MA642: Real Analysis -1

(Assignment 4: Functions of several variables)

January - April, 2023

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There exists a one-one continuous function from $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ onto $\mathbb{R}^2$.
   (b) There exists a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable only at $(1, 0)$.
   (c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f_x(0, 0) = 0$. Then there exists some $\delta > 0$ such that $f(x, 0)$ is continuous on $(-\delta, \delta)$.
   (d) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with $f(0, 0) = (1, 1)$ and $[f'(0, 0)] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then there cannot exist a differentiable function $g : \mathbb{R}^2 \to \mathbb{R}^2$ with $g(1, 1) = (0, 0)$ and $(f \circ g)(x, y) = (y, x)$ for all $(x, y) \in \mathbb{R}^2$.
   (e) A continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ cannot be one-one and onto if $\det[f'(x, y)] = 0$ for some $(x, y) \in \mathbb{R}^2$.
   (f) The equation $\sin(xyz) = z$ defines x implicitly as a differentiable function of y and z locally around the point $(x, y, z) = (\frac{\pi}{2}, 1, 1)$.

2. Let Ω be an open subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^m$ and $g : \Omega \to \mathbb{R}^m$ be continuous at $\mathbf{x}_0 \in \Omega$. If for each $\varepsilon > 0$, there exist $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0)$ such that $f(\mathbf{x}) = g(\mathbf{y})$, then show that $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

3. Let $A(\neq \emptyset) \subset \mathbb{R}^n$ be such that every continuous function $f : A \to \mathbb{R}$ is bounded. Show that A is a closed and bounded subset of $\mathbb{R}^n$.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and let $f(\mathbf{x}) = T(\mathbf{x}) \cdot \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Find $f'(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$.

5. Examine the differentiability of f at $\mathbf{0}$, where
   (a) $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \le \|\mathbf{x}\|_2^2$ for all $\mathbf{x} \in \mathbb{R}^n$.
   (b) $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$.
   (c) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

6. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in \Omega$, let $f(\mathbf{x}_0) = 0$ and let $g : \Omega \to \mathbb{R}$ be continuous at $\mathbf{x}_0$. Prove that $fg : \Omega \to \mathbb{R}$, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in \Omega$, is differentiable at $\mathbf{x}_0$.

7. Let Ω be a nonempty open subset of $\mathbb{R}^n$ and let $g : \Omega \to \mathbb{R}^n$ be continuous at $\mathbf{x}_0 \in \Omega$. If $f : \Omega \to \mathbb{R}$ is such that $f(\mathbf{x}) - f(\mathbf{x}_0) = g(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0)$ for all $\mathbf{x} \in \Omega$, then show that f is differentiable at $\mathbf{x}_0$.

8. The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0, 0)$ in the directions of $(1, 2)$ and $(2, 1)$ are 1 and 2 respectively. Find $f_x(0, 0)$ and $f_y(0, 0)$.

9. Let $A \in GL(\mathbb{R}^n)$ and $\alpha \ge 2$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|f(x)\| \le k\|x\|^\alpha$, for some $k > 0$. Prove/disprove that the map $g = f + A$ is continuously differentiable at $\mathbf{0}$ and g is invertible in the neighborhood of $\mathbf{0}$.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable such that $f(1, 1) = 1$, $f_x(1, 1) = 2$ and $f_y(1, 1) = 5$. If $g(x) = f(x, f(x, x))$ for all $x \in \mathbb{R}$, determine $g'(1)$.

11. Prove that a differentiable function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^m$ is homogeneous of degree $\alpha \in \mathbb{R}$ (*i.e.* $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and for all $t > 0$) iff $f'(\mathbf{x})(\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

12. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is satisfying $f(rx) = r^{\frac{3}{2}} f(x)$ for all $(x, r) \in \mathbb{R}^n \times (0, \infty)$. Whether f is differentiable at $\mathbf{0}$?

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $f_x(a, b) = f_y(a, b)$ for all $(a, b) \in \mathbb{R}^2$ and $f(a, 0) > 0$ for all $a \in \mathbb{R}$. Show that $f(a, b) > 0$ for all $(a, b) \in \mathbb{R}^2$.

14. Let Ω be an open subset of $\mathbb{R}^n$ such that $\mathbf{a}, \mathbf{b} \in \Omega$ and $S = \{(1 - t)\mathbf{a} + t\mathbf{b} : t \in [0, 1]\} \subset \Omega$. If $f : \Omega \to \mathbb{R}^m$ is differentiable at each point of S, then show that there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(\mathbf{b}) - f(\mathbf{a}) = L(\mathbf{b} - \mathbf{a})$.

15. Let $f(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Show that there exist open sets U and V in $\mathbb{R}^2$ containing $(0, 1)$ and $(2, 0)$ respectively such that $f : U \to V$ is one-one and onto.

16. Let $f(x, y) = (3x - y^2, 2x + y, xy + y^3)$ and $g(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Examine whether $(f \circ g^{-1})'(2, 0)$ exists (with a meaningful interpretation of g^{-1}) and find $(f \circ g^{-1})'(2, 0)$ if it exists.

17. For $n \geq 2$, let $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\}$ and let $f(\mathbf{x}) = \|\mathbf{x}\|_2^2 \mathbf{x}$ for all $\mathbf{x} \in B$. Show that $f : B \to B$ is differentiable and invertible but that $f^{-1} : B \to B$ is not differentiable at $\mathbf{0}$.

18. Using implicit function theorem, show that the system of equations
$$x^3(y^3 + z^3) = 0,$$
$$(x - y)^3 - z^2 = 7,$$
can be solved locally near the point $(1, -1, 1)$ for y and z as a differentiable function of x.

19. Using implicit function theorem, show that in a neighbourhood of any point $(x_0, y_0, u_0, v_0) \in \mathbb{R}^4$ which satisfies the equations
$$x - e^u \cos v = 0,$$
$$v - e^y \sin x = 0,$$
there exists a unique solution $(u, v) = \varphi(x, y)$ satisfying $\det[\varphi'(x, y)] = v/x$.

20. Show that around the point $(0, 1, 1)$, the equation $xy - z \log y + e^{xz} = 1$ can be solved locally as $y = f(x, z)$ but cannot be solved locally as $z = g(x, y)$.

21. Find the 3rd order Taylor polynomial of $f(x, y, z) = x^2 y + z$ about the point $(1, 2, 1)$.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Show that f is not one-one.