Assignment 3

1. (a) If Λ' is a compactly supported distribution, does it imply Λ is also a compactly supported distribution ?

   (b) Is it necessary that every compactly supported distribution is of finite order ?

2. Suppose f is a continuous function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} f\varphi = 0$, whenever $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Show that $f = 0$.

3. Let $\Lambda = \Lambda_f$, where f is a continuous function on $\mathbb{R}^n$. Show that $\operatorname{supp}\Lambda_f = \operatorname{supp} f$. Does it hold true for locally integrable functions ?

4. Show that there exists $\psi \in \mathcal{D}(\mathbb{R})$ such that $\varphi = \psi^{(k)}$ (the k derivative) if and only if $\int_{\mathbb{R}} p(x)\varphi(x)dx = 0$ for each polynomial p of degree at most $k-1$.

5. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ satisfies $\Lambda' = 0$, then show that $\Lambda = \Lambda_c$, for some constant c.

6. Show that every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ can be expressed as $\varphi = \psi' + c\varphi_o$, where φ_o is fixed test function in $\mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi_o \neq 0$.

7. Show that every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ can be expressed as $\varphi = x\,\psi + c\varphi_o$, where φ_o is fixed test function in $\mathcal{D}(\mathbb{R})$ with $\varphi_o(0) \neq 0$. For $\Lambda \in \mathcal{D}'(\mathbb{R})$, deduce that $x\,\Lambda = 0$ implies $\Lambda = c\delta_o$.

8. Find all those $f \in C^\infty(\mathbb{R})$ such that $f\delta_o' = 0$.

9. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ is compactly supported then show that Λ' is also compactly supported.

10. Show that $\langle \Lambda, \varphi \rangle = \sum\limits_{n=1}^{\infty} \varphi^{(n)}(n)$ defines a distribution on $\mathbb{R}$. Is Λ compactly supported ?

11. Let $H = \chi_{(-\infty,0)}$ and h_n be sequence of differentiable functions such that $h_n \to H$ in $\mathcal{D}'(\mathbb{R})$. Show that $h_n' \to \delta_o$ in $\mathcal{D}'(\mathbb{R})$. Does the conclusion remains same if $H = \chi_{(-\infty,0]}$?

12. Let $\Lambda_n \in \mathcal{D}'(\mathbb{R})$ be defined by $\langle \Lambda_n, \varphi \rangle = n\left(\varphi(\frac{1}{n}) - \varphi(-\frac{1}{n})\right)$, What distribution is $\lim \Lambda_n$?

13. For $a > 0$, define
$$\langle \Lambda_a, \varphi \rangle = \left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) \frac{\varphi(x)}{|x|} dx + \int_{-a}^{a} \frac{\varphi(x) - \varphi(0)}{|x|} dx.$$
Show that Λ is a distribution on $\mathcal{D}(\mathbb{R})$. Find the $\lim\limits_{a\to 0} \Lambda_a$ in $\mathcal{D}'(\mathbb{R})$. What is the distributional derivative of $\lim\limits_{a\to 0} \Lambda_a$?

14. For $\Lambda \in \mathcal{D}'(\mathbb{R})$, define $\langle G, \varphi \rangle = \int_{\mathbb{R}} \langle \Lambda, \varphi_y \rangle dy$, where for $\varphi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi_y(x) = \varphi(x,y)$. Show that $G \in \mathcal{D}'(\mathbb{R}^2)$.

15. Let $\Lambda_i \in \mathcal{D}'(\mathbb{R})$, $i = 1,2$ be such that $\langle \Lambda_1, \varphi \rangle = 0$ if and only if $\langle \Lambda_2, \varphi \rangle = 0$. Show that $\langle \Lambda_1, \varphi \rangle = c\langle \Lambda_2, \varphi \rangle$.

16. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ be such that $\Lambda^k = 0$, then show that Λ is a polynomial of degree at most $k-1$.

17. Let $\Omega = (0,\infty)$. Define $\langle \Lambda, \varphi \rangle = \sum\limits_{n=1}^{\infty} \varphi^{(n)}(\frac{1}{n})$, where $\varphi \in \mathcal{D}(\Omega)$. Show that Λ is a distribution of infinite order. Further, show that Λ cannot be extended to a distribution on $\mathbb{R}$.

18. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ is of order N, then show that $\Lambda = f^{(N+2)}$ in $\mathcal{D}'(\mathbb{R})$ for some continuous function f. If $\Lambda = \delta_o$, then what are possibilities for f ?

19. For $k \in \mathbb{N}$, define $f_k = k\chi_{\left(\frac{1}{k}, \frac{2}{k}\right)}$. Show that $f_k \to \delta_o$ in $\mathcal{D}'(\mathbb{R})$. Further, show that $f_k^2(x) \to 0$ point-wise but f_k^2 does not converge in the sense of distribution.

20. Let
$$f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ x^2 + 2x & \text{if } 1 \leq x \leq 2, \\ 2x & \text{if } x \geq 2. \end{cases}$$
Find the distributional derivative of f.

21. Let
$$f(t) = \begin{cases} e^{-t} & \text{if } t > 0, \\ -e^{t} & \text{if } t < 0. \end{cases}$$
Show that $f'' = 2\delta_o' + f$. Deduce that the Fourier transform of f is given by $\hat{f}(x) = -\frac{2i\,x}{1+x^2}$.

22. If $H = \chi_{-\infty,0)}$, then show that

$$(a)\ \ H * \varphi(x) = \int_{-\infty}^{x} \varphi(t)dt \qquad (b)\ \ \delta_o' * H = \delta_o \qquad (c)\ \ 1 * \delta_o' = 0$$
$$(d)\ \ 1 * (\delta_o' * H) = 1 * \delta_o = 1 \qquad (e)\ \ (1 * \delta_o') * H = 0 * H = 0.$$

23. Let $\{x_k\}$ be sequence of real numbers such that $\lim |x_k| = \infty$. Show that $\delta_{(x - x_k)} \to 0$ in the sense of distribution.