Assignment 2

1. For $1 \le p < \infty$, let $f \in L^p(\mathbb{R})$. Define $F(x) = \int_x^{x+1} f(t)dt$. Show that $F \in C_o(\mathbb{R})$. Does the above conclusion true if $f \in L^\infty(\mathbb{R})$?

2. Let $1 \le p < \infty$. For $f \in L^p(\mathbb{R})$ and $h \in \mathbb{R}$, define $\Delta_h f(x) = \frac{f(x+h)-f(x)}{h}$. Show that there exists $g \in L^p(\mathbb{R})$ such that $\lim\limits_{h\to 0} \|\Delta_h f - g\|_p = 0$ if and only if f is absolutely continuous on every bounded interval in $\mathbb{R}$ (expect on a null set) and its point-wise derivative $f' \in L^p(\mathbb{R})$. Does the above conclusion holds true for $f \in L^\infty(\mathbb{R})$?

3. If $f \in L^1(\mathbb{R})$, show that $2\hat{f}(\xi) = \int_{\mathbb{R}}[f(x) - f(x - \frac{\pi}{\xi})]e^{-i\xi x}dx$. Further, use this identity to prove the Riemann-Lebesgue lemma.

4. Let $f, g \in L^1(\mathbb{R})$. Show that $\int_{\mathbb{R}} f(y)\hat{g}(y)dy = \int_{\mathbb{R}} \hat{f}(\xi)g(\xi)d\xi$. If $\hat{f} \in L^1(\mathbb{R})$, using this identity, deduce the Fourier inversion formula for f.

5. For $n \in \mathbb{N}$, define a function f on $\mathbb{R}$ by $f(x) = \frac{x^n}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$. Show that $\hat{f}(\xi) = P_n(\xi)e^{\frac{-\xi^2}{2}}$, where P_n is a polynomial of degree n.

6. A continuous function f on $\mathbb{R}$ is said to be of **moderate decrease** if there exists $A > 0$ such that $|f(x)| \le \frac{A}{1+x^2}$. Suppose f is of moderate decrease and satisfying $\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}dy = 0$ for all $x \in \mathbb{R}$, then show that$f = 0$.

7. Let f be a function of moderate decrease on $\mathbb{R}$. Define $f * K_\lambda(x) = \frac{1}{2\pi}\int_\lambda^\lambda (1 - \frac{|\xi|}{\lambda})\hat{f}(\xi)d\xi$. Prove that $f * K_\lambda$ converges uniformly to f.

8. Let $\{k_\lambda\}$ be a family of good kernels in $L^1(\mathbb{R})$. If $f \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, then show that $f * k_\lambda$ converges uniformly to f on every compact subset of $\mathbb{R}$.

9. Let $1 \le p \le 2$. Show that the space $\{f \in L^p(\mathbb{R}) : \text{supp}\hat{f} \text{ is compact}\}$ is dense in $L^p(\mathbb{R})$.

10. Let $X = \{\hat{f} : f \in L^1(\mathbb{R})\}$. Show that X is dense in $C_o(\mathbb{R})$.

11. Let f be twice continuously differentiable and compactly supported function on $\mathbb{R}$. Show that there exists $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\hat{g} = f$.

12. For $f \in L^2(\mathbb{R})$, define $\tau_x f(y) = f(y - x)$. Show that $X = \{\tau_x f : x \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$ if and only if $\hat{f}(\xi) \ne 0$ for almost all ξ.

13. Let $f \in L^1(\mathbb{R})$ be a compactly supported function. Show that $\hat{f}$ is a real analytic function on $\mathbb{R}$. Does $\hat{f} \in L^1(\mathbb{R})$? What happen if $f \in C_c^2(\mathbb{R})$?

14. Let $f \in L^1(\mathbb{R})$ and $f \ge 0$. Show that $\|\hat{f}\|_\infty = \hat{f}(0) = \|f\|_1$.

15. Let $f \in L^1(\mathbb{R})$ be continuous at $x = 0$. If $\hat{f}(\xi) \ge 0$, for all $\xi \in \mathbb{R}$, then show that $\hat{f} \in L^1(\mathbb{R})$ and $f(0) = \int_{\mathbb{R}} \hat{f}(\xi)d\xi$.

16. For $n \in \mathbb{N}$, write $g_n = \chi_{[-1,1]} * \chi_{[-n,n]}$. Show that g_n is the Fourier transform of $f_n \in L^1$, where $f_n(x) = \dfrac{\sin x \sin nx}{\pi^2 x^2}$. Further, by proving $\|f_n\|_1 \to \infty$, conclude that the mapping $f \to \hat{f}$ maps $L^1(\mathbb{R})$ into a proper subspace of $C_o(\mathbb{R})$.

17. For $f \in L^1(\mathbb{R})$, define $f_\lambda(x) = \lambda f(\lambda x)$. If $\varphi_\lambda(t) = 2\pi \sum\limits_{j=-\infty}^{\infty} f_\lambda(t + 2\pi j)$, then show that $\lim\limits_{\lambda\to\infty} \|\varphi_\lambda\|_{L^1(S^1)} = \|f\|_{L^1(\mathbb{R})}$.