of zeros of the equation $z^7 - 4z^3 + z - 1 = 0$ in the unit disc $|z| < 1$.
Take $f(z) = -4z^3$; $g(z) = z^7 + z - 1$. Then $|f(z)| = 4$ and $|g(z)| \leq 3$ when $|z| = 1$. Since $f$ has three zeros inside $|z| = 1$, by $Rouch\acute{e}'s$ theorem, the equation $z^7 - 4z^3 + z - 1 = 0$ has three zeros in the unit disc $|z| < 1$.