Engineering Optimization

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Are you using optimization?

The word “optimization” may be very familiar or may be quite new to you.

……. but whether you know about optimization or not, you are using optimization in many occasions of your day to day life …….

…………….Examples……………….

Optimization in real life

Newspaper hawker

Cooking

Forensic artist

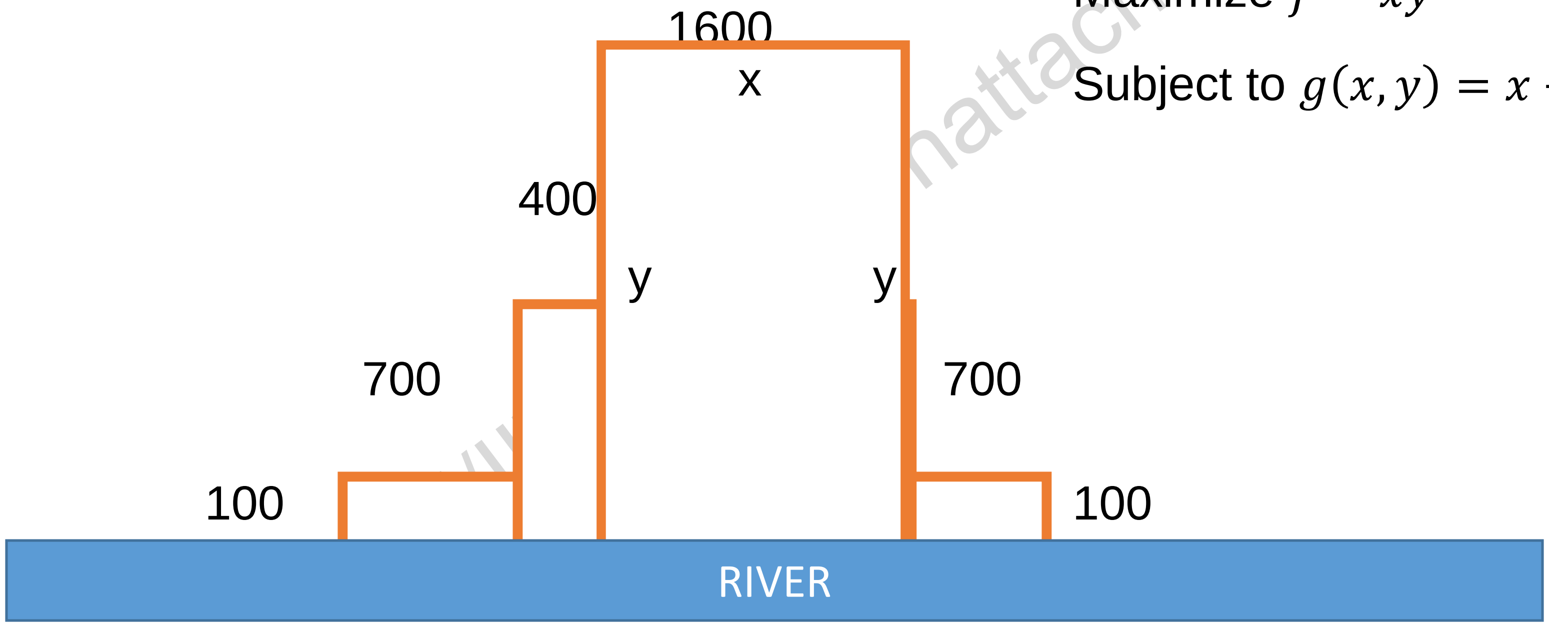
Ant colony

Books

- K. Deb., Optimization for Engineering Design: Algorithms and Examples, PHI Pvt Ltd., 1998.
- J. S. Arora, Introduction to Optimum Design, McGraw Hill International Edition, 1989.
- S.S. Rao, Engineering optimization: Theory and Practice, New age international (P) Ltd. 2001
- D. E. Goldberg, Genetic Algorithms in search and optimization, Pearson publication, 1990.
- K. Deb, Multi-Objective Optimization Using Evolutionary Algorithms, Chichester, UK : Wiley

Example

A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

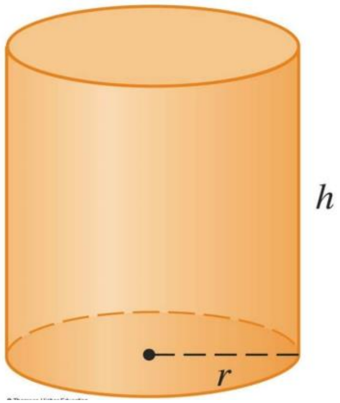


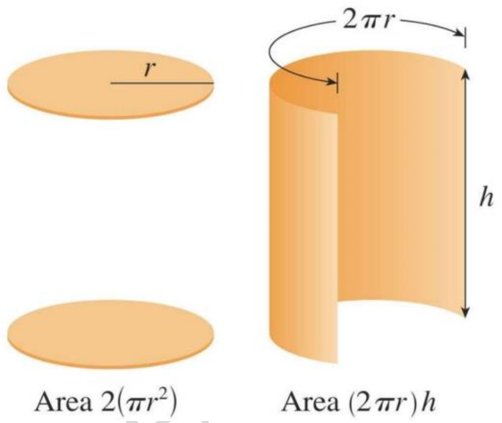
Maximize $f = xy$

Subject to $g(x, y) = x + 2y = 2400$

Example

A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.





Minimize: $A = 2\pi r^2 + 2\pi rh$

Constraint: $\pi r^2 h = 1500$

Dimension is in cm

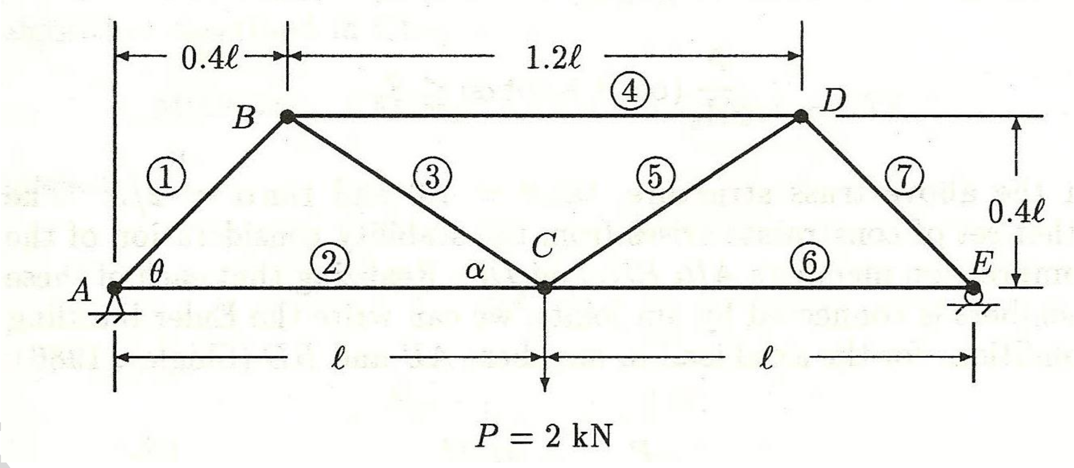
Example

Objectives

Topology: Optimal connectivity of the structure

Minimum cost of material: optimal cross section of all the members

We will consider the second objective only



The design variables are the cross sectional area of the members, i.e. A1 to A7

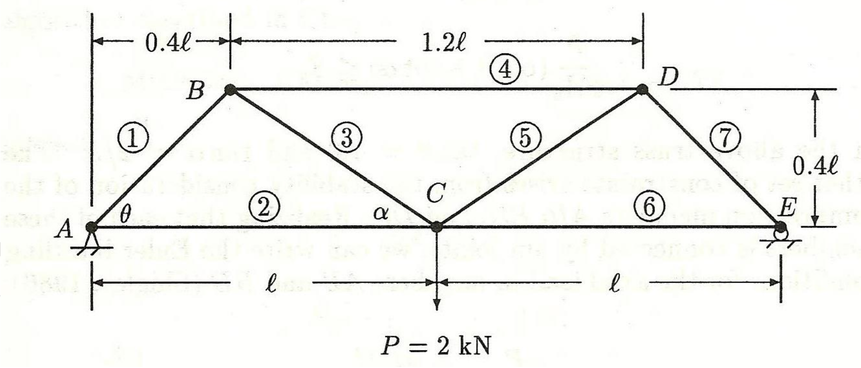
Using symmetry of the structure
A7=A1, A6=A2, A5=A3

You have only four design variables, i.e., A1 to A4

Optimization formulation

Objective

$$\text{Minimize} \quad 1.132A_1\ell + 2A_2\ell + 1.789A_3\ell + 1.2A_4\ell$$



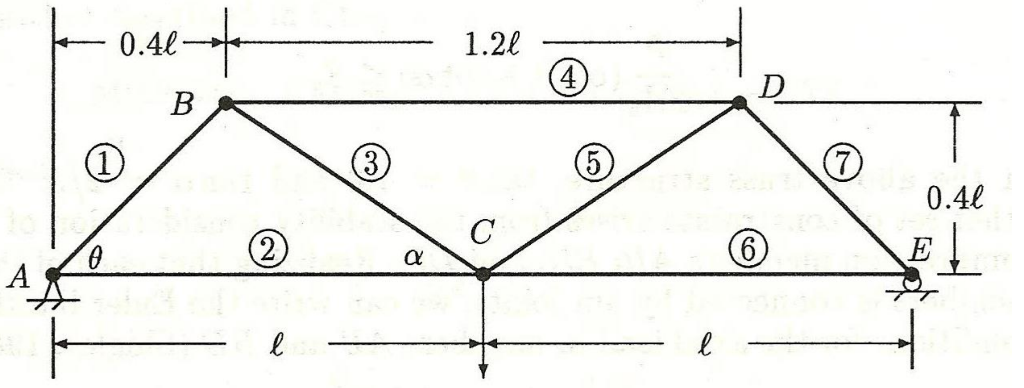
What are the constraints?

One essential constraint is non-negativity of design variables, i.e.
A1, A2, A3, A4 >= 0

Is it complete now?

Optimization formulation

Member	Force	Member	Force
AB	$-\frac{P}{2}\csc\theta$	BC	$+\frac{P}{2}\csc\alpha$
AC	$+\frac{P}{2}\cot\theta$	BD	$-\frac{P}{2}(\cot\theta+\cot\alpha)$



First set of constraints

$$\frac{P\csc\theta}{2A_1} \le S_{yc},$$

$$\frac{P\cot\theta}{2A_2} \le S_{yt},$$

$$\frac{P\csc\alpha}{2A_3} \le S_{yt},$$

$$\frac{P}{2A_4}(\cot\theta+\cot\alpha) \le S_{yc}$$

Another constraint is buckling of compression members

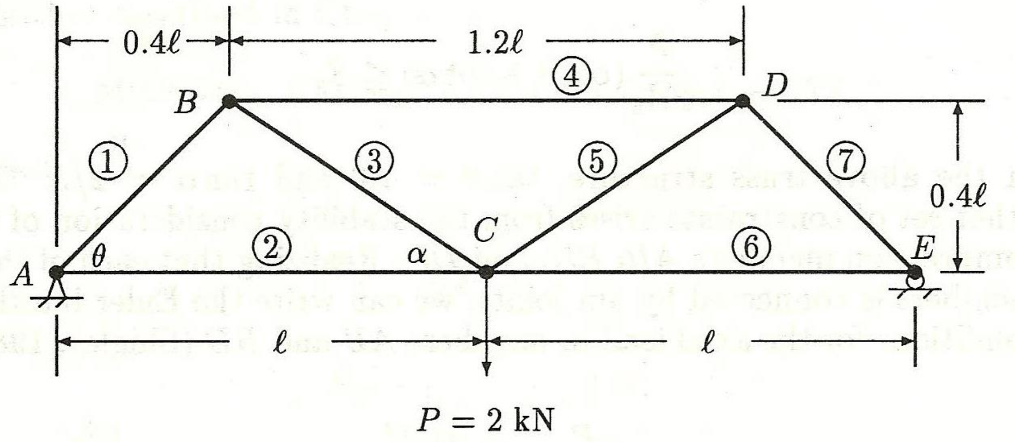
$$\frac{P}{2\sin\theta} \le \frac{\pi E A_1^2}{1.281\ell^2}$$

$$\frac{P}{2}(\cot\theta+\cot\alpha) \le \frac{\pi E A_4^2}{5.76\ell^2}$$

Another constraint may be the minimization of deflection at C

$$\frac{P\ell}{E}\left(\frac{0.566}{A_1}+\frac{0.500}{A_2}+\frac{2.236}{A_3}+\frac{2.700}{A_4}\right) \le \delta_{\max}$$

Optimization formulation



Minimize $1.132A_1\ell + 2A_2\ell + 1.789A_3\ell + 1.2A_4\ell$

subject to

$$S_{yc} - \frac{P}{2A_1 \sin\theta} \geq 0,$$

$$S_{yt} - \frac{P}{2A_2 \cot\theta} \geq 0,$$

$$S_{yt} - \frac{P}{2A_3 \sin\alpha} \geq 0,$$

$$S_{yc} - \frac{P}{2A_4}(\cot\theta + \cot\alpha) \geq 0,$$

$$\frac{\pi E A_1^2}{1.281\ell^2} - \frac{P}{2\sin\theta} \geq 0,$$

$$\frac{\pi E A_4^2}{5.76\ell^2} - \frac{P}{2}(\cot\theta + \cot\alpha) \geq 0,$$

$$\delta_{\max} - \frac{P\ell}{E}\left(\frac{0.566}{A_1} + \frac{0.500}{A_2} + \frac{2.236}{A_3} + \frac{2.700}{A_4}\right) \geq 0,$$

$$10 \times 10^{-6} \leq A_1, A_2, A_3, A_4 \leq 500 \times 10^{-6}.$$

What is Optimization?

- Optimization is the act of obtaining the best result under a given circumstances.

- Optimization is the mathematical discipline which is concerned with finding the maxima and minima of functions, possibly subject to constraints.

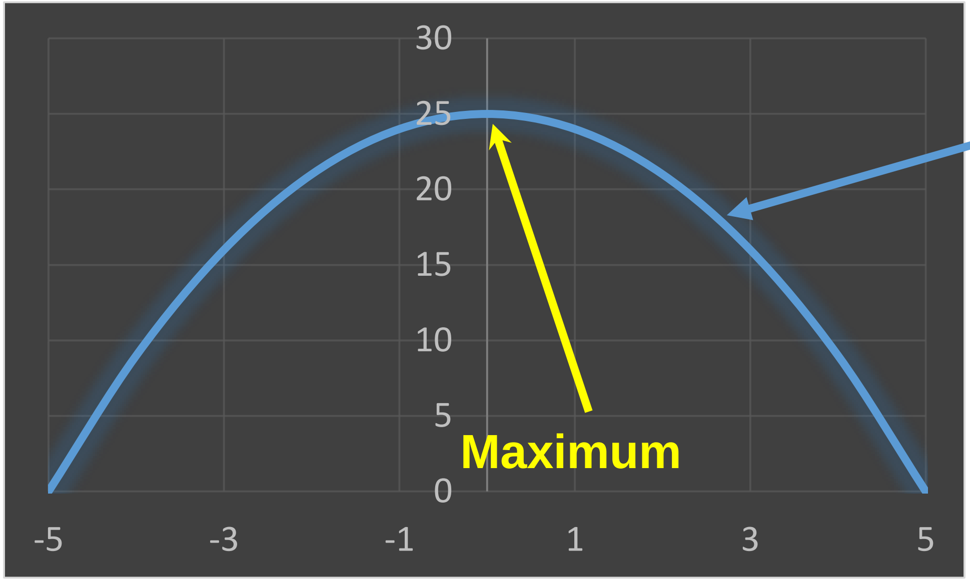
Introduction to optimization

$f = (x-5)^2$ Equation of the line

How to find out the minimum of the function

$f' = 2 \times (x-5) = 0$

$x^* = 5$ Optimal solution

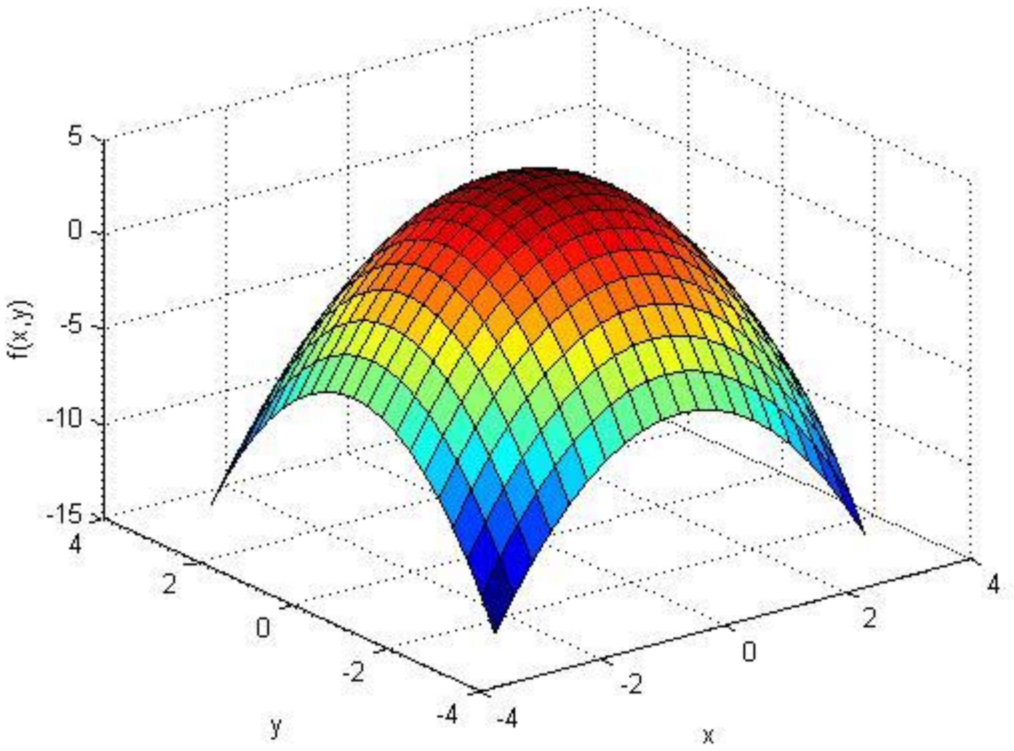


$f = 25 + x^2$ Equation of the line

$f' = 2x = 0$

$x^* = 0$ Optimal solution

Introduction to optimization



Equation of the surface

$$f(x,y) = -(x^2 + y^2) + 4$$

In this case, we can obtain the optimal solution by taking derivatives with respect to variable x and y and equating them to zero

$$\frac{\partial f}{\partial x} = -2x = 0 \qquad \Rightarrow x^* = 0$$

$$\frac{\partial f}{\partial y} = -2y = 0 \qquad \Rightarrow y^* = 0$$

Optimal solution is $(0,0)$

Single variable optimization

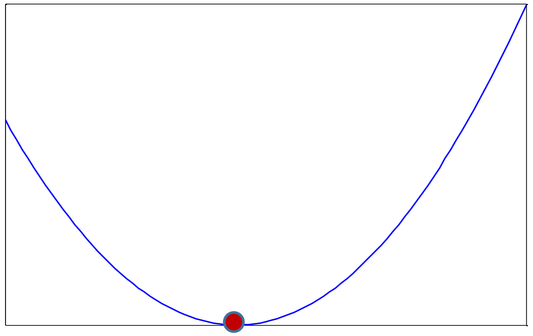
Objective function is defined as

Minimization/Maximization f(x)

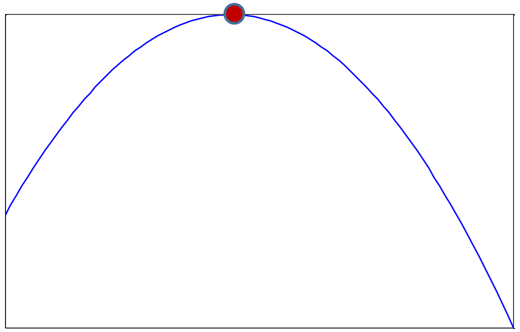
Single variable optimization

Stationary points

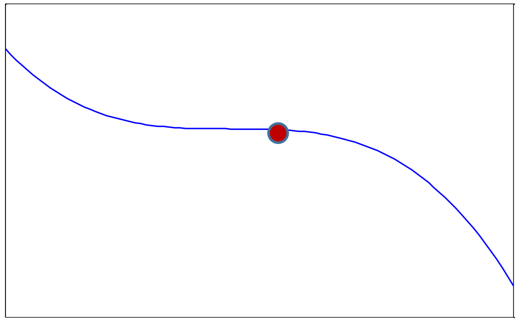
For a continuous and differentiable function *f(x), a stationary point x* is a point at which the* slope of the function is zero, *i.e.* $f'(x) = 0$ *at* $x = x^*$,



Minima



Maxima



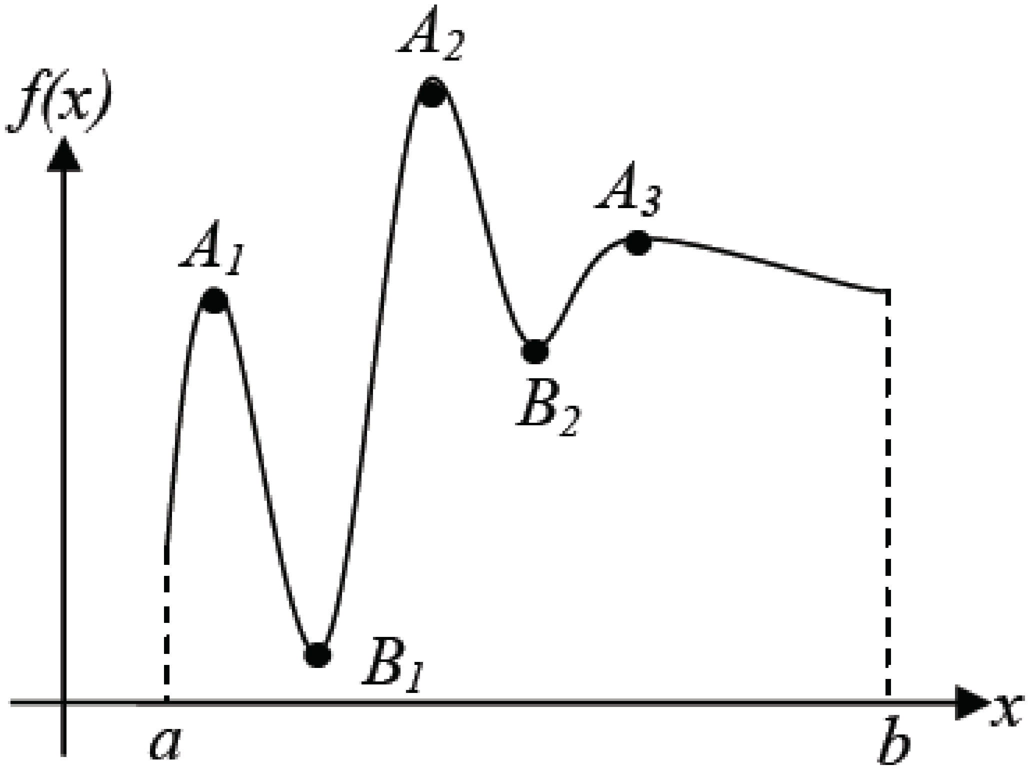
Inflection point

Global minimum and maximum

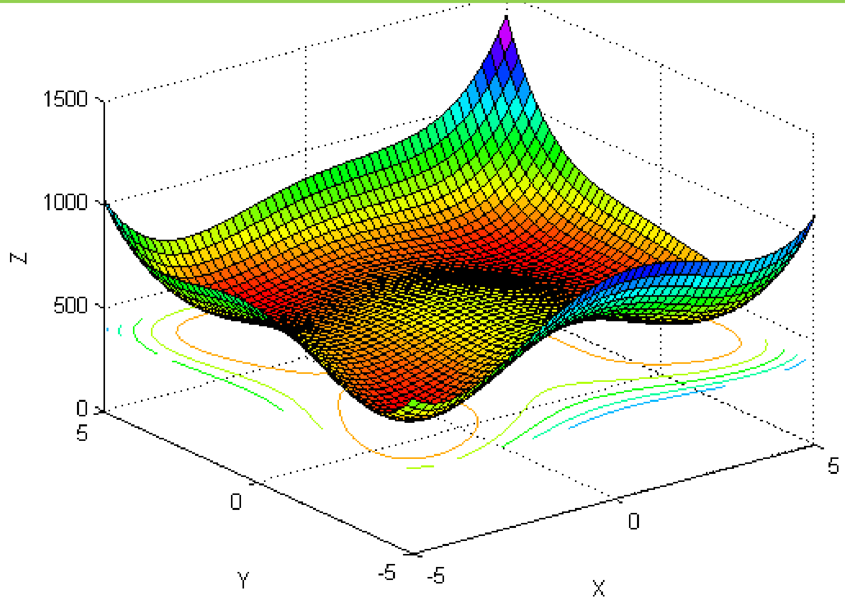
A function is said to have a *global or absolute minimum at* $x = x^*$ *if* $f(x^*) \le f(x)$ *for all x in the domain over which f(x) is defined.*

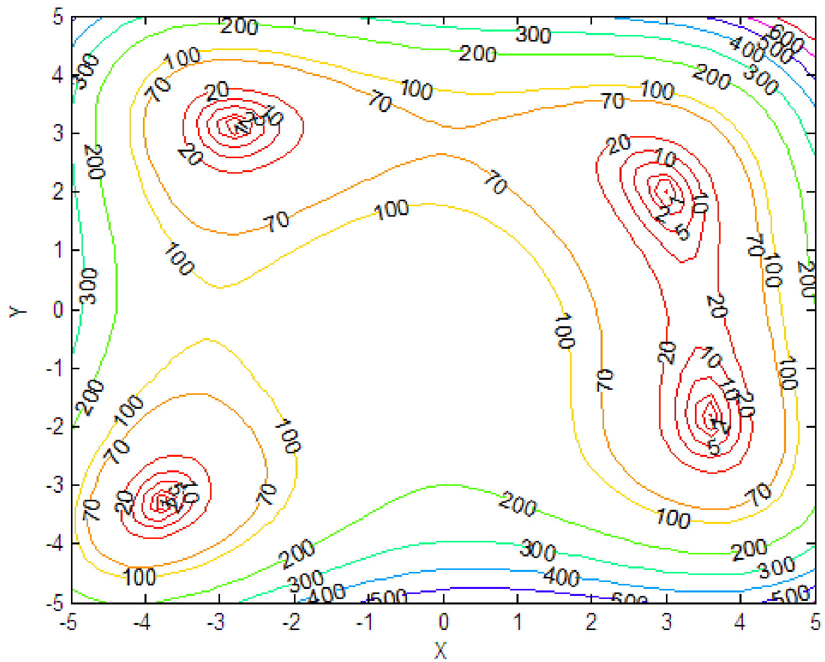
A function is said to have a *global or absolute maximum at* $x = x^*$ *if* $f(x^*) \ge f(x)$ *for all x in the domain over which f(x) is defined.*

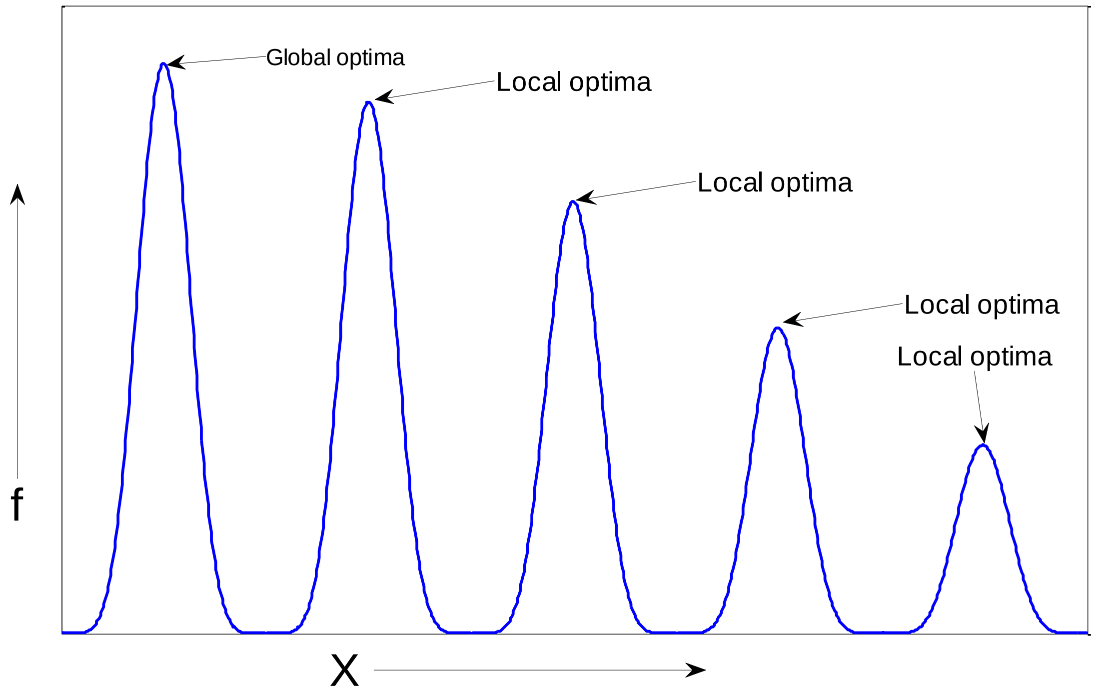
A_1, A_2, A_3 = Relative maxima
A_2 = Global maximum
B_1, B_2 = Relative minima
B_1 = Global minimum



Introduction to optimization







Necessary and sufficient conditions for optimality

Necessary condition

If a function $f(x)$ is defined in the interval $a \leq x \leq b$ and has a relative minimum at $x = x^*$, Where $a \leq x^* \leq b$ and if $f^/(x)$ exists as a finite number at $x = x^*$, then $f^/(x^*) = 0$

Proof

$$f^/(x^*) = \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h}$$

Since x^* is a relative minimum $\quad f(x^*) \leq f(x^* + h)$

For all values of h sufficiently close to zero, hence

$$\frac{f(x^* + h) - f(x^*)}{h} \geq 0 \qquad \text{if } h \geq 0$$

$$\frac{f(x^* + h) - f(x^*)}{h} \leq 0 \qquad \text{if } h \leq 0$$

Necessary and sufficient conditions for optimality

Thus

$f^{/}(x^{*}) \geq 0$ If h tends to zero through +ve value

$f^{/}(x^{*}) \leq 0$ If h tends to zero through -ve value

Thus only way to satisfy both the conditions is to have

$$f^{/}(x^{*}) = 0$$

Note:

- This theorem can be proved if x^{*} is a relative maximum
- Derivative must exist at x^{*}
- The theorem does not say what happens if a minimum or maximum occurs at an end point of the interval of the function
- It may be an inflection point also.

Sufficient conditions for optimality

Sufficient condition

Suppose at point x^* , the first derivative is zero and first nonzero higher derivative is denoted by n, then

1. *If n is odd,* x^* *is an inflection point*
2. *If n is even,* x^* *is a local optimum*
    1. *If the derivative is positive,* x^* *is a local minimum*
    2. *If the derivative is negative,* x^* *is a local maximum*

Sufficient conditions for optimality

Proof

Apply Taylor's series

$$f(x^* + h) = f(x^*) + hf'(x^*) + \frac{h^2}{2!}f''(x^*) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(x^*) + \frac{h^n}{n!}f^n(x^*)$$

Since $f'(x^*) = f''(x^*) = \cdots = f^{n-1}(x^*) = 0$

$$f(x^* + h) - f(x^*) = \frac{h^n}{n!}f^n(x^*)$$

When n is even $\frac{h^n}{n!} \geq 0$

Thus if $f'(x^*)$ is positive $f(x^* + h) - f(x^*)$ is positive **Hence it is local minimum**

Thus if $f'(x^*)$ negative $f(x^* + h) - f(x^*)$ is negative **Hence it is local maximum**

When n is odd $\frac{h^n}{n!}$ changes sign with the change in the sign of h.
Hence it is an inflection point

Sufficient conditions for optimality

Take an example

$$f(x) = x^3 - 10x - 2x^2 - 10$$

Apply necessary condition $f'(x) = 3x^2 - 10 - 4x = 0$

Solving for x $x^* = 2.61 \; and \; -1.28$ These two points are stationary points

Apply sufficient condition $f''(x) = 6x - 4$

$f''(2.61) = 11.66$ *positive and n is odd* $f''(-1.28) = -11.68$ *negative and n is odd*

$x^* = 2.61$ is a minimum point

$x^* = -1.28$ is a maximum point

Multivariable optimization without constraints

Minimize $f(X)$ Where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Necessary condition for optimality

If $f(X)$ has an extreme point (maximum or minimum) at $X = X^*$ and if the first partial Derivatives of $f(X)$ exists at X^*, then

$$\frac{\partial f(X^*)}{\partial x_1} = \frac{\partial f(X^*)}{\partial x_2} = \cdots = \frac{\partial f(X^*)}{\partial x_n} = 0$$

Multivariable optimization without constraints

Sufficient condition for optimality

The sufficient condition for a stationary point X^* to be an extreme point is that the matrix of second partial derivatives of $f(X)$ evaluated at X^* is

(1) positive definite when X^* is a relative minimum
(2) negative definite when X^* is a relative maximum
(3) neither positive nor negative definite when X^* is neither a minimum nor a maximum

Proof Taylor series of two variable function

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y} + \frac{1}{2!}\left(\Delta x^2\frac{\partial^2 f}{\partial x^2} + 2\Delta x\Delta y\frac{\partial^2 f}{\partial x\partial y} + \Delta y^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

$$f(x+\Delta x, y+\Delta y) = f(x,y) + [\Delta x \quad \Delta y]\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{bmatrix} + \frac{1}{2!}[\Delta x \quad \Delta y]\begin{bmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} + \cdots$$

Multivariable optimization without constraints

$$f(X^* + h) = f(X^*) + h^T \nabla f(X^*) + \frac{1}{2!} h^T H h + \cdots$$

Since X^* is a stationary point, the necessary condition gives that $\nabla f(X^*) = 0$

Thus

$$f(X^* + h) - f(X^*) = \frac{1}{2!} h^T H h + \cdots$$

Now, X^* will be a minima, if $h^T H h$ is positive

X^* will be a maxima, if $h^T H h$ is negative

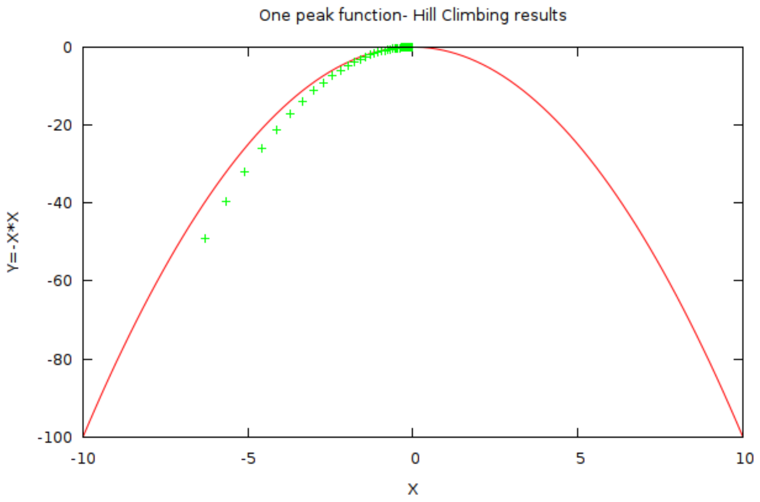
$h^T H h$ will be positive if H is a positive definite matrix

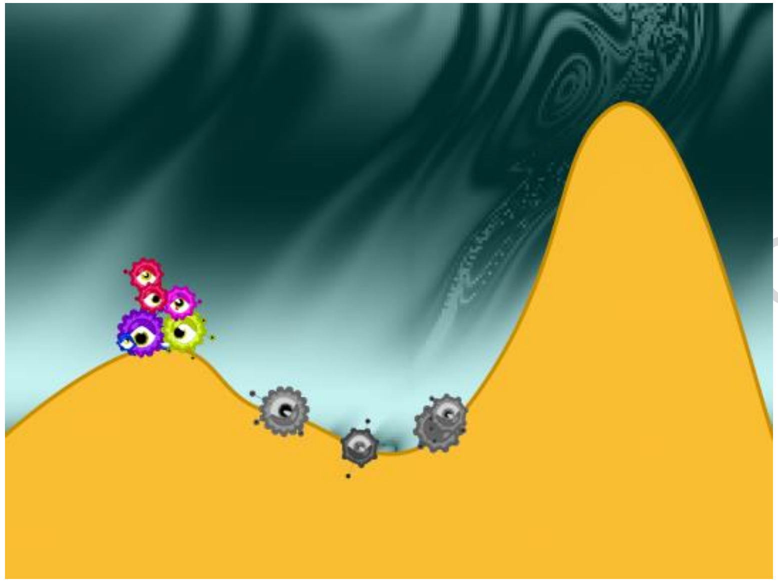
$h^T H h$ will be negative if H is a negative definite matrix

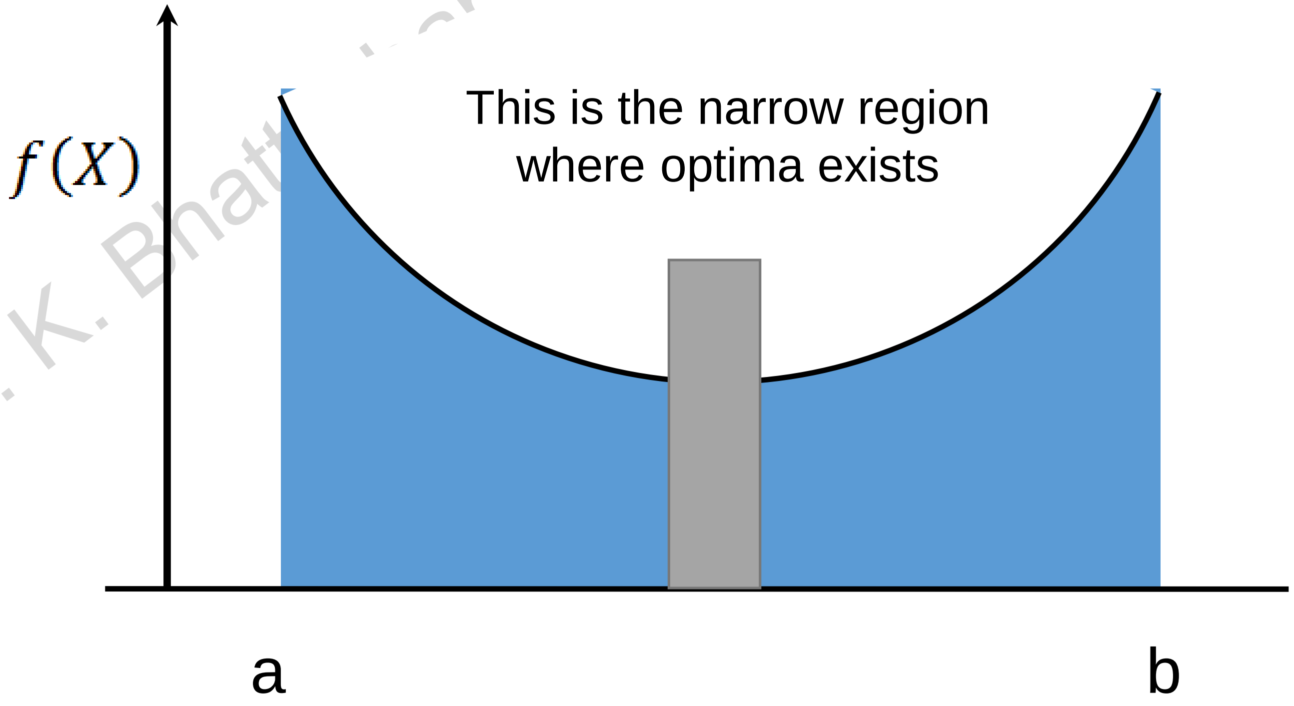
A matrix H will be positive definite if all the eigenvalues are positive, i.e. all the λ values are positive which satisfies the following equation

$$|A - \lambda I| = 0$$

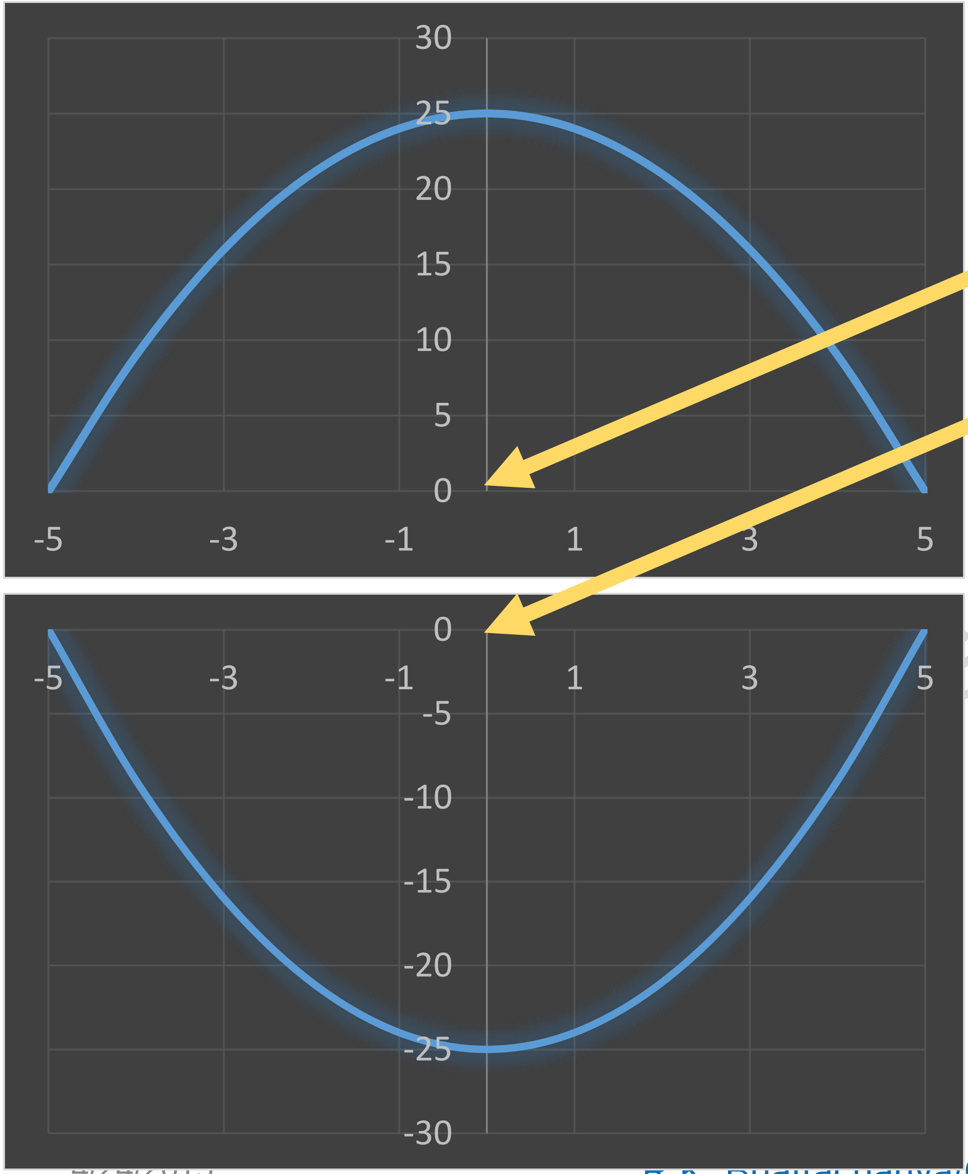
Line search techniques







Unimodal and duality principle

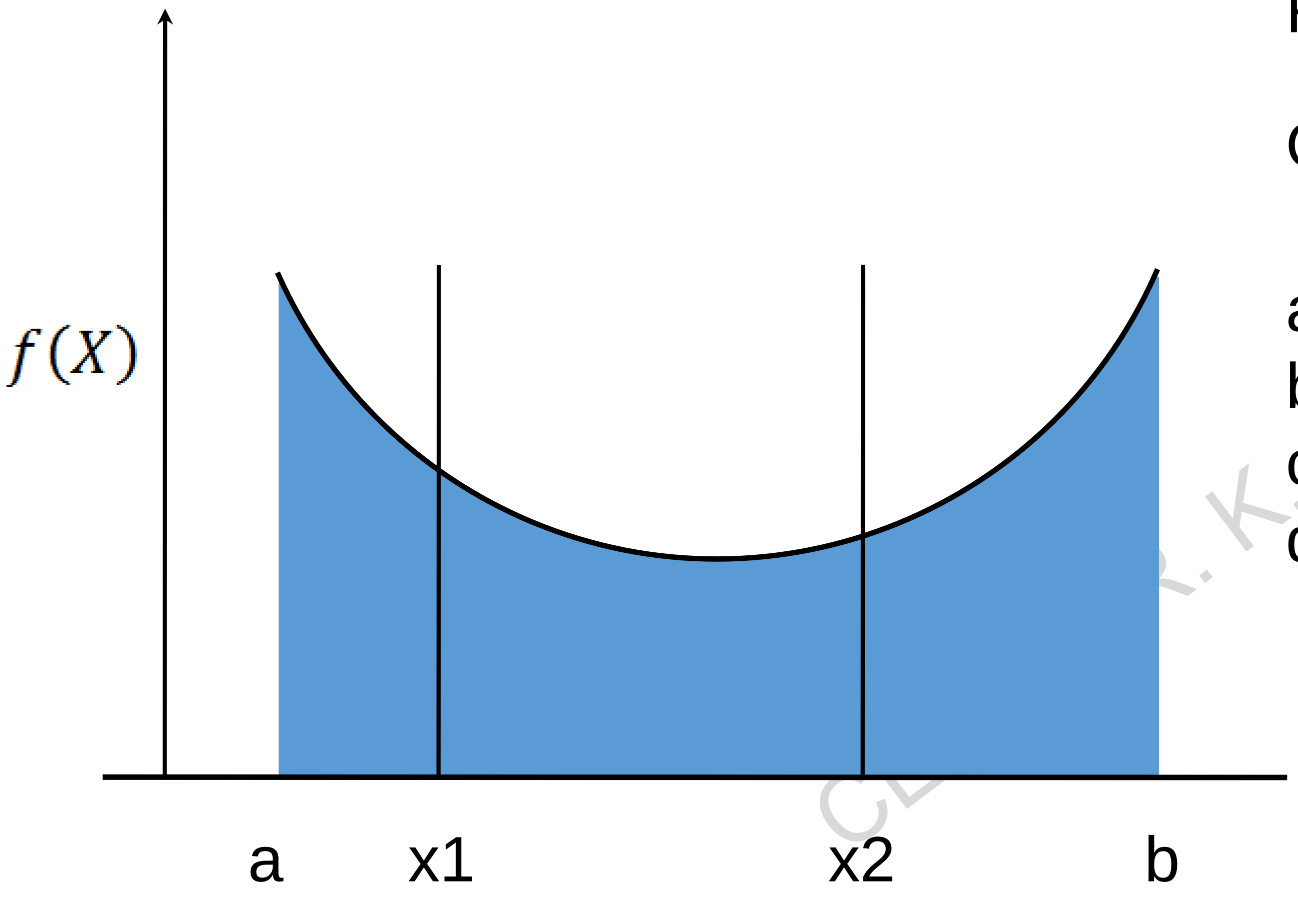


Optimal solution $x^* = 0$

Optimal solution $x^* = 0$

Minimization $f(x)$ = Maximization $-f(x)$

Quiz



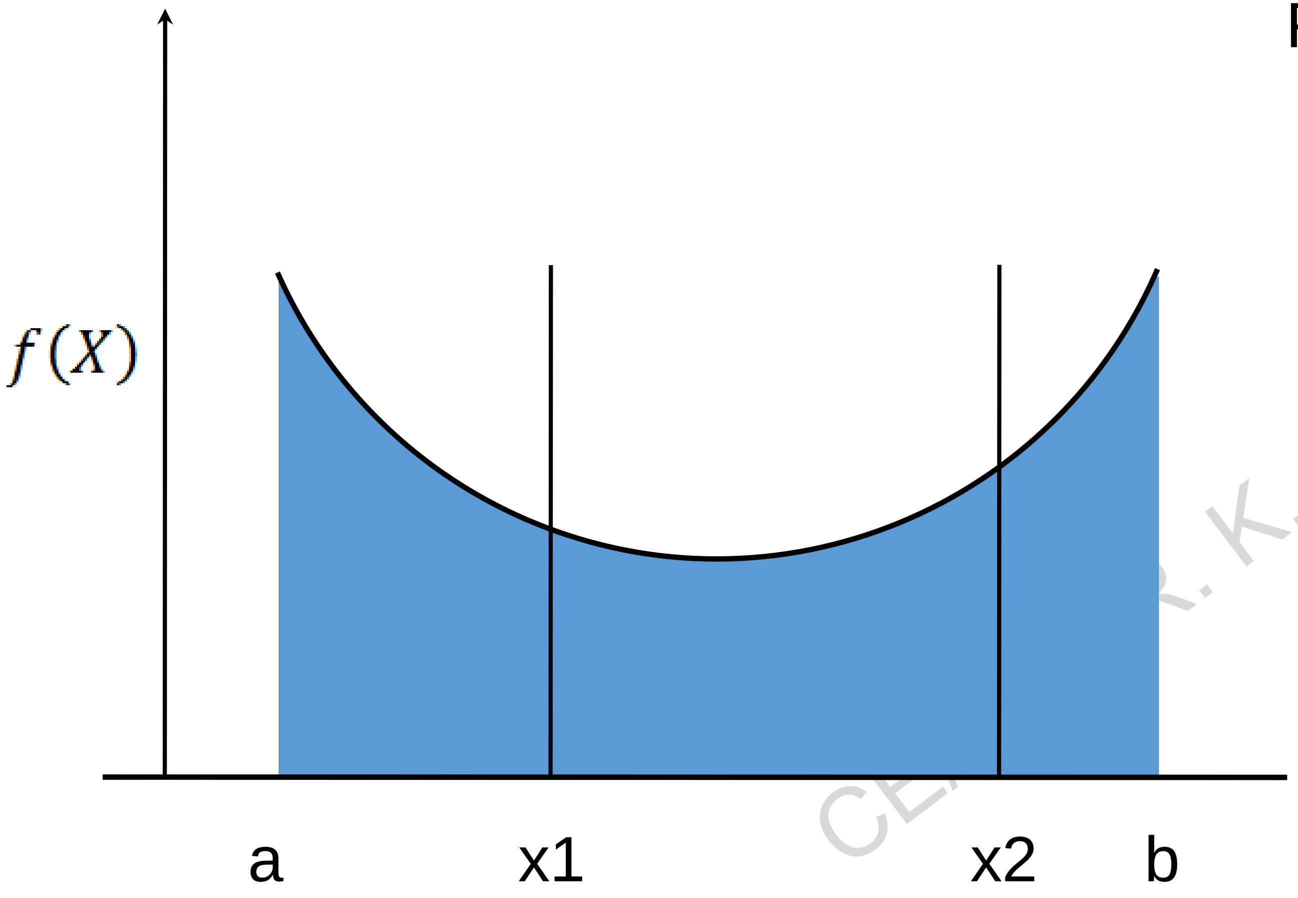
For the unimodal function $if\ f(X_1) > f(X_2)$

Optima is not

a. Between $[a, X_1]$
b. Between $[X_1, X_2]$
c. Between $[X_2, b]$
d. Between $[a, b]$

Ans: a

Quiz



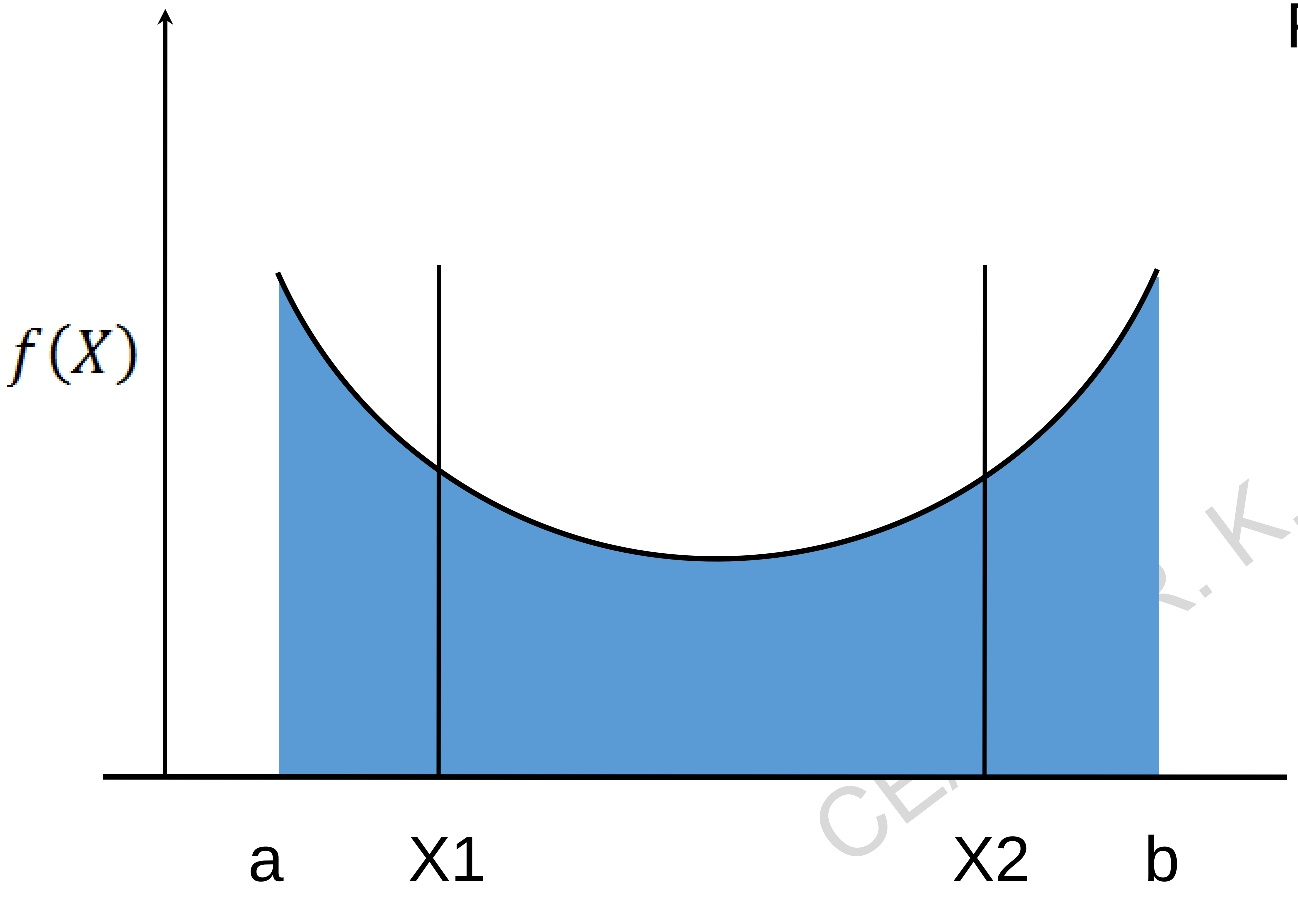
For the unimodal function $if\ f(X_2) > f(X_1)$

Optima is not

a. Between $[a, X_1]$
b. Between $[X_1, X_2]$
c. Between $[X_2, b\ ]$
d. Between $[a, b\ ]$

Ans: c

Quiz



For the unimodal function $if\ f(X_2) = f(X_1)$

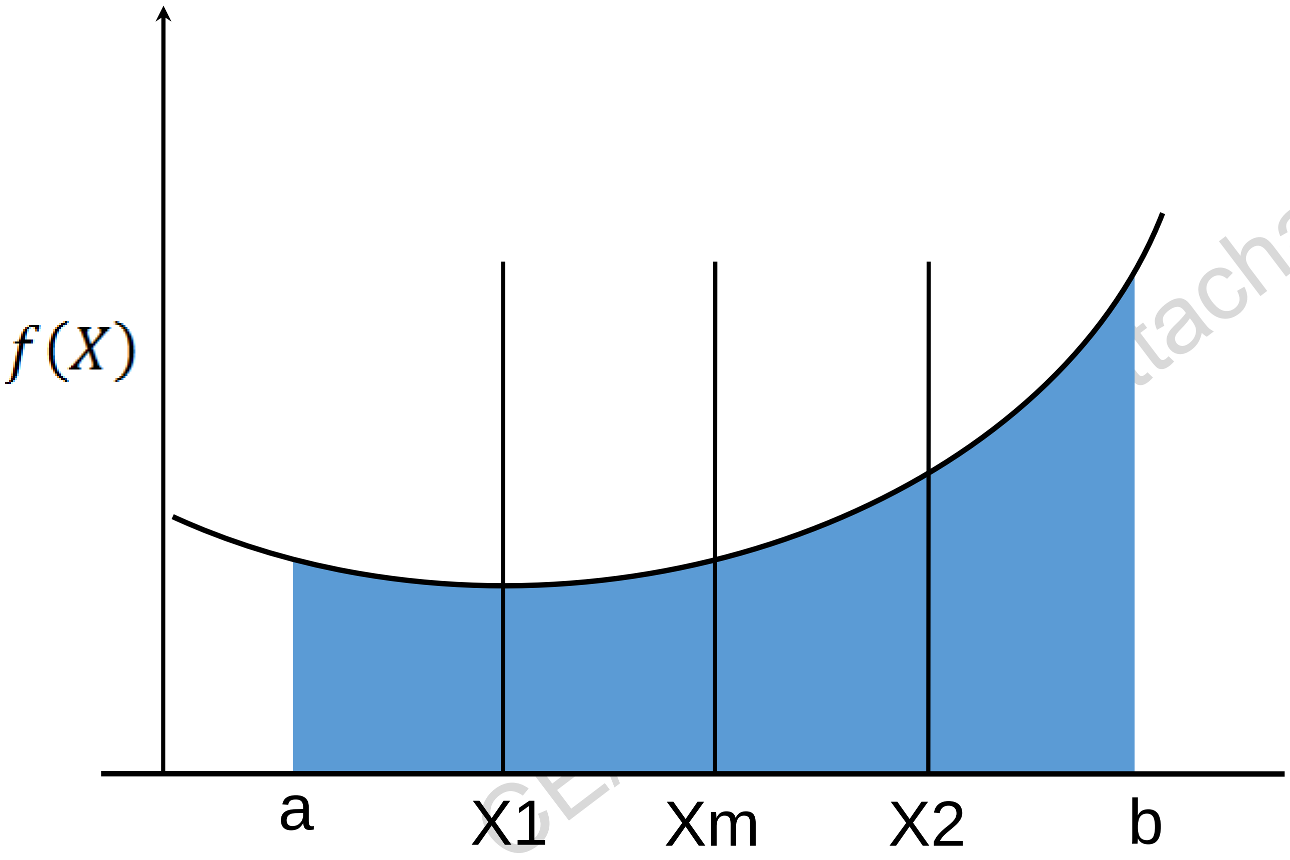
Optima is not

a. Between $[a, X_1]$
b. Between $[X_1, X_2]$
c. Between $[X_2, b]$
d. Between $[a, b]$

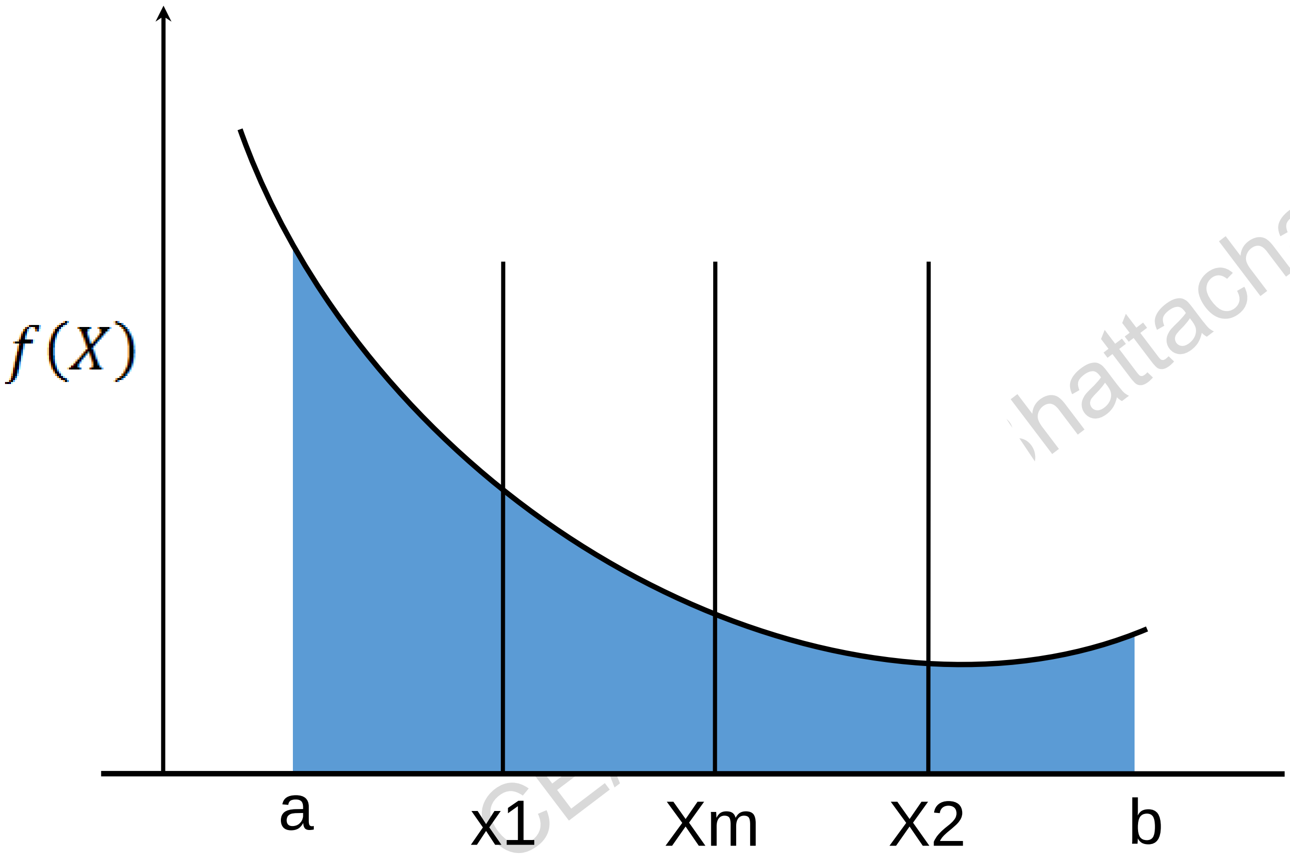
Ans: a, c

Interval halving method

$$if\ f(X_1) < f(X_m)$$



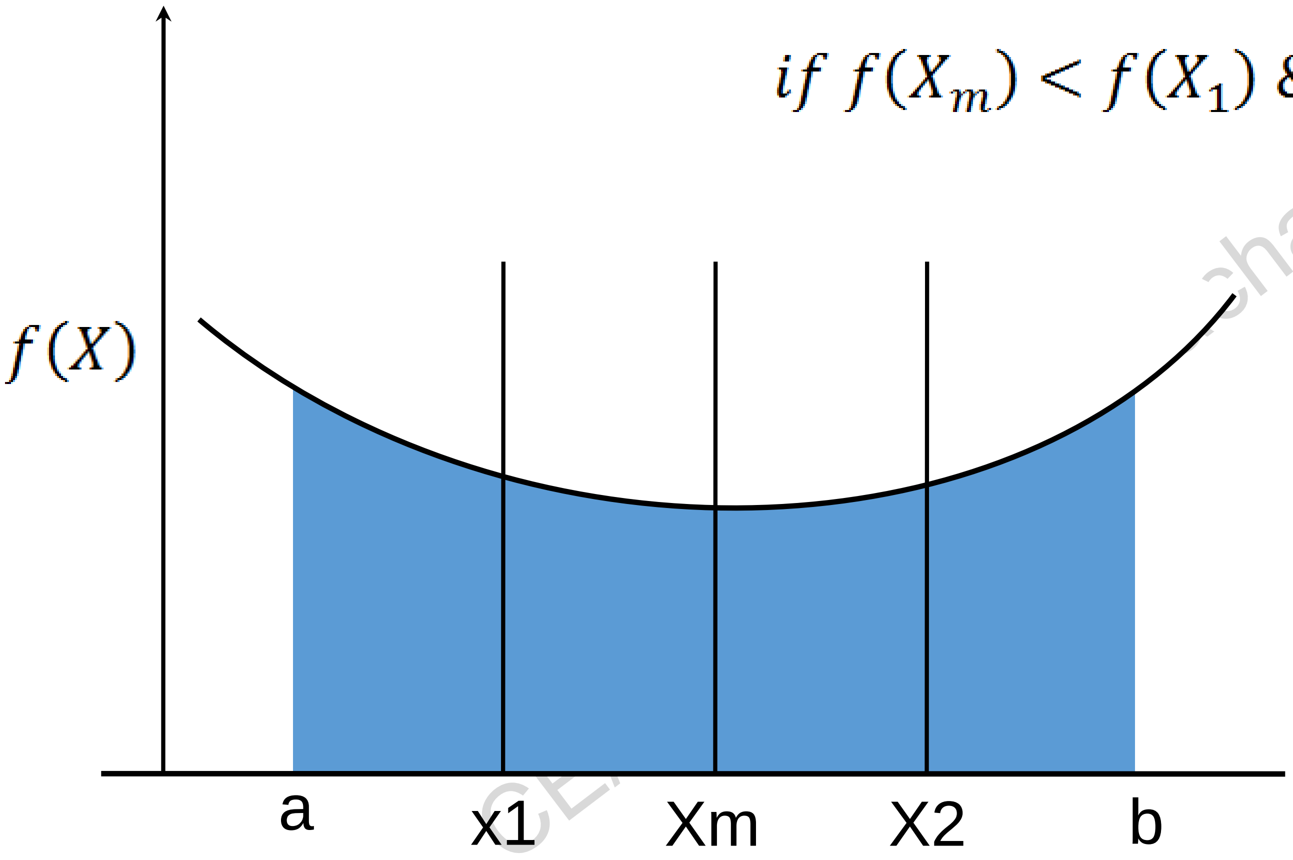
Interval halving method



$if\ f(X_2) < f(X_m)$

Interval halving method

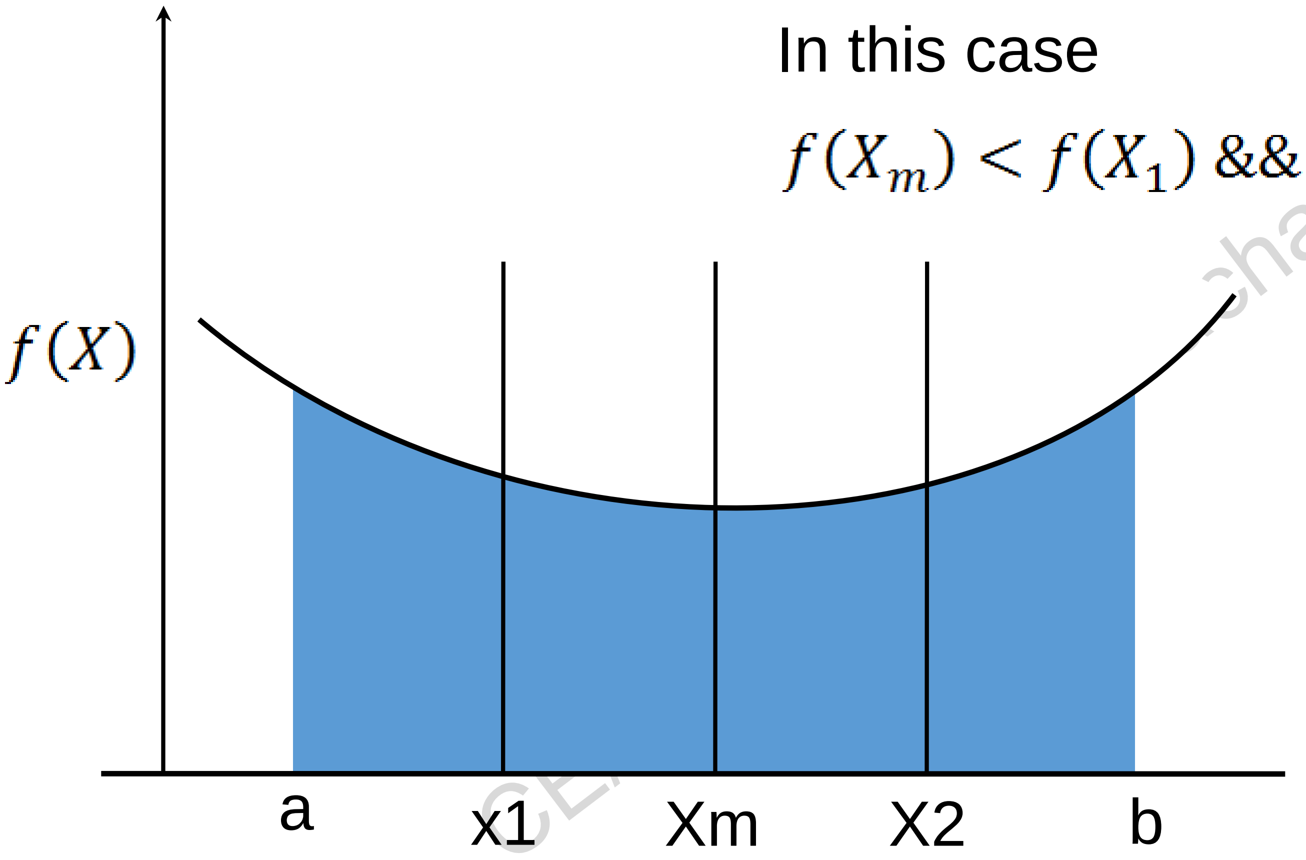
$$if\ f(X_m) < f(X_1)\ \&\&\ f(X_m) < f(X_2)$$



Interval halving method

In this case

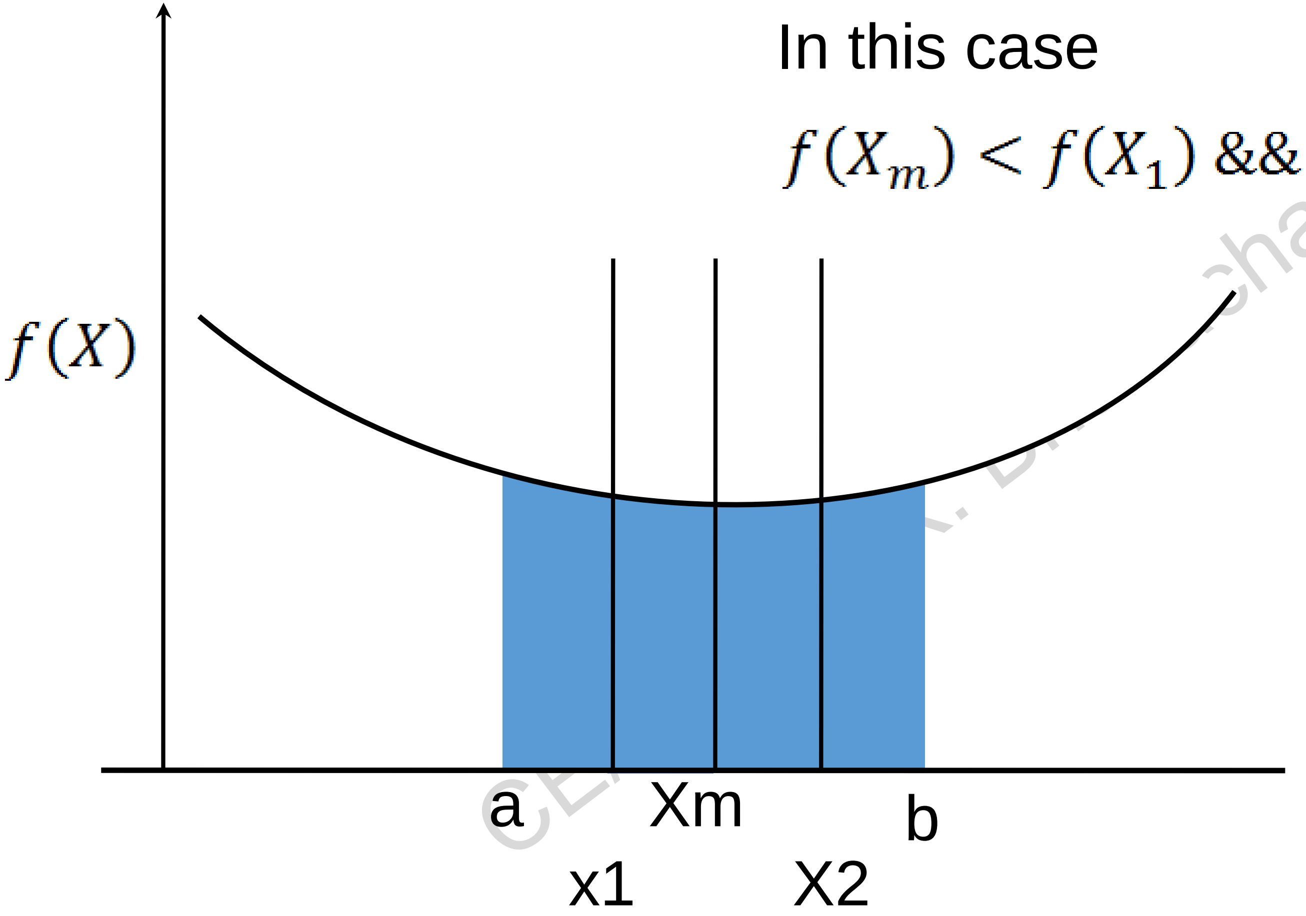
$$f(X_m) < f(X_1) \;\&\&\; f(X_m) < f(X_2)$$



Interval halving method

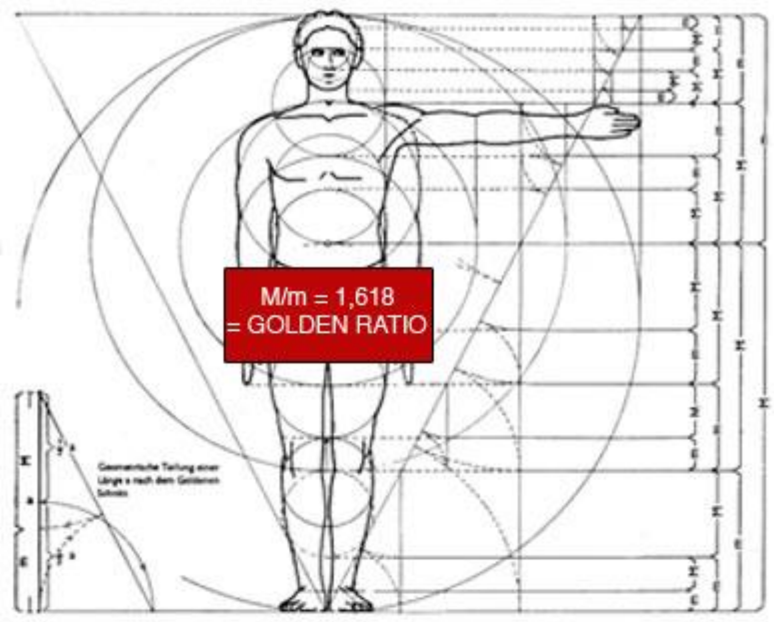
In this case

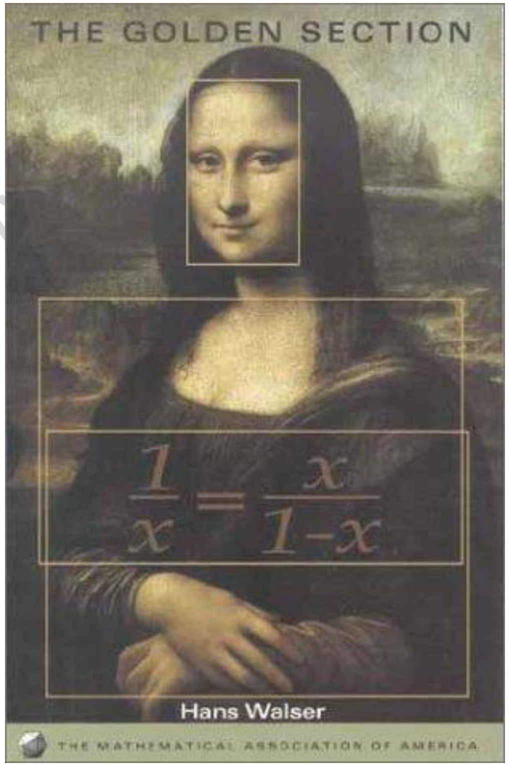
$$f(X_m) < f(X_1) \;\&\&\; f(X_m) < f(X_2)$$



CONTINUE

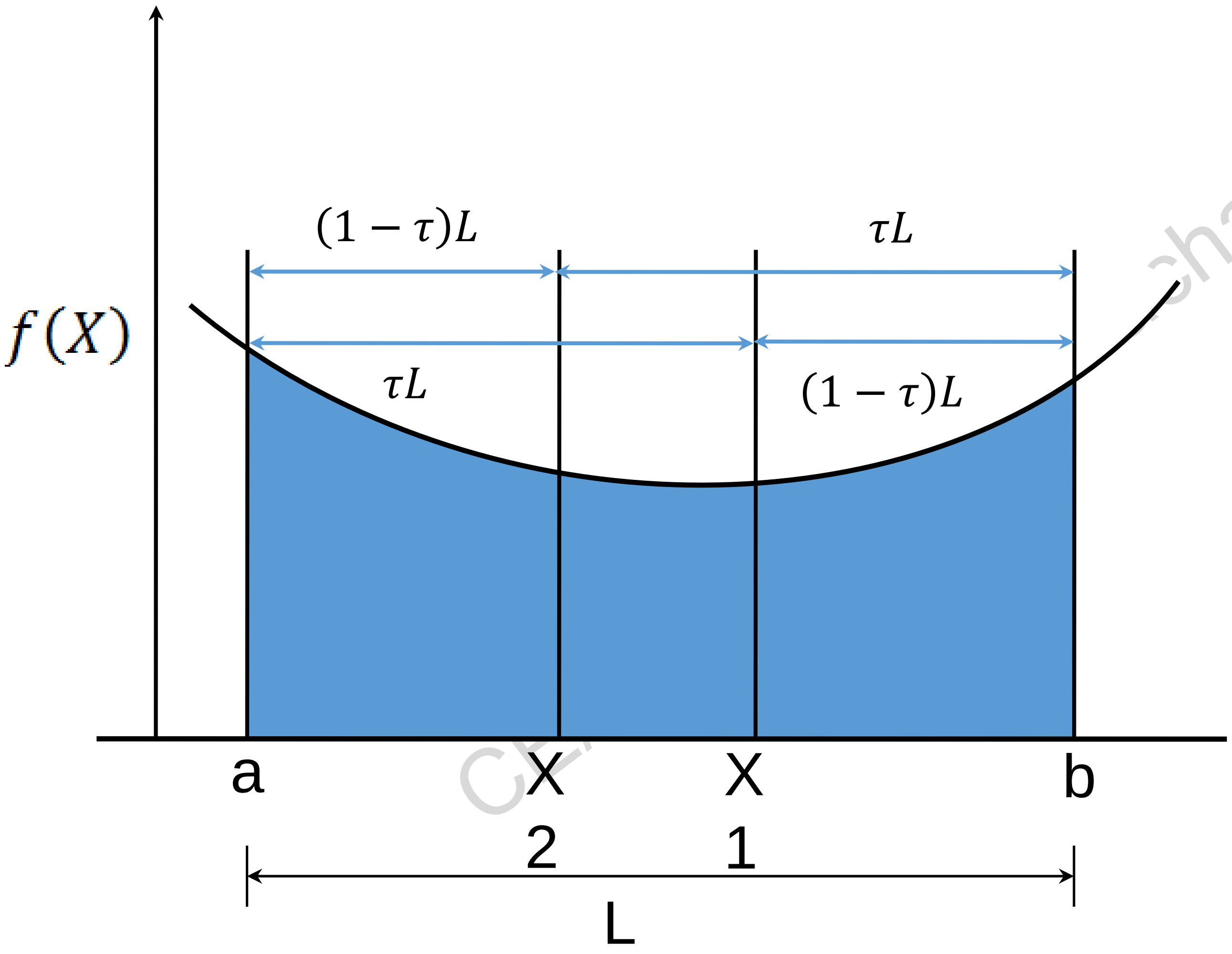
Golden ratio





Golden ratio 0.618=1/1.618

Golden Section Search Method

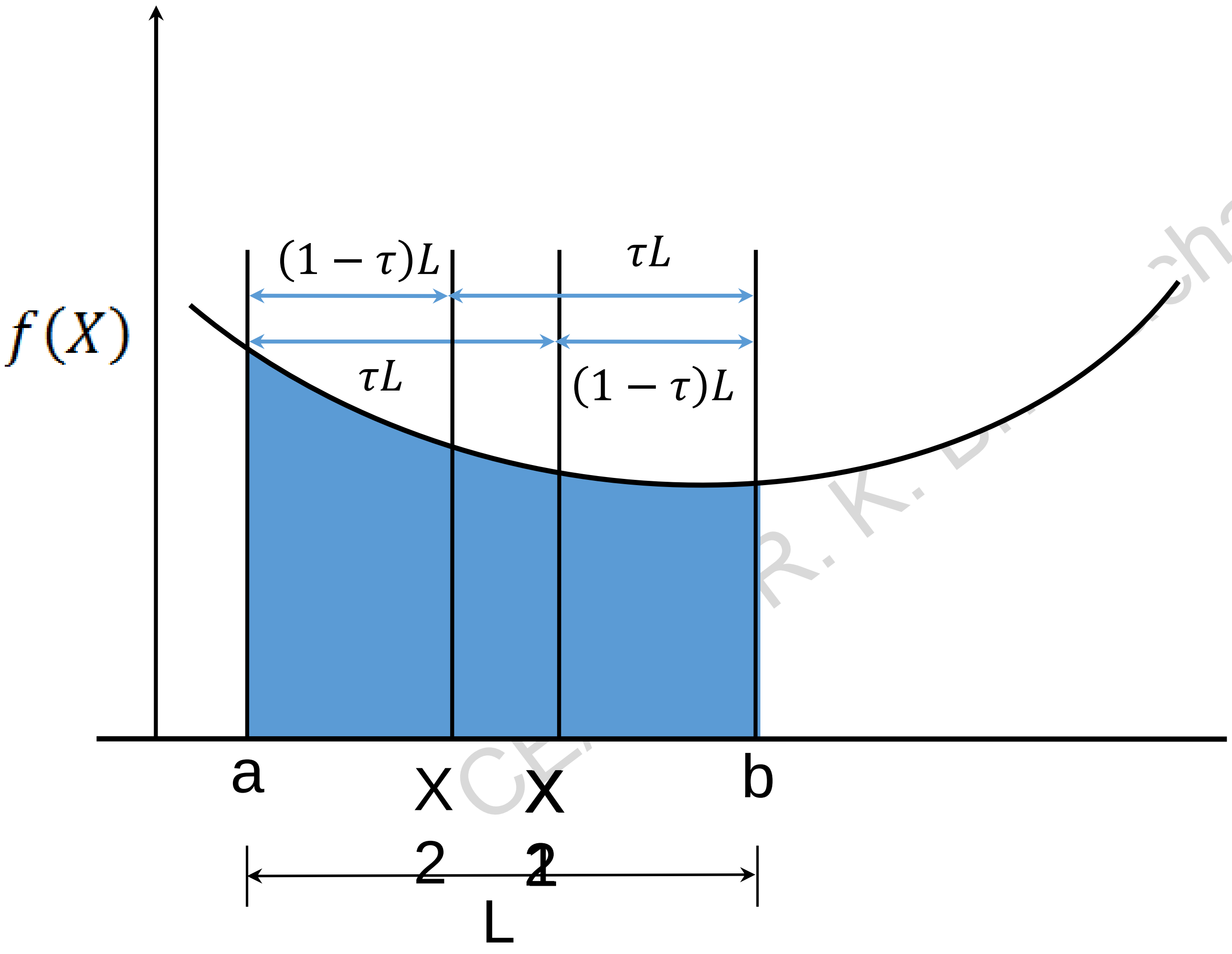


Apply region elimination rules

Suppose

$$f(X_1) > f(X_2)$$

Golden Section Search Method

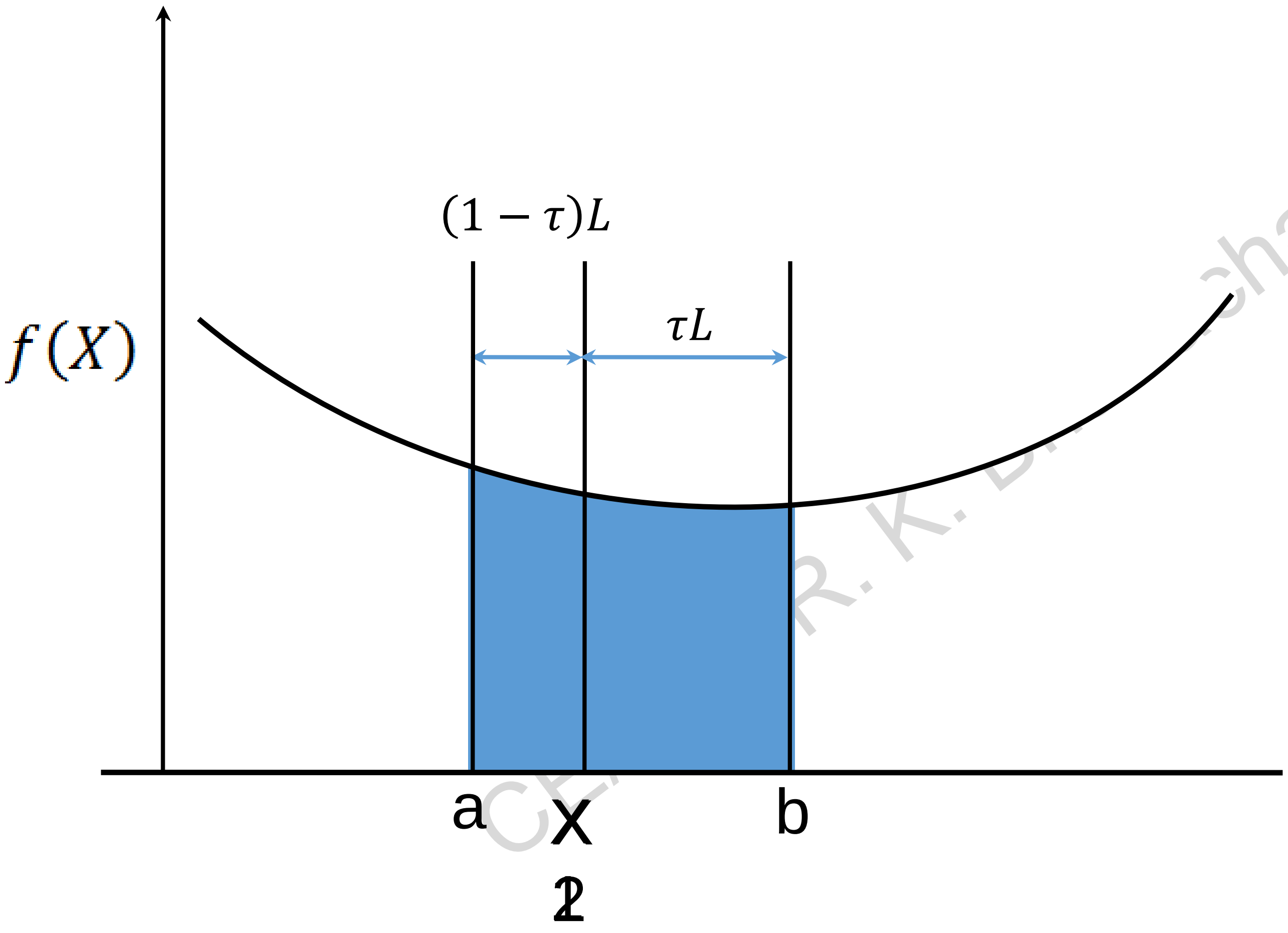


Apply region elimination rules

Suppose

$$f(X_1) < f(X_2)$$

Golden Section Search Method



Continue iteration

Golden Section Search Method

$c = a + \tau(b - a) \qquad (1)$

$d = b - \tau(b - a) \qquad (2)$

If $f(d) < f(c)$

$d = a + \tau(c - a) \qquad (3)$

Putting (1) in (3), we have

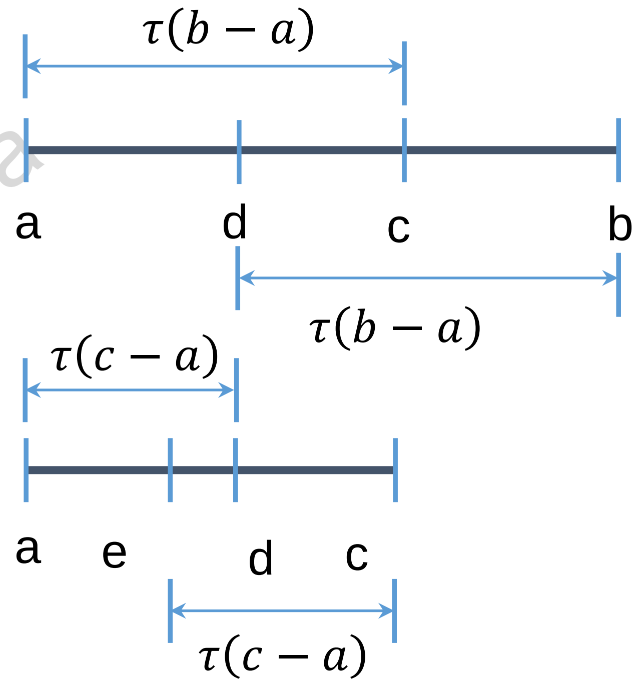
$d = a + \tau(a + \tau(b - a) - a)$

$d = a + \tau^2(b - a) \qquad (4)$

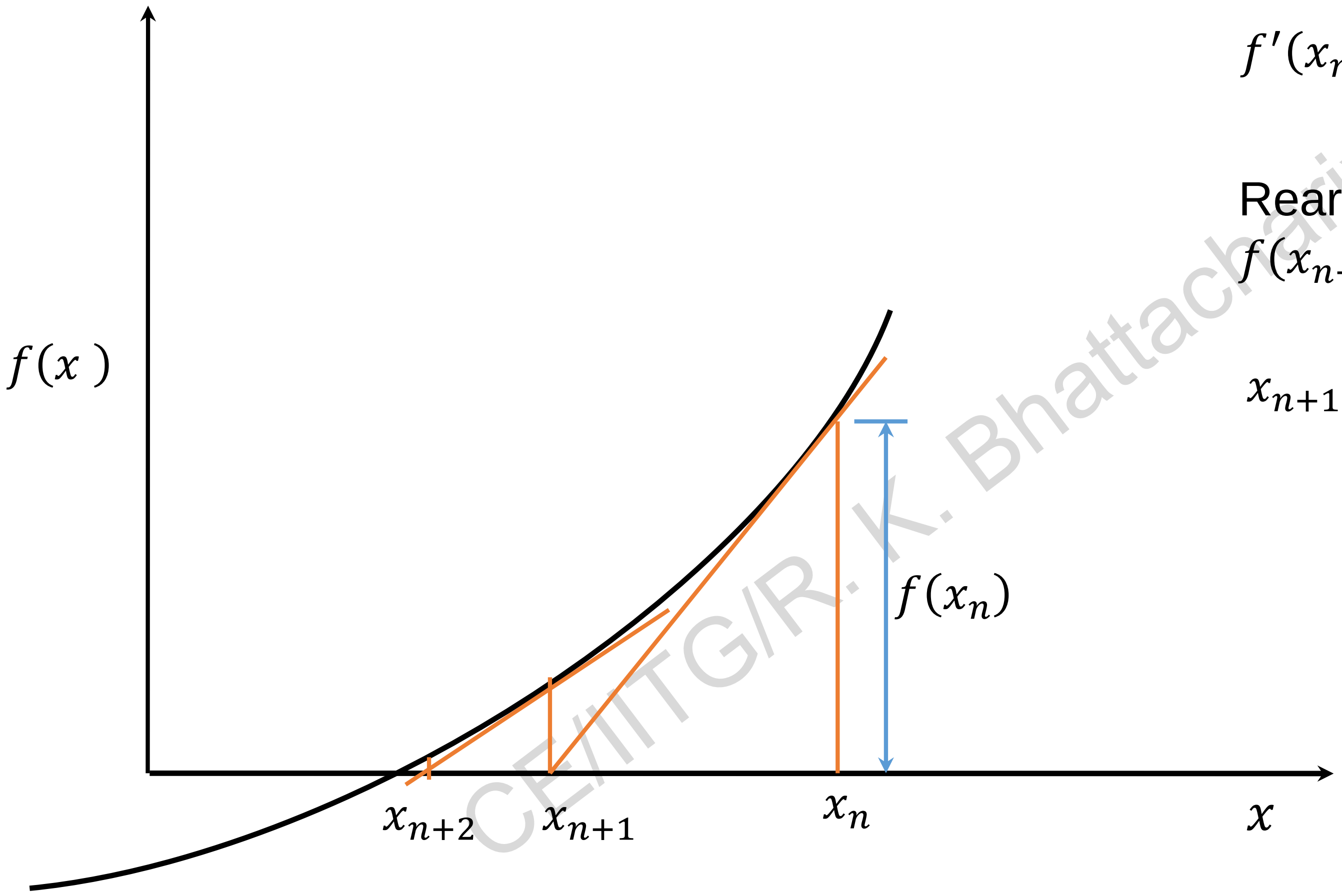
Equating (4) and (2), we have

$b - \tau(b - a) = a + \tau^2(b - a)$

$\tau^2 + \tau - 1 = 0$ Solving τ=0.618, -1.618 0.618 is the golden



Newton-Raphson method



$$f'(x_n) = \frac{f(x_n) - f(x_{n+1})}{x_n - x_{n+1}}$$

Rearranging and putting $f(x_{n+1})$=0

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Continue iteration

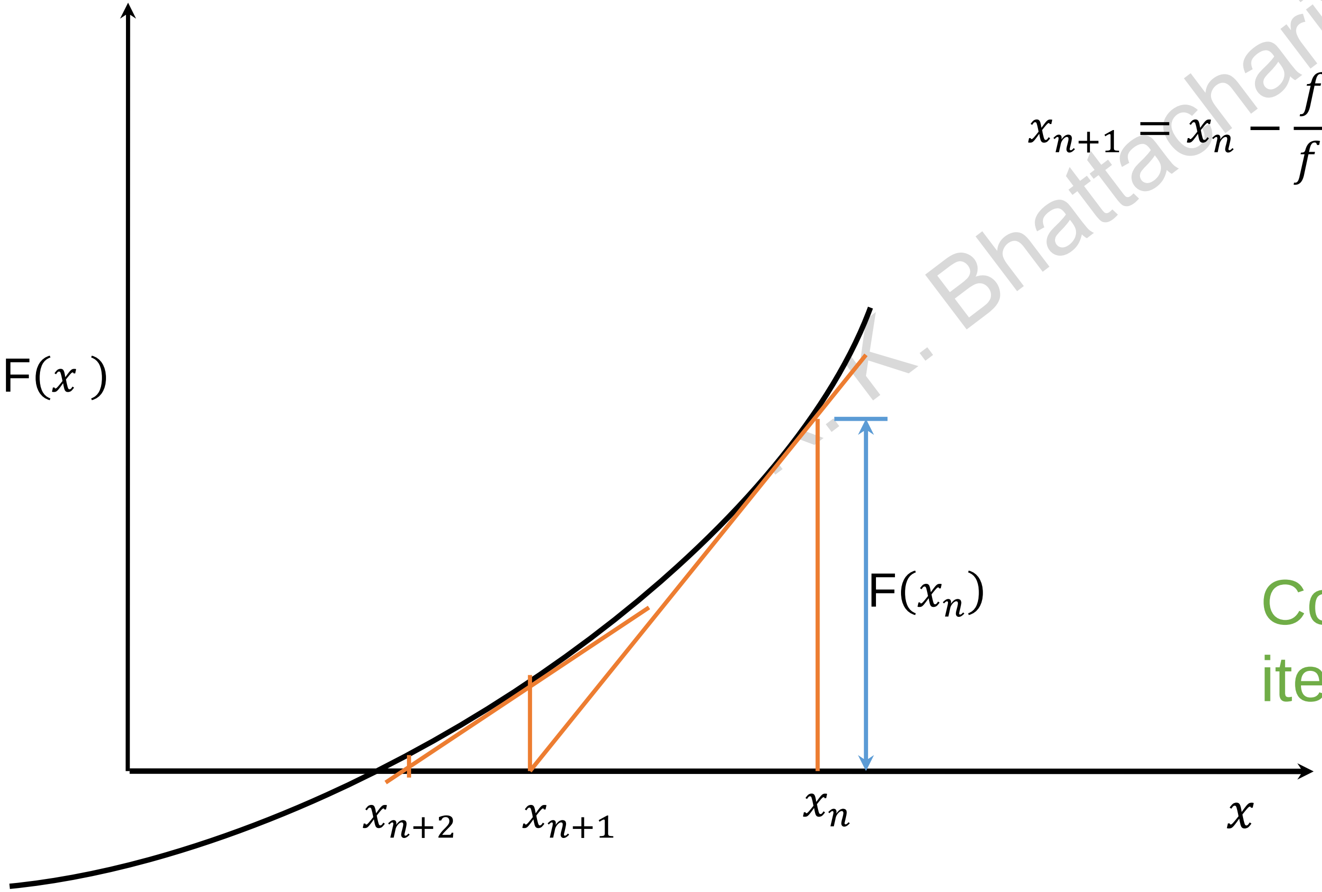
Newton-Raphson method

Incase optimization problem, $f'(x) = 0$

Considering $F(x) = f'(x)$

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$



Continue iteration

QUIZ

1. If $f(x)$ is an unimodal convex function in the interval $[a, b]$, then $f'(a) \times f'(b)$ is
a) Positive
b) Negative
c) It may be negative or may be positive
d) None of the above

Ans. b

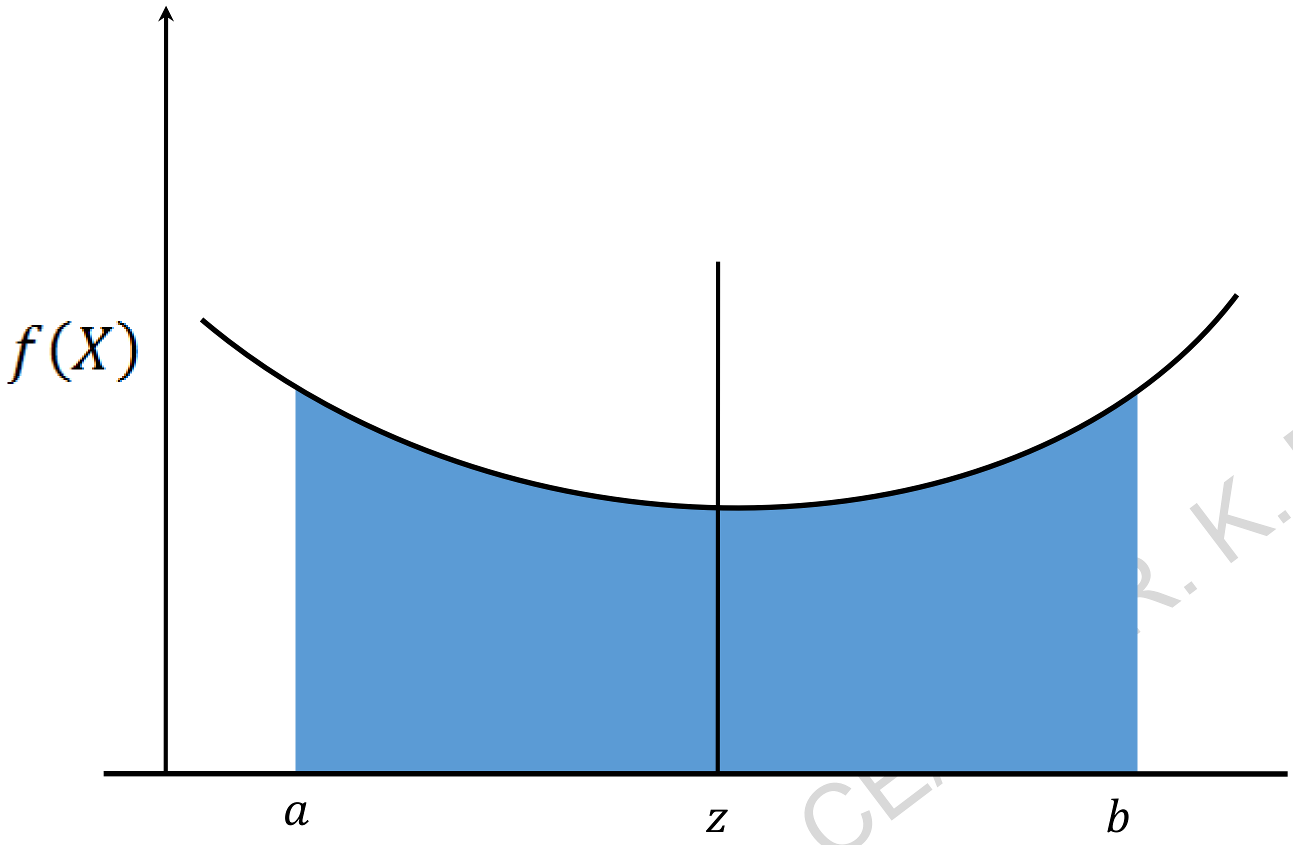
2. For the same function, take any point c between $[a, b]$. If $f'(c)$ is less than 0, then minima is not within the range
a) $[a, c]$
b) $[c, b]$
c) $[a, b]$
d) None of the above

Ans. a

2. For the same function, take any point c between $[a, b]$. If $f'(c)$ is greater than 0, then minima not within the range
a) $[a, c]$
b) $[c, b]$
c) $[a, b]$
d) None of the above

Ans. b

Bisection method



Take a point $z = \frac{a+b}{2}$

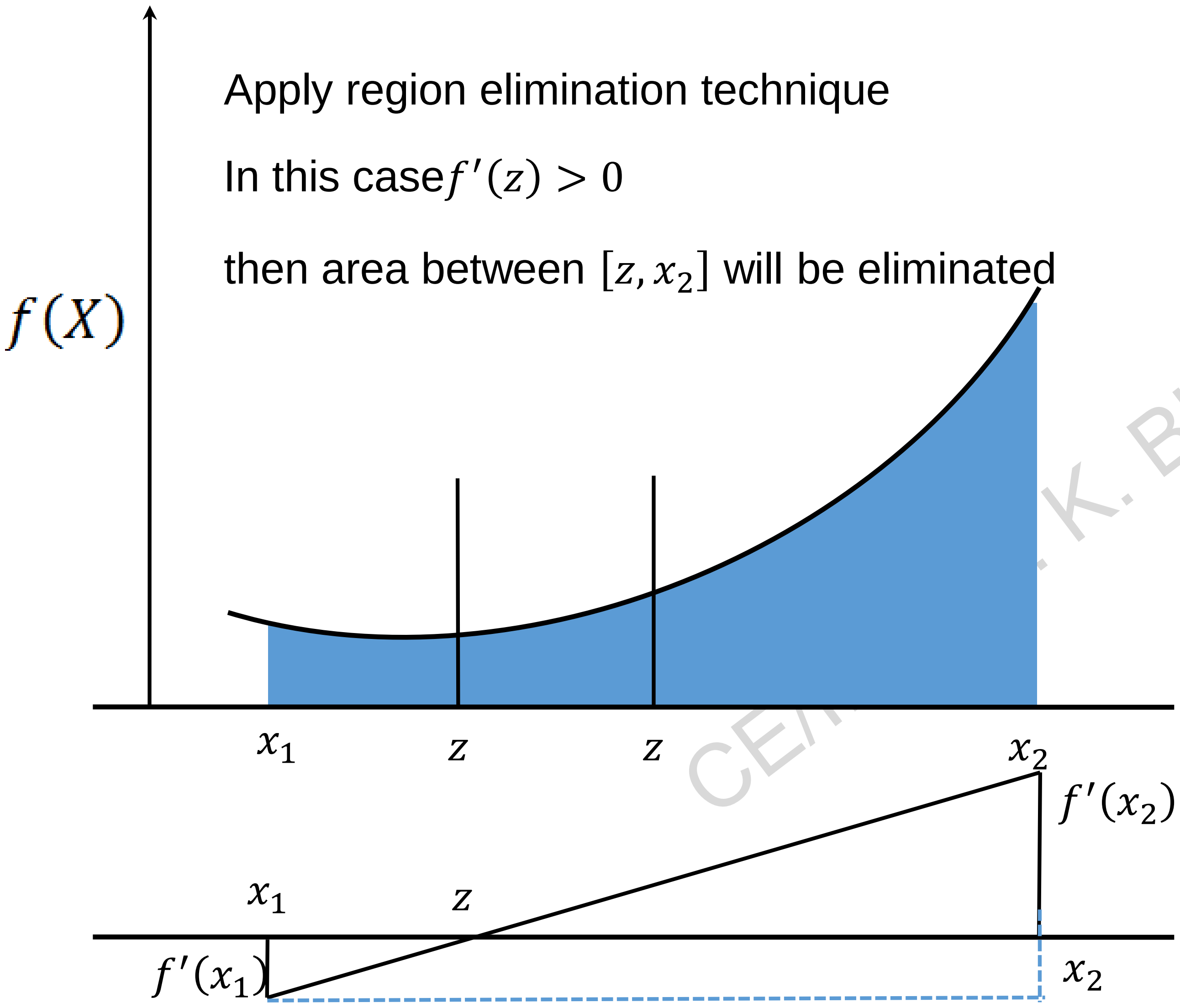
$if\, f'(z) < 0$ then area between $[a, z]$ will be eliminated

$if\, f'(z) > 0$ then area between $[z, b]$ will be eliminated

Disadvantage

- Magnitude of the derivatives is not considered

Bisection method

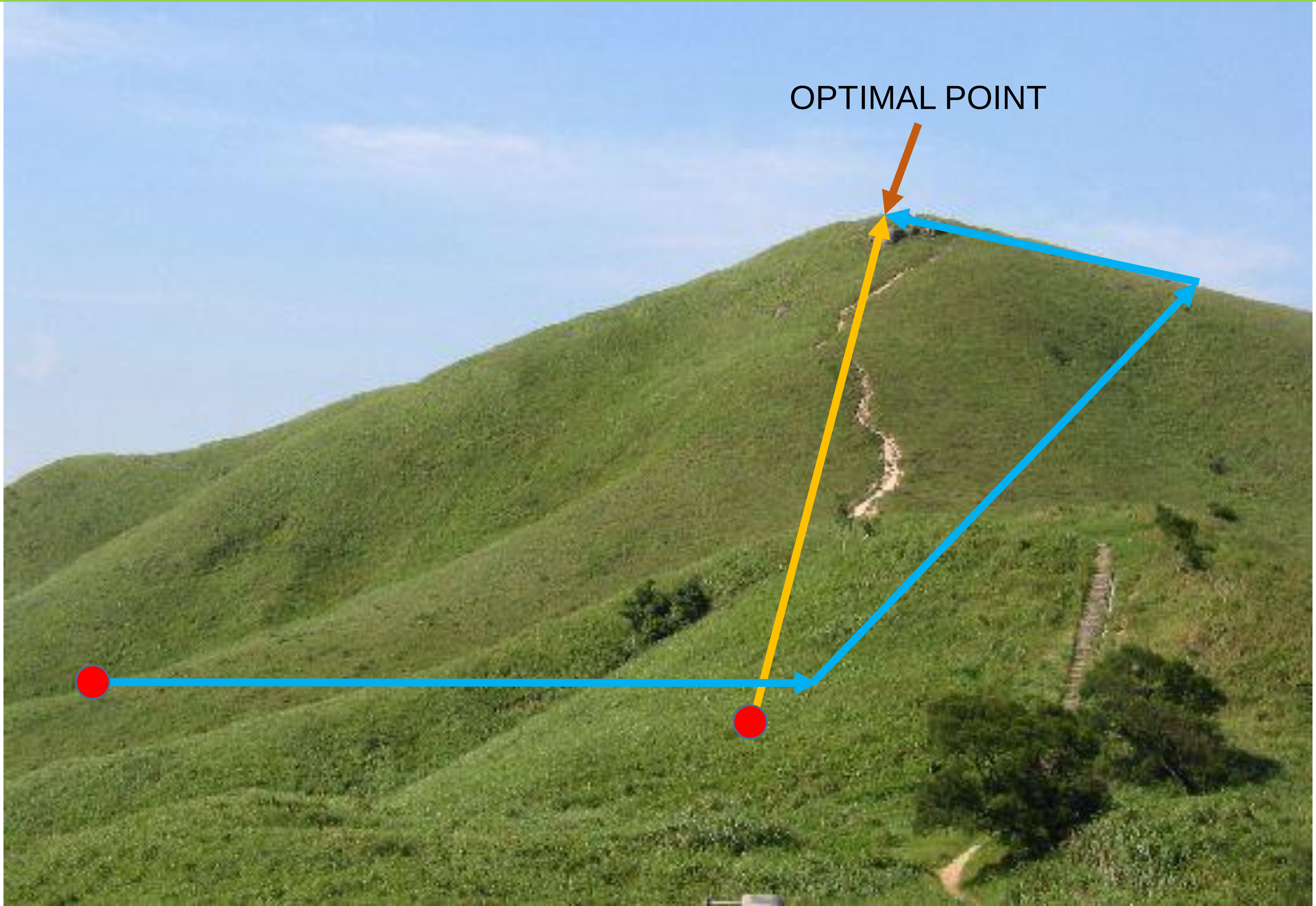


Considering similar triangle

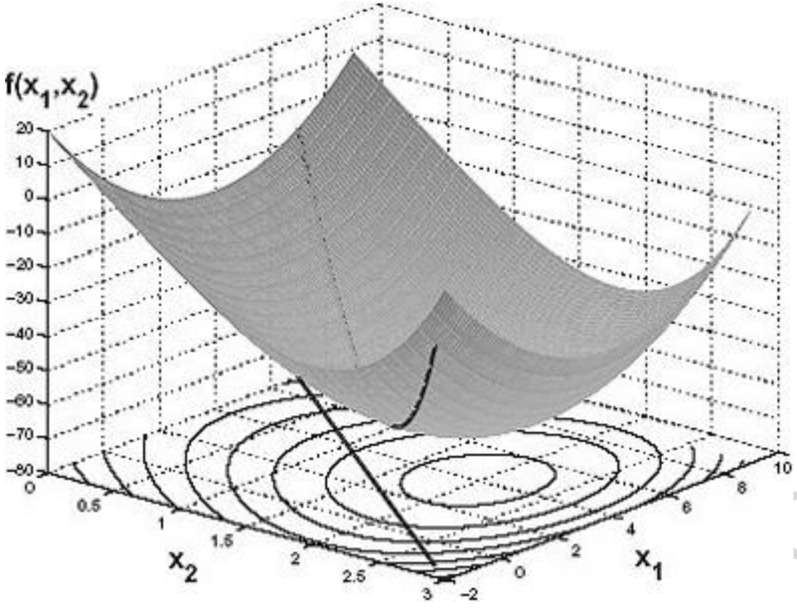
$$\frac{f'(x_2)}{x_2 - z} = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1}$$

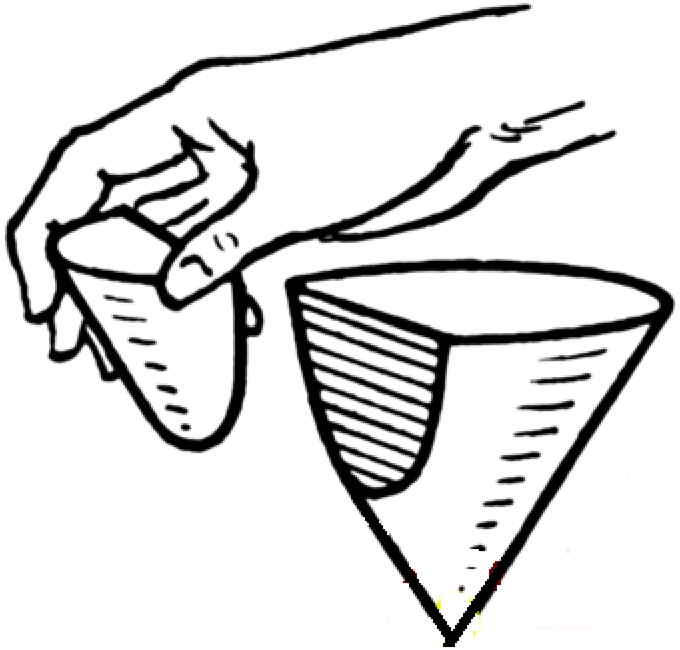
$$z = x_2 - \frac{f'(x_2)}{\frac{f'(x_2) - f'(x_1)}{x_2 - x_1}}$$

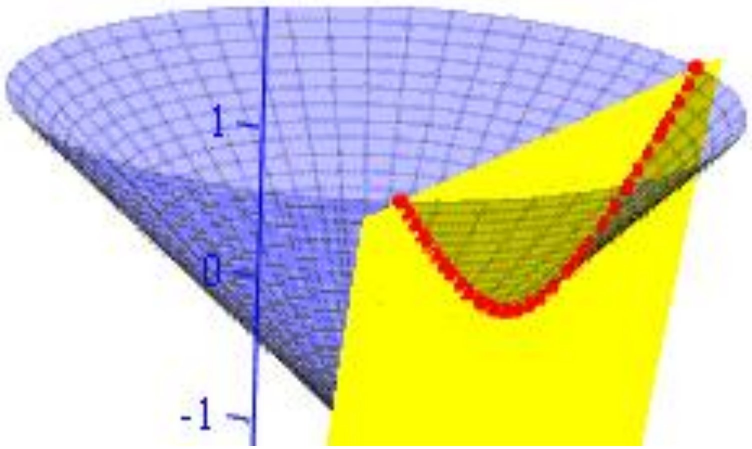
Multivariable problem



Multivariable problem







Multivariable problem

A multivariable problem can be converted to a single variable problem using the following equation

$$x^{t+1} = x^t + \alpha d^t$$

Take an example $f(x, y) = -(x^2 - y^2) + 4$

$$X^0 = \begin{pmatrix}1\\1\end{pmatrix} \quad d = \begin{pmatrix}1\\0\end{pmatrix} \quad X^1 = X^0 + \alpha d$$

$$X^1 = \begin{pmatrix}1\\1\end{pmatrix} + \alpha\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1+\alpha\\1\end{pmatrix}$$

Putting in equation (1)

$$f(\alpha) = -((1+\alpha)^2 - 1^2) + 4$$

Taking first derivative

$$f'(\alpha) = -2 - 2\alpha = 0$$

$$\alpha^* = -1 \qquad X^1 = \begin{pmatrix}1+\alpha\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$X^2 = X^1 + \alpha d$$

$$X^2 = \begin{pmatrix}0\\1\end{pmatrix} + \alpha\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1+\alpha\end{pmatrix}$$

Putting in equation (1)

$$f(\alpha) = -(0 + (1+\alpha)^2) + 4$$

Taking first derivative

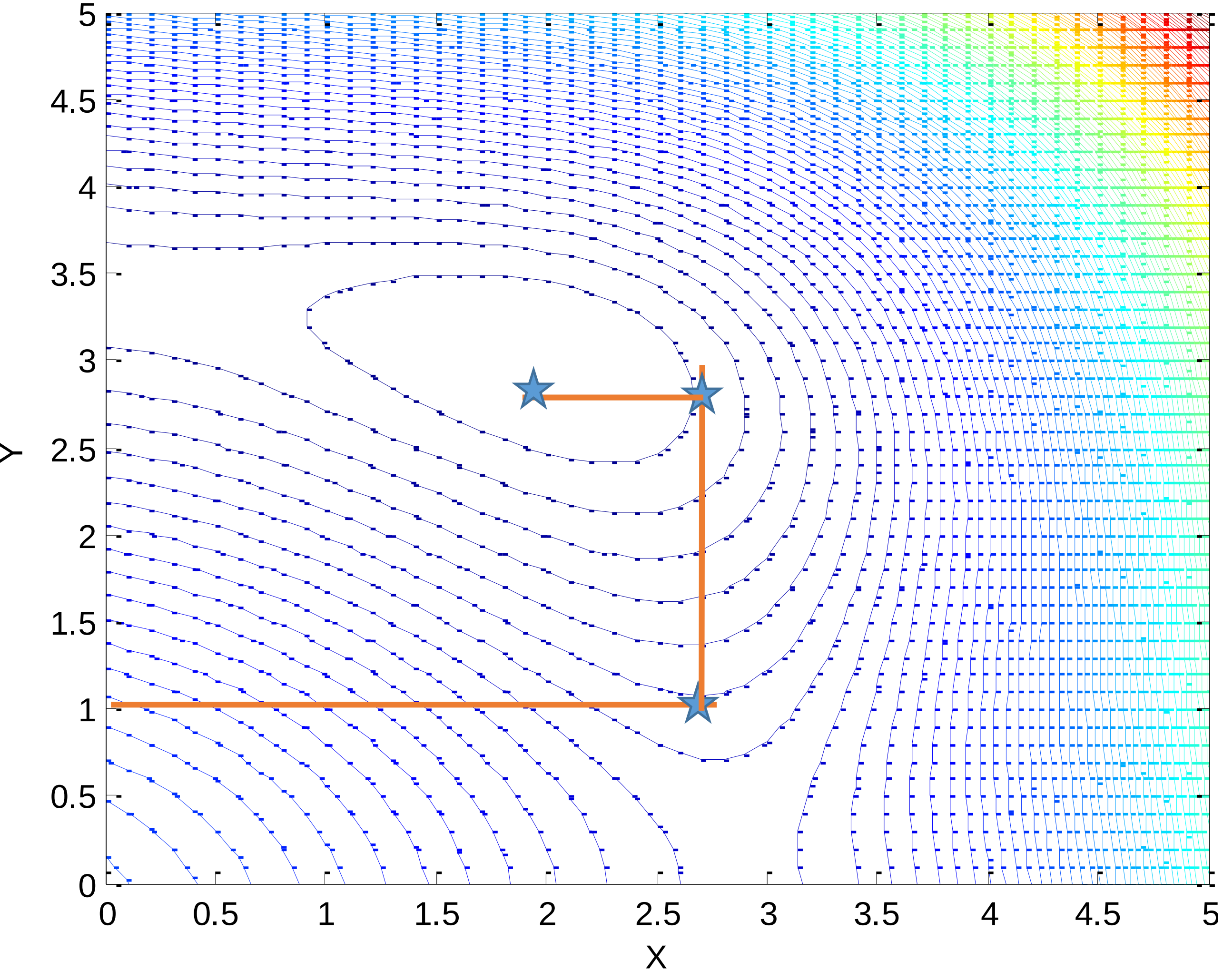
$$f'(\alpha) = -2 - 2\alpha = 0$$

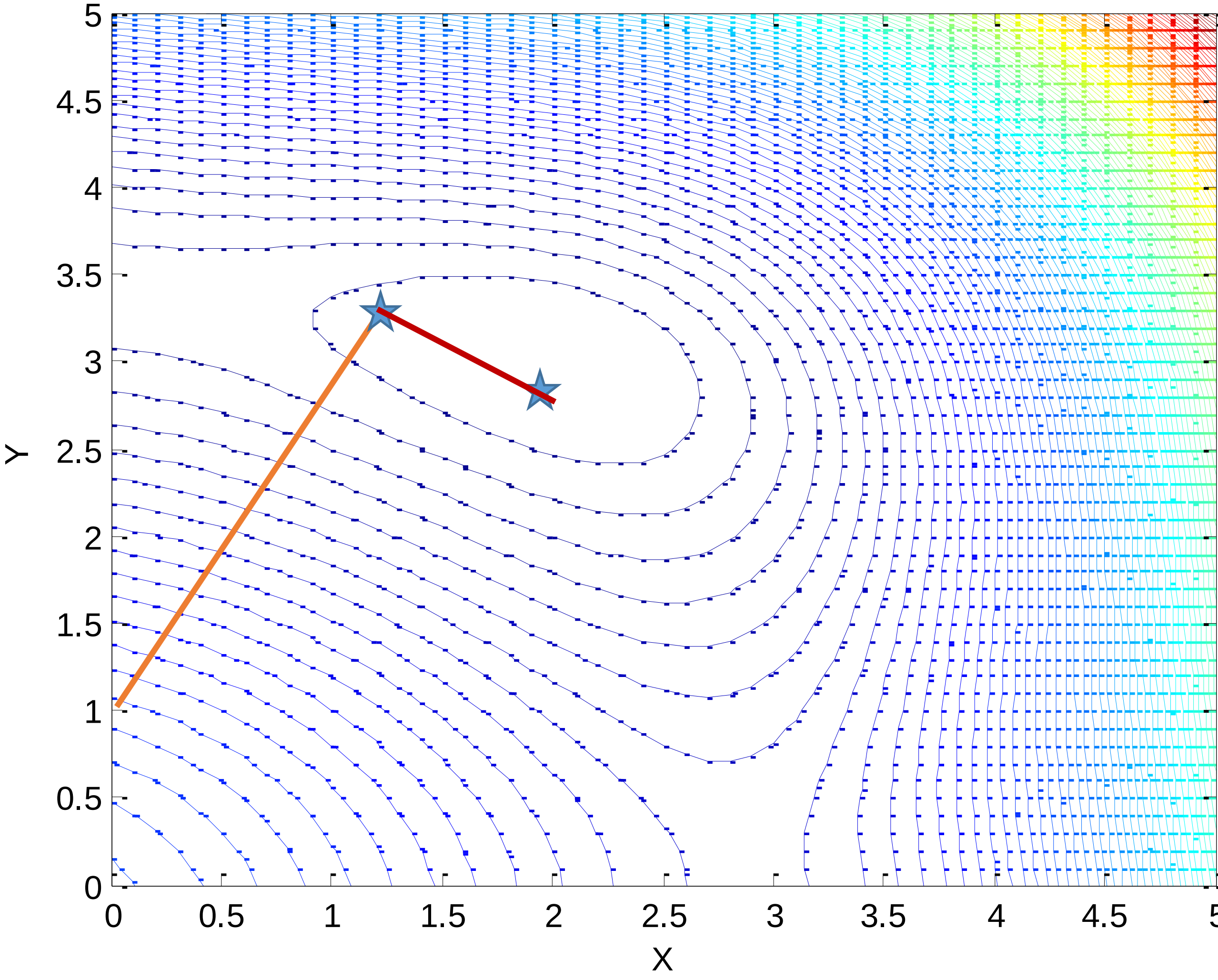
$$\alpha^* = -1$$

$$X^2 = \begin{pmatrix}0\\1+\alpha\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

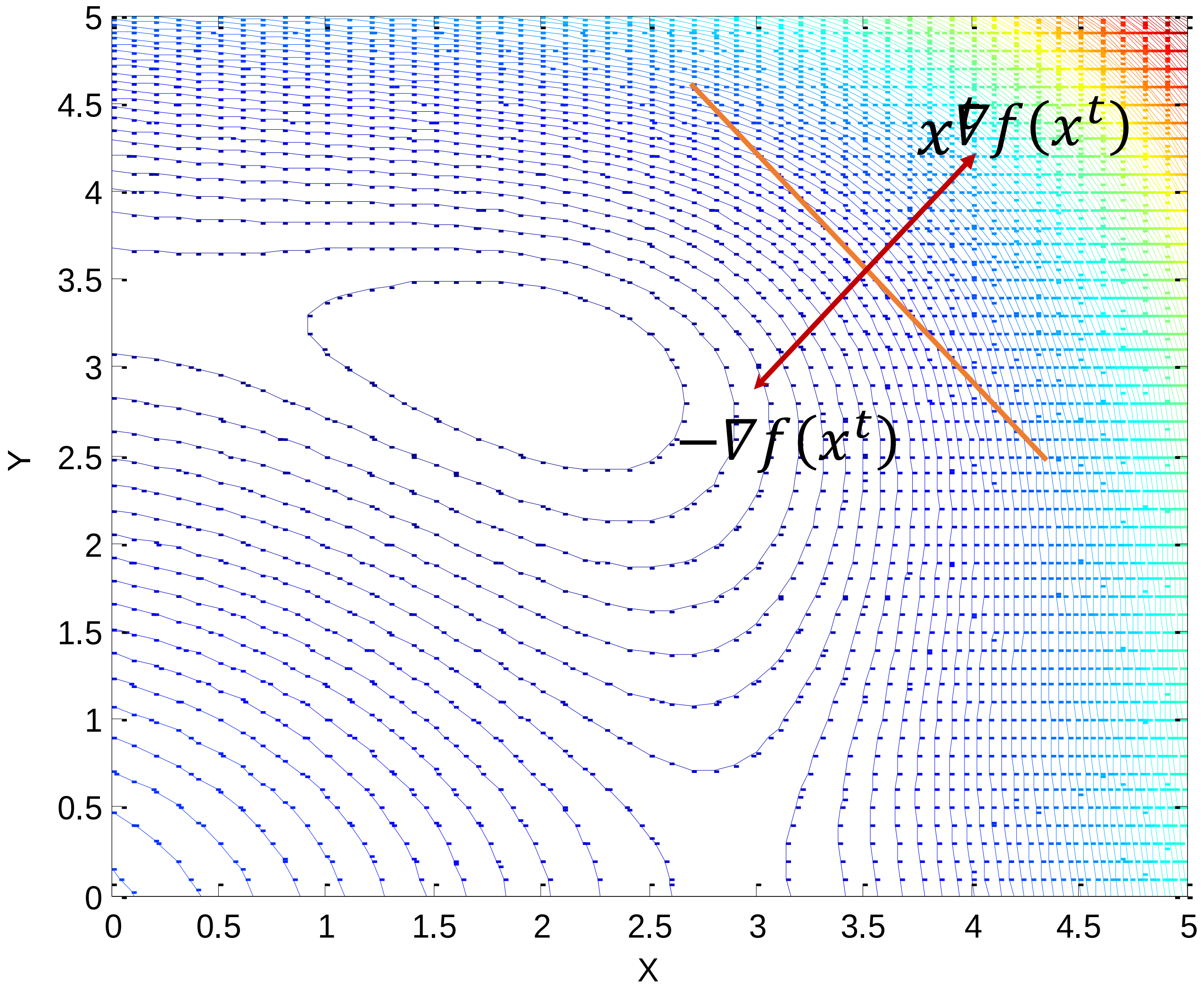
OPTIMAL SOLUTION

Multivariable problem





Multivariable problem



Descent direction

A search direction d^t is a descent direction at point x^t if the condition $\nabla f(x^t).d^t < 0$ is satisfied in the vicinity of the point x^t.

$$f(x^{t+1}) = f(x^t + \alpha d^t)$$
$$= f(x^t) + \alpha \nabla^T f(x^t).d^t$$

The $f(x^{t+1}) < f(x^t)$

When $\alpha \nabla^T f(x^t).d^t < 0$

Or, $\nabla^T f(x^t).d^t < 0$

Steepest descent direction

Newton's method for multi-variable problem

Taylor series $f(X+h) = f(X) + h^T \nabla f(X) + \frac{1}{2!} h^T H h + \cdots$

$$f(X_{i+1}) = f(X_i) + \nabla f(X_i)^T (X_{i+1} - X_i) + \frac{1}{2!} (X_{i+1} - X_i)^T H (X_{i+1} - X_i) + \cdots$$

By setting partial derivative of the equation to zero for minimization of $f(X)$, we have

$$\nabla f = 0 + \nabla f(X_i) + H(X_{i+1} - X_i) = 0$$

$$\boldsymbol{X_{i+1} = X_i - H^{-1} \nabla f}$$

Since higher order derivative terms have been neglected, the above equation can be iteratively used to find the value of the optimal solution

$$\frac{\partial (X^T A X)}{\partial X} = AX + A^T X$$

In this case

$$\frac{\partial (X^T A X)}{\partial X} = 2AX$$

$$\frac{\partial (AX)}{\partial X} = A^T$$

$$\frac{\partial (X^T A)}{\partial X} = A$$

$$\frac{\partial (A^T X)}{\partial X} = A$$

Transformation method

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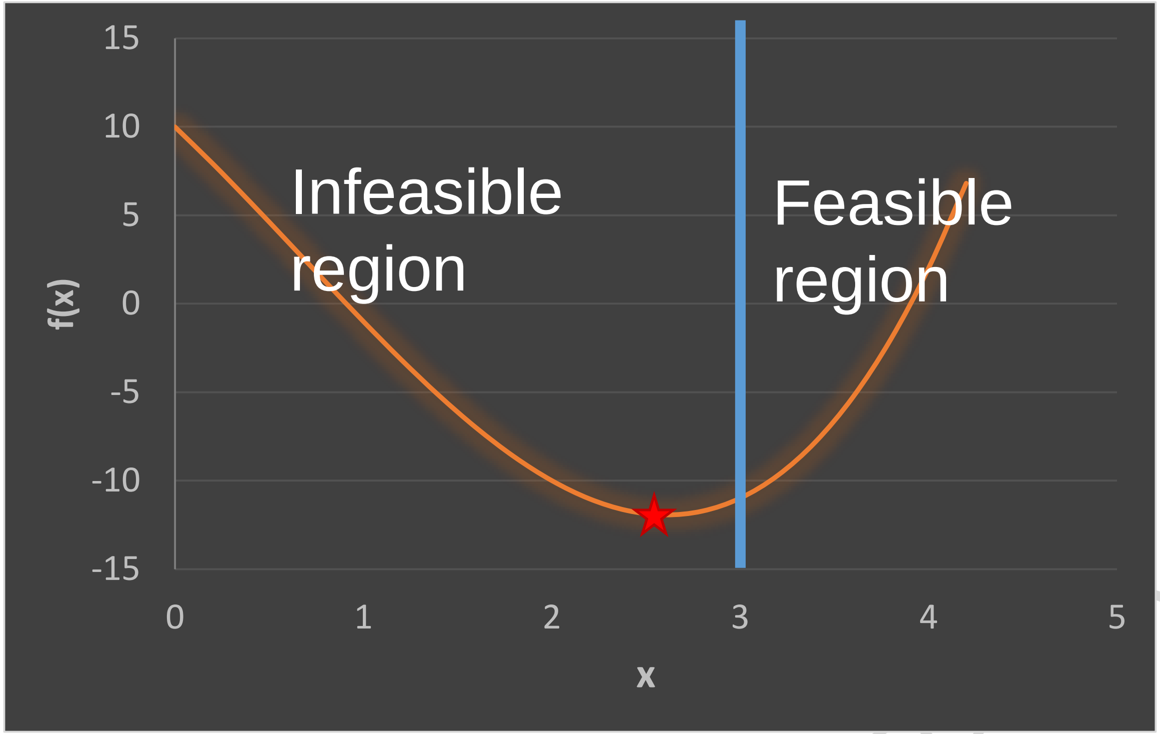


Minimize

$f(x) = x^3 - 10x - 2x^2 + 10$

Subject to $g(x) = x \geq 3$

Or, $g(x) = x - 3 \geq 0$

The problem can be written as

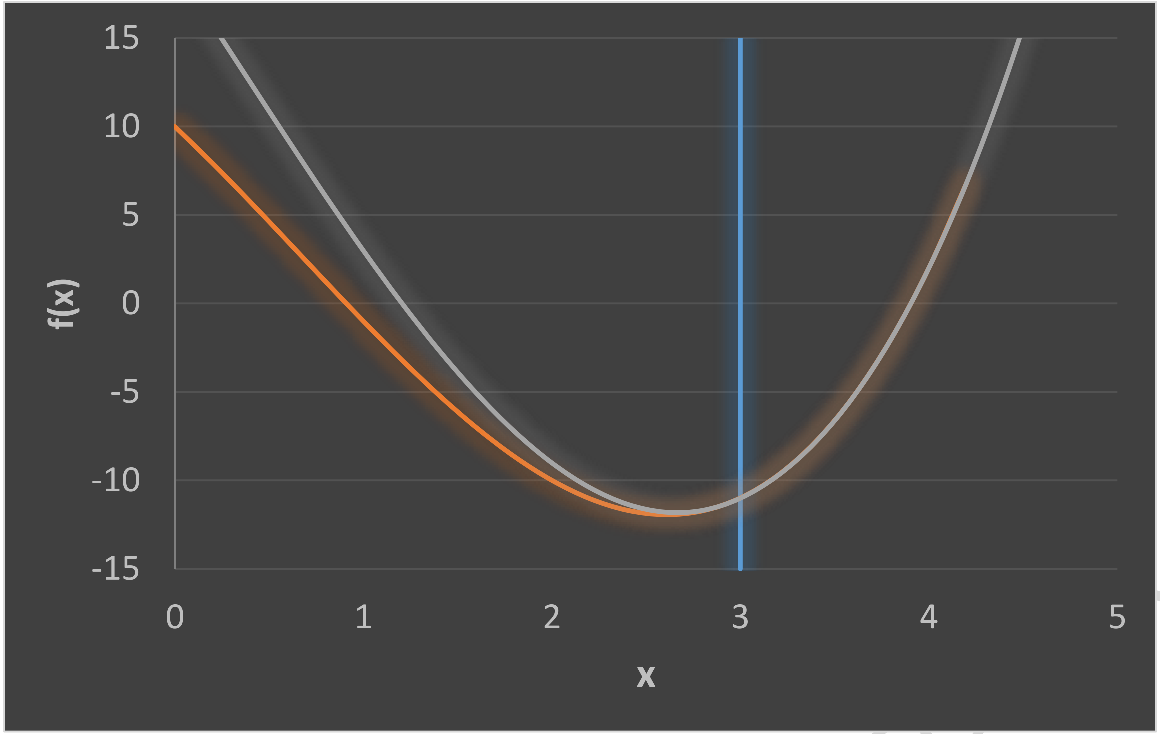
$\mathrm{F}(x, R) = f(x) + R\langle g(x)\rangle^2$

Where,

$\langle g(x)\rangle = 0$ if $x \geq 3$

$\langle g(x)\rangle = g(x)$ otherwise

The bracket operator $\langle \quad \rangle$ can be implemented using $min(g, 0)$ function



Minimize

$$f(x) = x^3 - 10x - 2x^2 + 10$$

Subject to $g(x) = x \geq 3$

Or, $g(x) = x - 3 \geq 0$

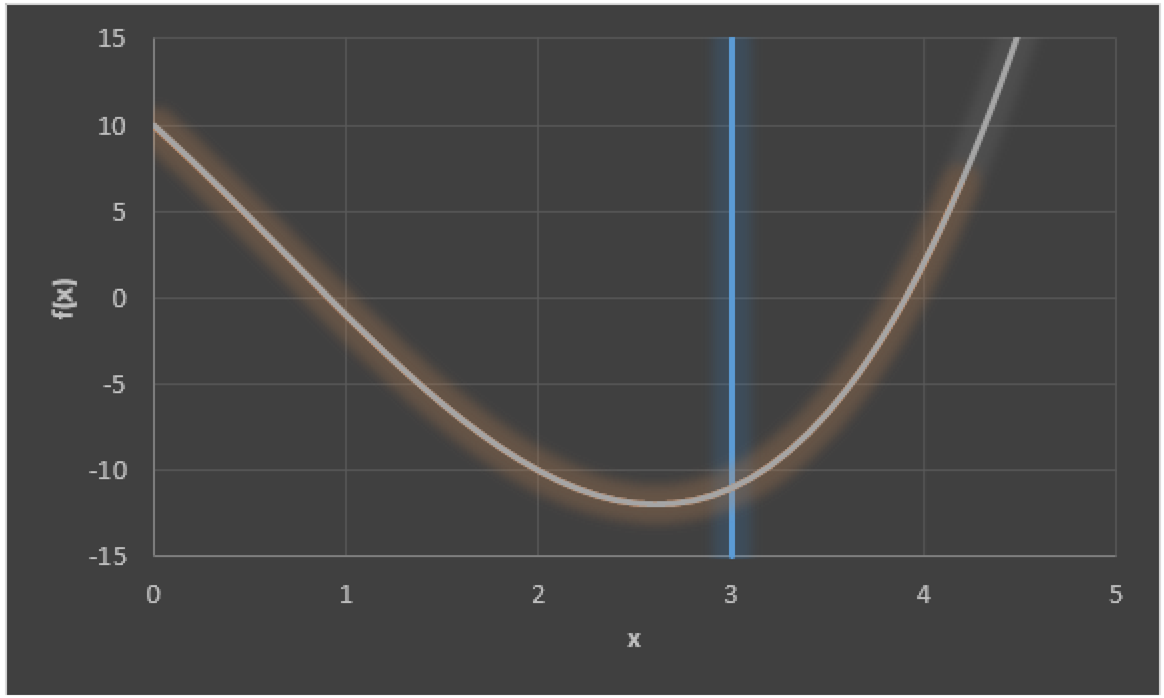
The problem can be written as

$$\text{F}(x, R) = (x^3 - 10x - 2x^2 + 10) + R\langle x - 3\rangle^2$$

$$\text{F}(x, R) = (x^3 - 10x - 2x^2 + 10) + R\big(min(x - 3,0)\big)^2$$

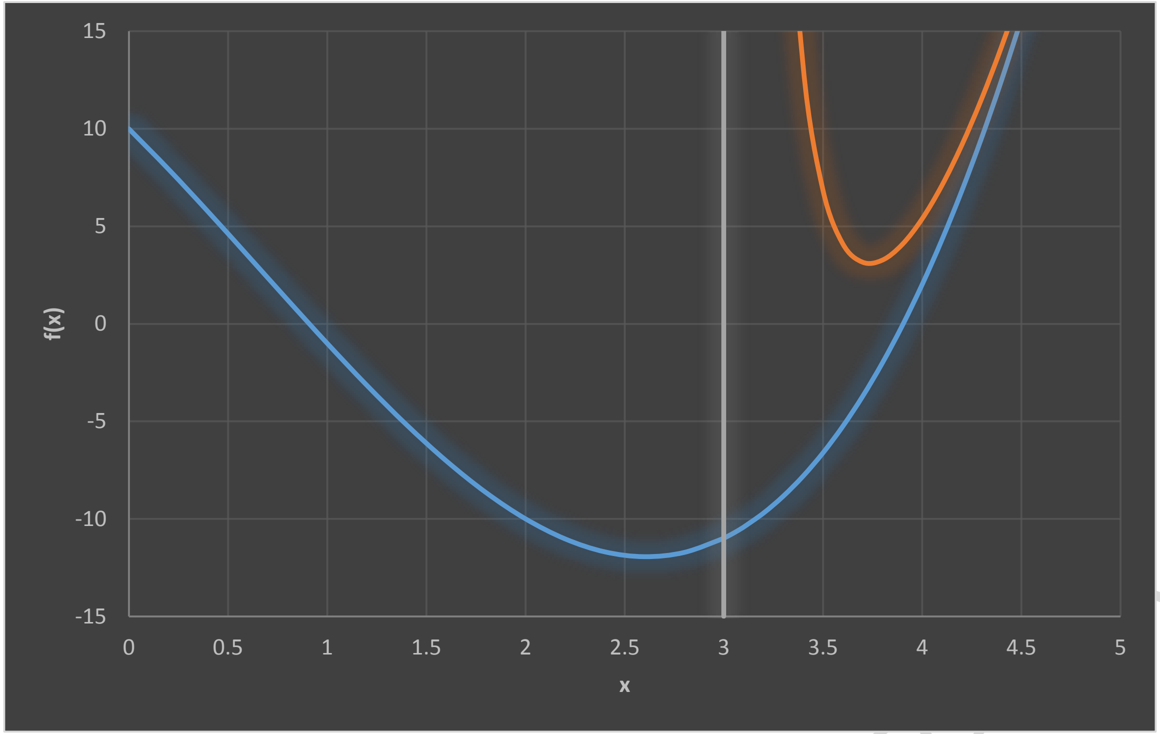
Minimize $F(x,R) = (x^3 - 10x - 2x^2 + 10) + R\left(min(x-3,0)\right)^2$

R 0



By changing R value, it is possible to avoid the infeasible solution

The minimization of the transformed function will provide the optimal solution which is in the feasible region only



Minimize

$$f(x) = x^3 - 10x - 2x^2 + 10$$

Subject to $g(x) = x \geq 3$

Or, $g(x) = x - 3 \geq 0$

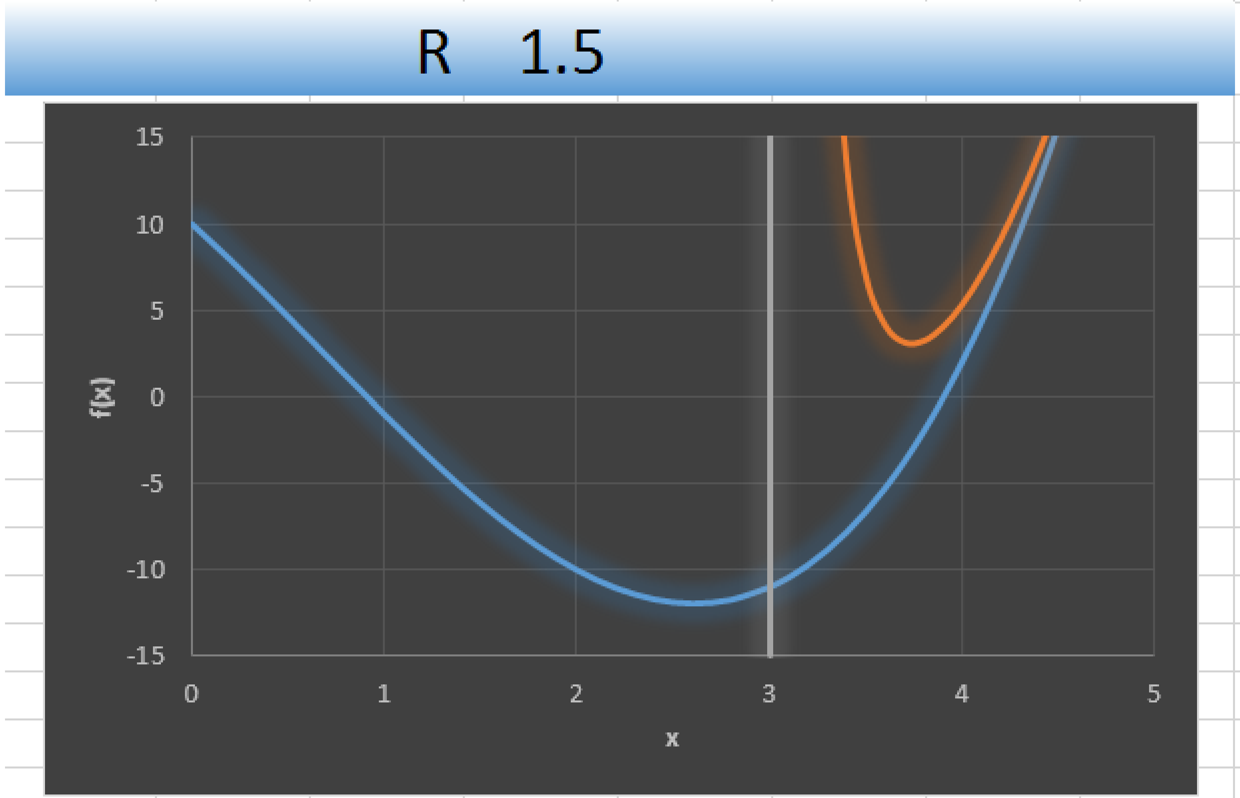
The problem can also be converted as

$$\text{F}(x, R) = (x^3 - 10x - 2x^2 + 10) + R\frac{1}{g(x)}$$

This term is added in feasible side only

$$\text{F}(x, R) = (x^3 - 10x - 2x^2 + 10) + R\frac{1}{(x-3)}$$

Minimize $\text{F}(x, R) = (x^3 - 10x - 2x^2 + 10) + R\frac{1}{(x-3)}$

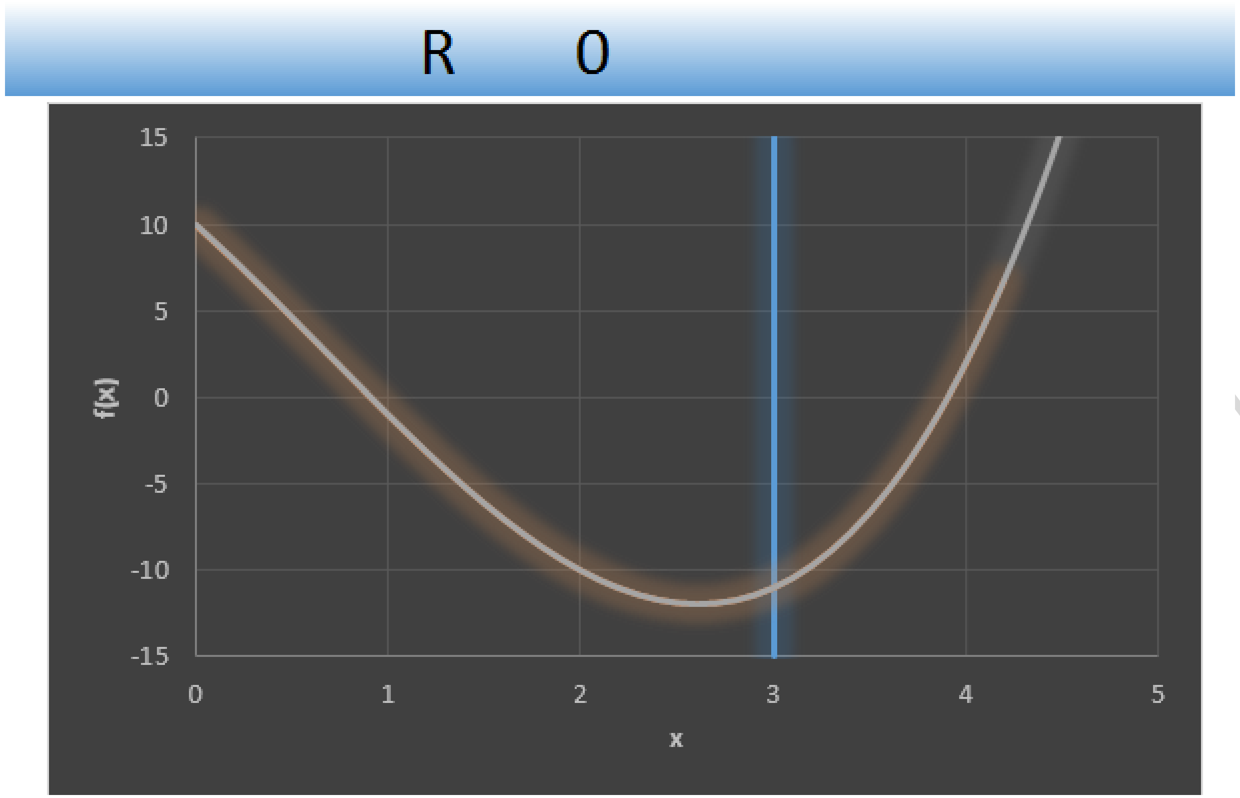


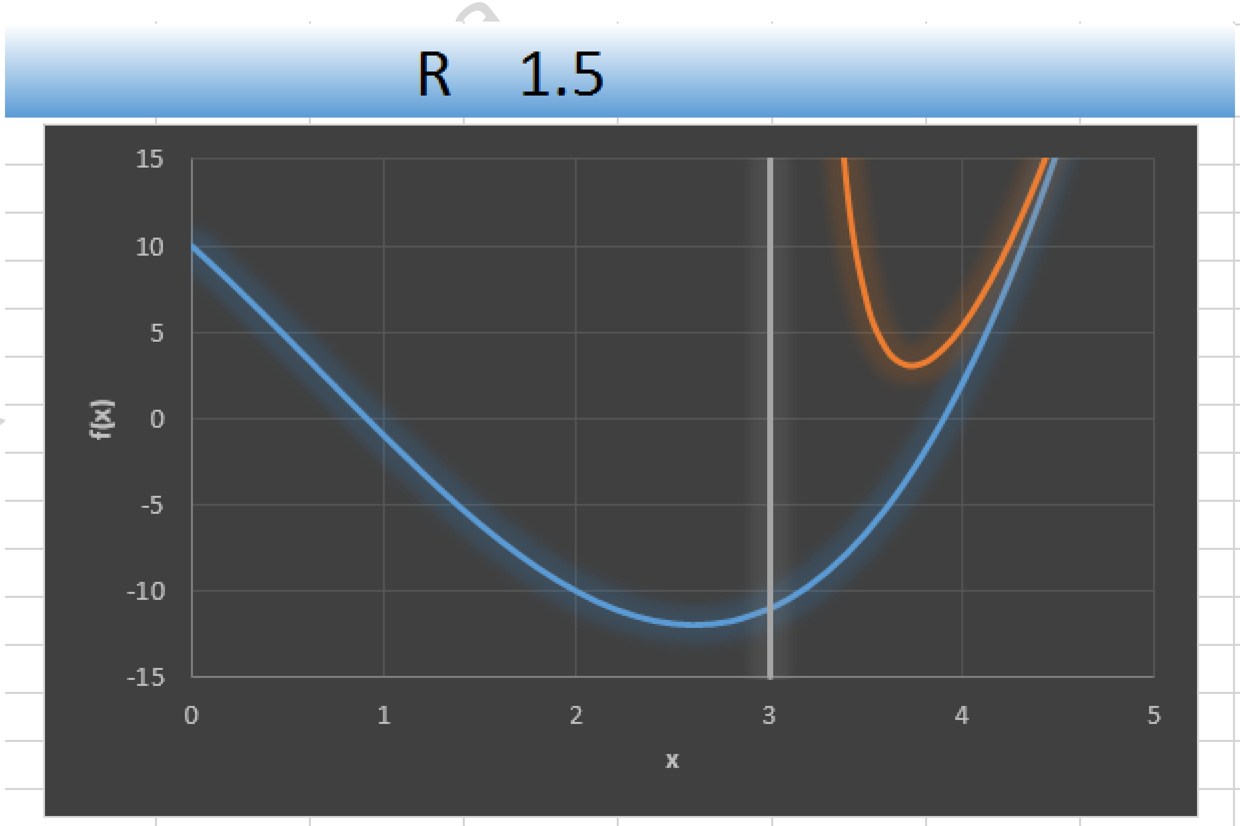
By changing R value, it is possible to avoid the infeasible solution

The minimization of the transformed function will provide the optimal solution which is in the feasible region only

Exterior penalty method

Interior penalty method





The transformation function can be written as

$$F(X,R) = f(X) + \Psi\big(g(X), h(X)\big)$$

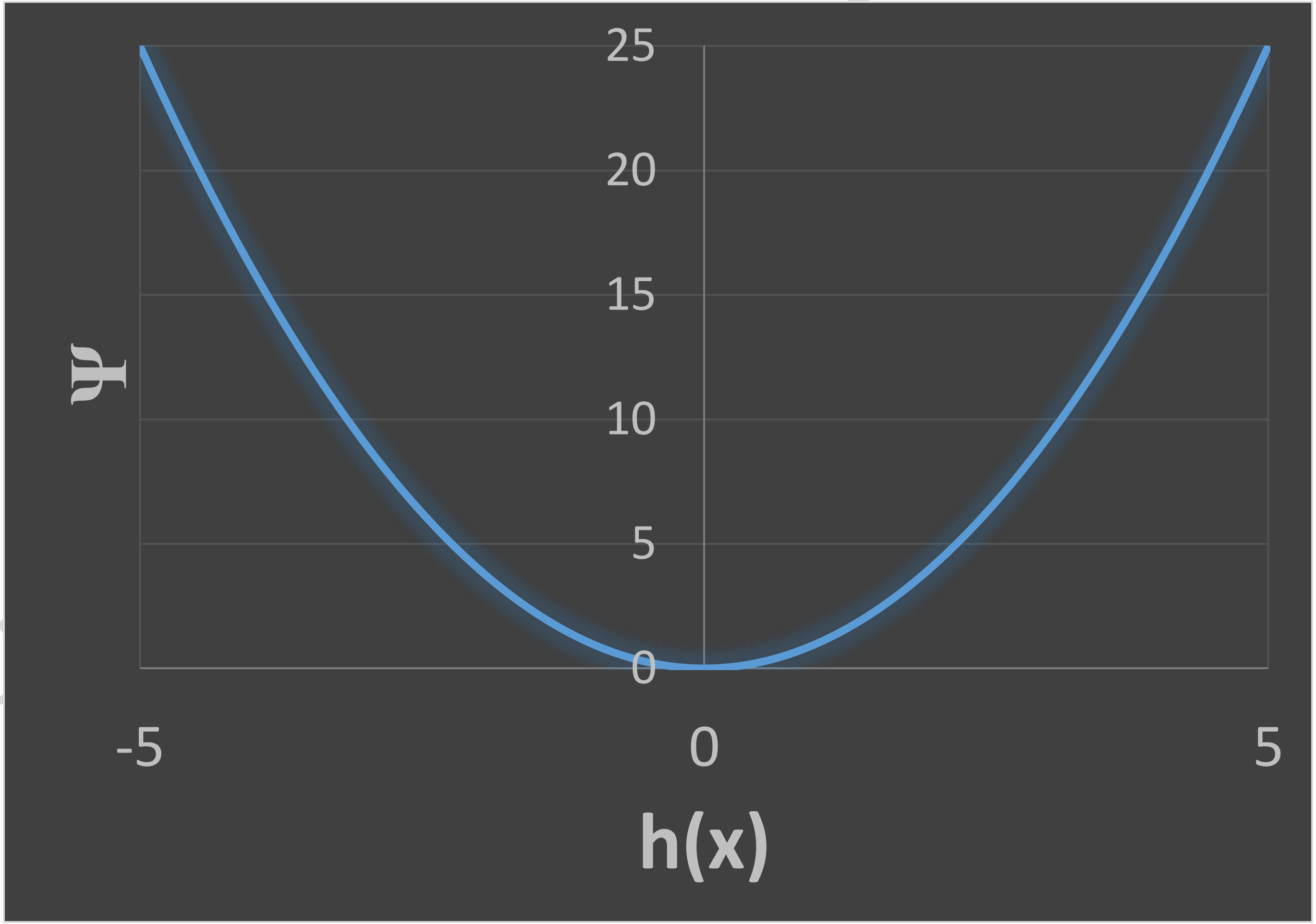
This term is called Penalty term

R is called penalty parameter

Penalty terms

Parabolic penalty

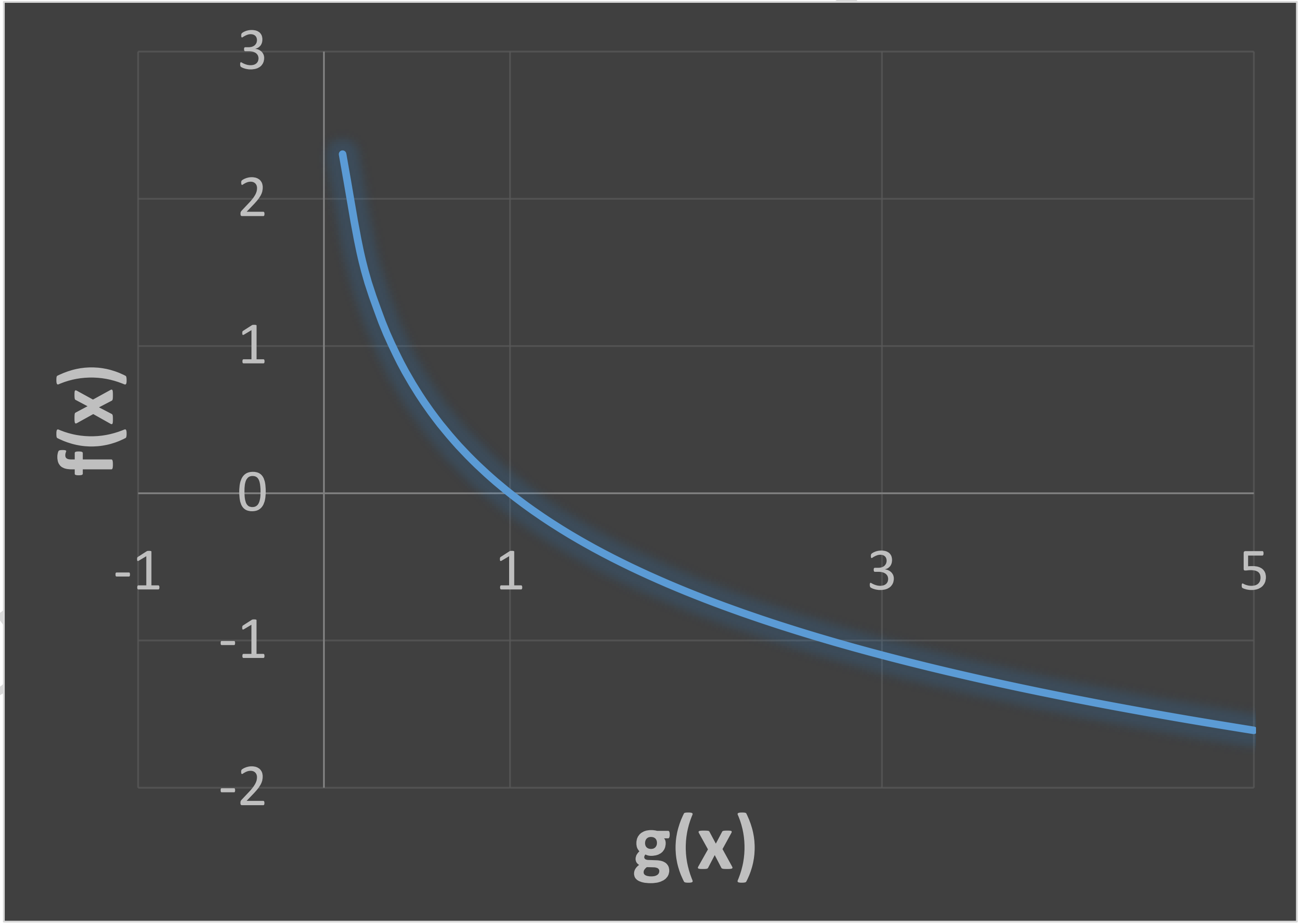
$$\Psi = R[h(x)]$$



Penalty terms

Log penalty

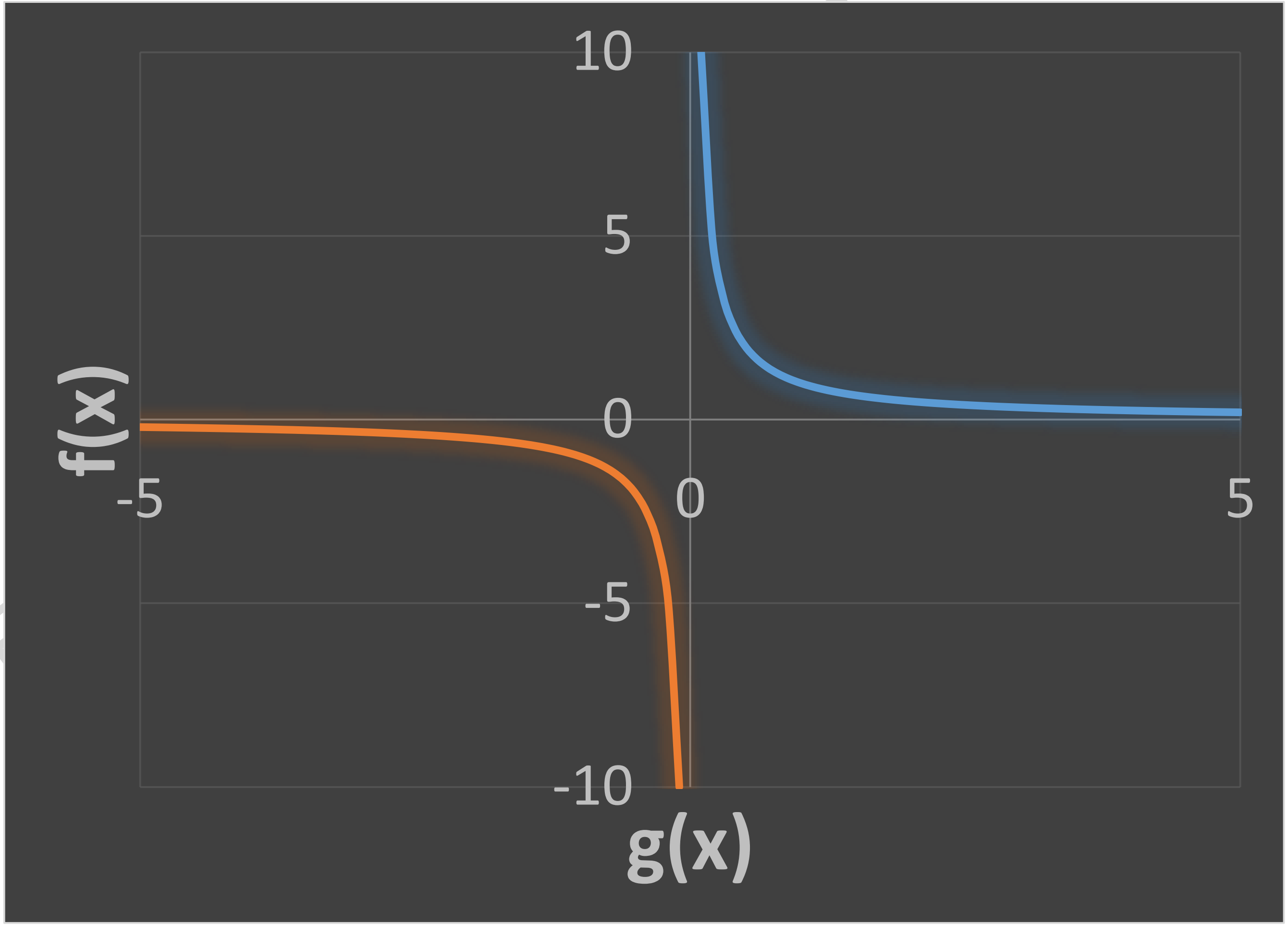
$$\Psi = -R ln[g(x)]$$



Penalty terms

Inverse penalty

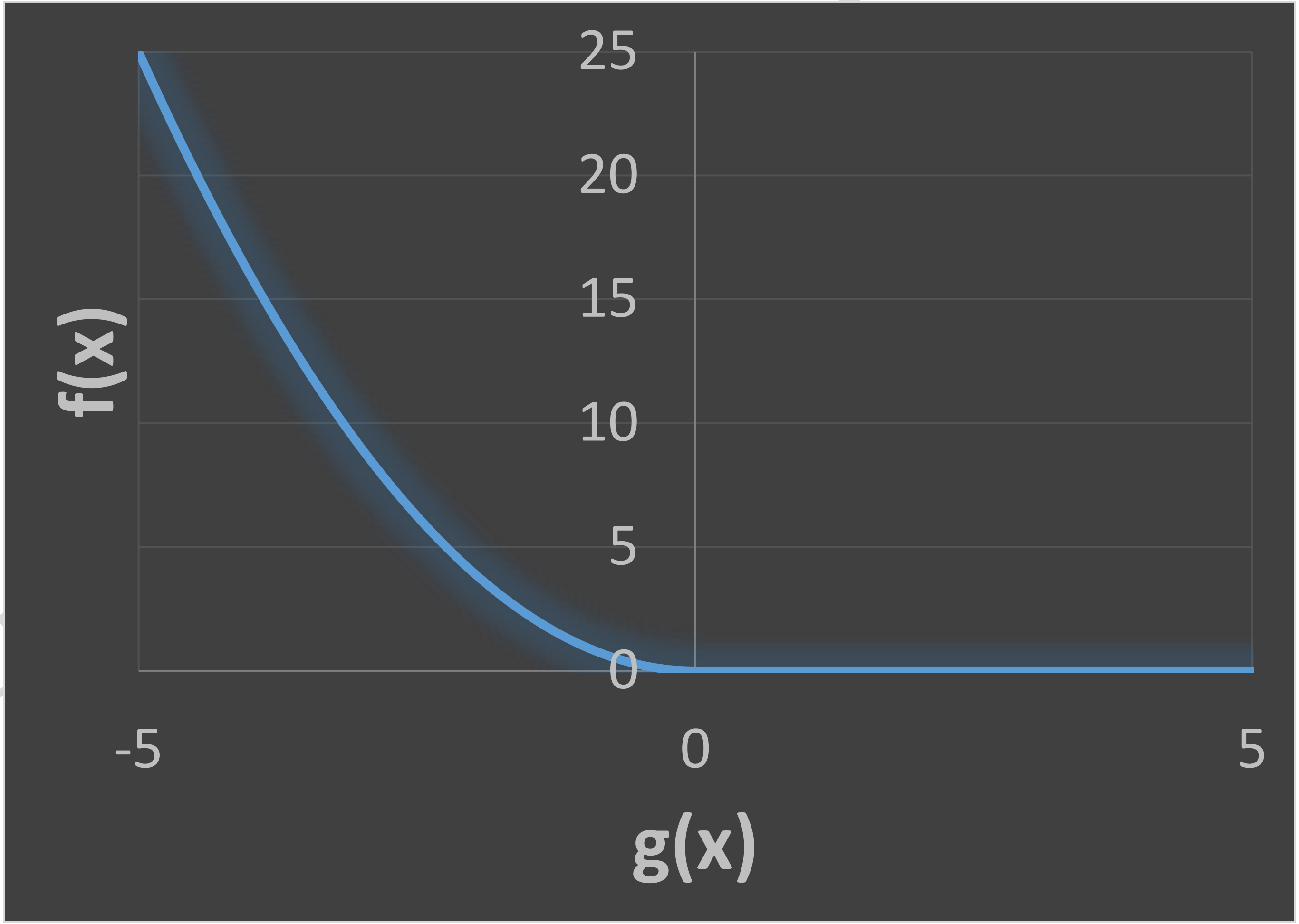
$$\Psi = -R\left[\frac{1}{g(x)}\right]$$



Penalty terms

Bracket operator

$$\Psi = R\langle g(x) \rangle$$



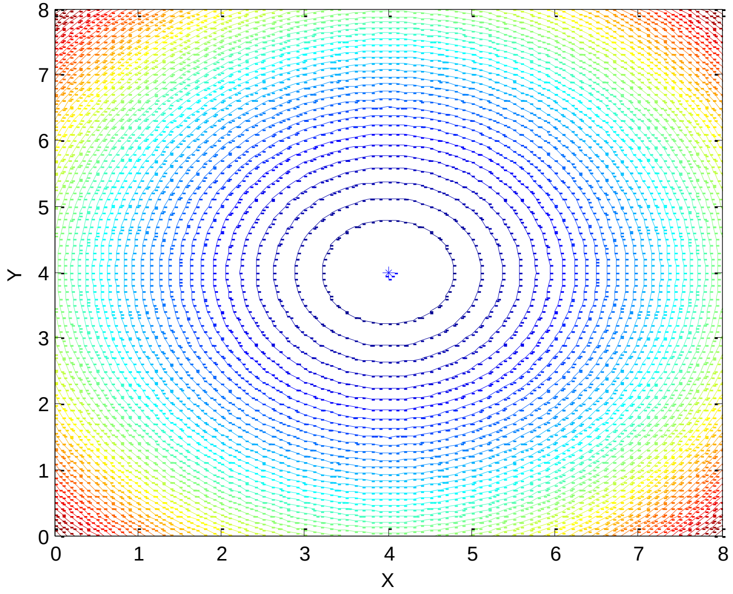
Take an example

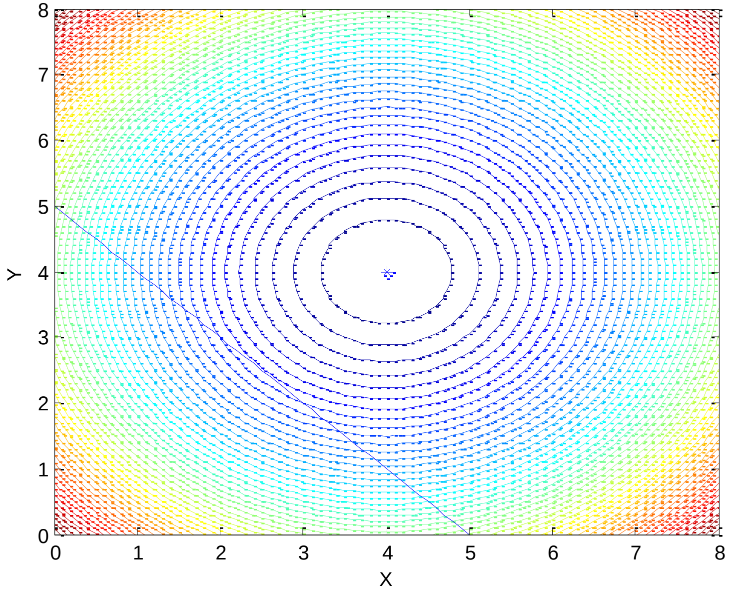
Minimize $f = (x_1 - 4)^2 + (x_2 - 4)^2$

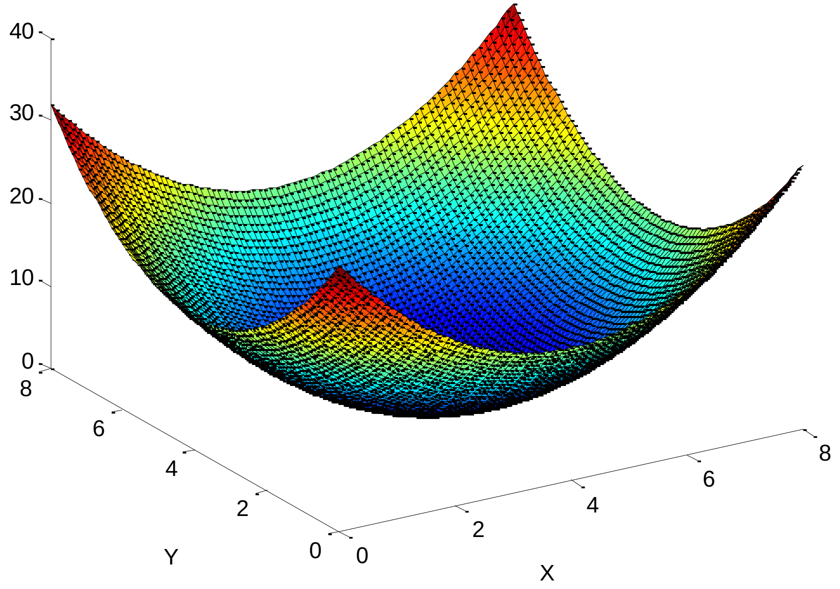
Subject to $g = x_1 + x_2 - 5$

The transform function can be written as

Minimize $\mathrm{F} = (x_1 - 4)^2 + (x_2 - 4)^2 + R(x_1 + x_2 - 5\ )^2$



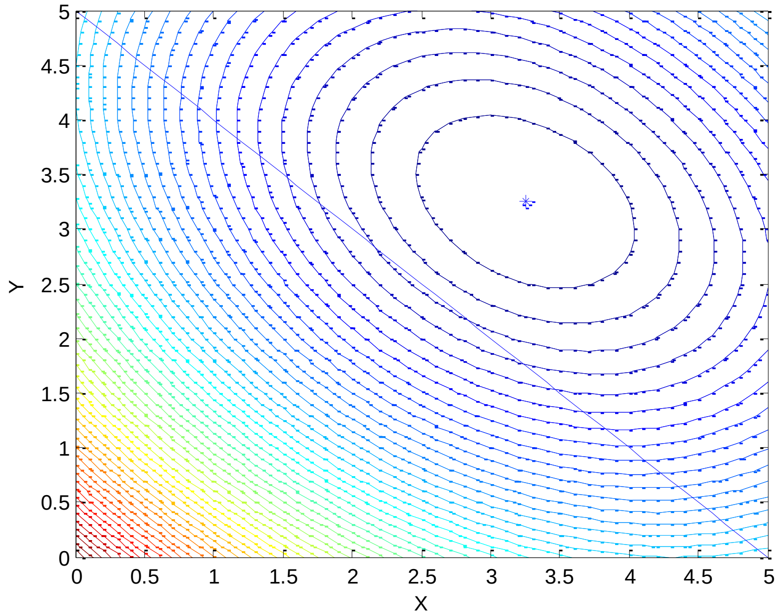




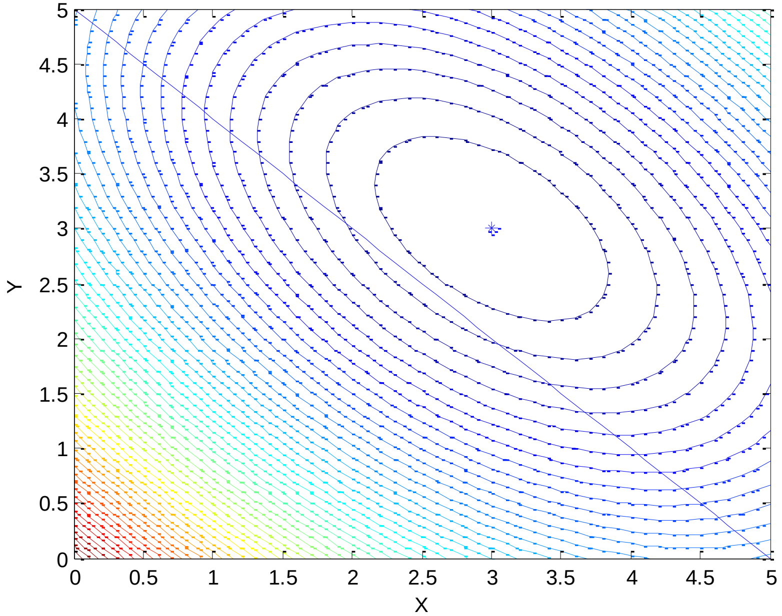
Minimize $f = (x_1 - 4)^2 + (x_2 - 4)^2$

Subject to $g = x_1 + x_2 - 5$

Minimize $\mathrm{F} = (x_1 - 4)^2 + (x_2 - 4)^2 + R(x_1 + x_2 - 5\ )^2$



$R = 0.5$



$R = 1$

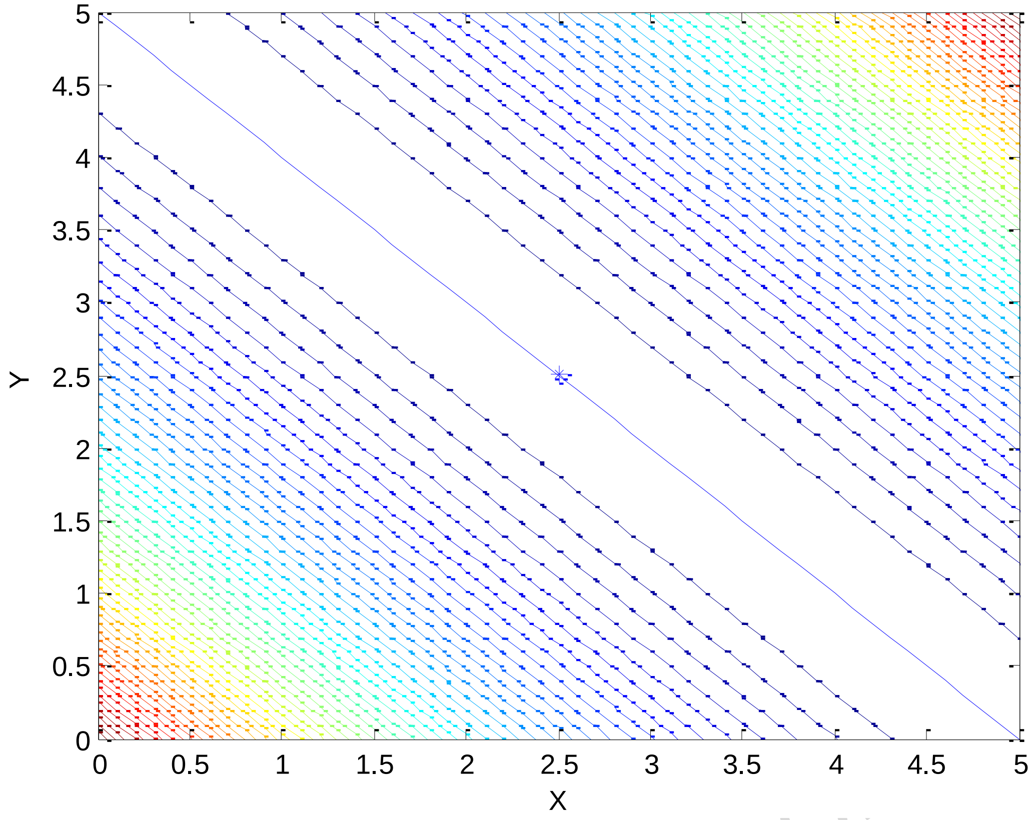
Optimal solution is

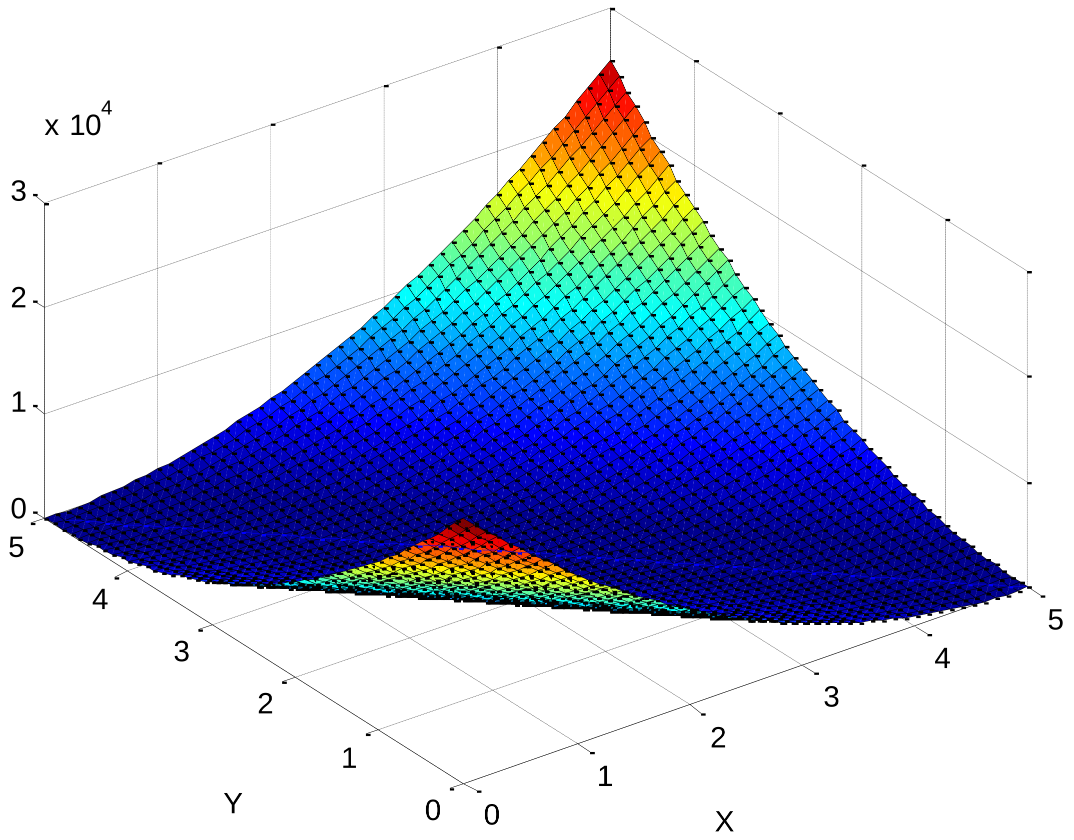
3.250	3.250

Optimal solution is

3.000	3.000

R	x1	x2	f(x)	h(x)	F
0	4.000	4.000	0.000	3.000	0.000
0.5	3.250	3.250	1.125	1.500	2.250
1	3.000	3.000	2.000	1.000	3.000
5	2.636	2.636	3.719	0.273	4.091
10	2.571	2.571	4.082	0.143	4.286
20	2.537	2.537	4.283	0.073	4.390
30	2.525	2.525	4.354	0.049	4.426
50	2.515	2.515	4.411	0.030	4.455
100	2.507	2.507	4.455	0.015	4.478
200	2.504	2.504	4.478	0.007	4.489
500	2.501	2.501	4.491	0.003	4.496
1000	2.501	2.501	4.496	0.001	4.498
10000	2.500	2.500	4.500	0.000	4.500





$R = 1000$

Optimal solution is 2.501 2.501

Quadratic approximation

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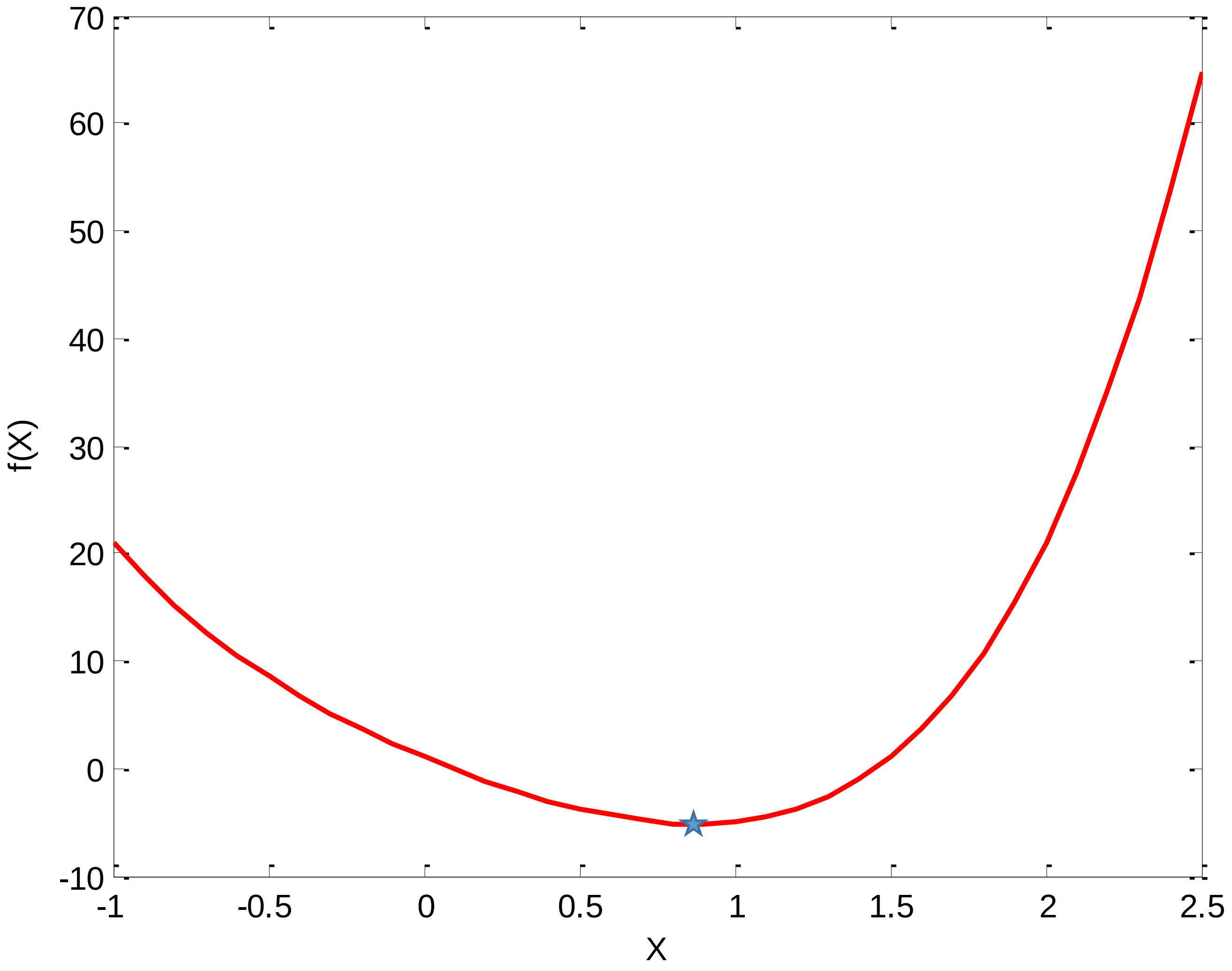
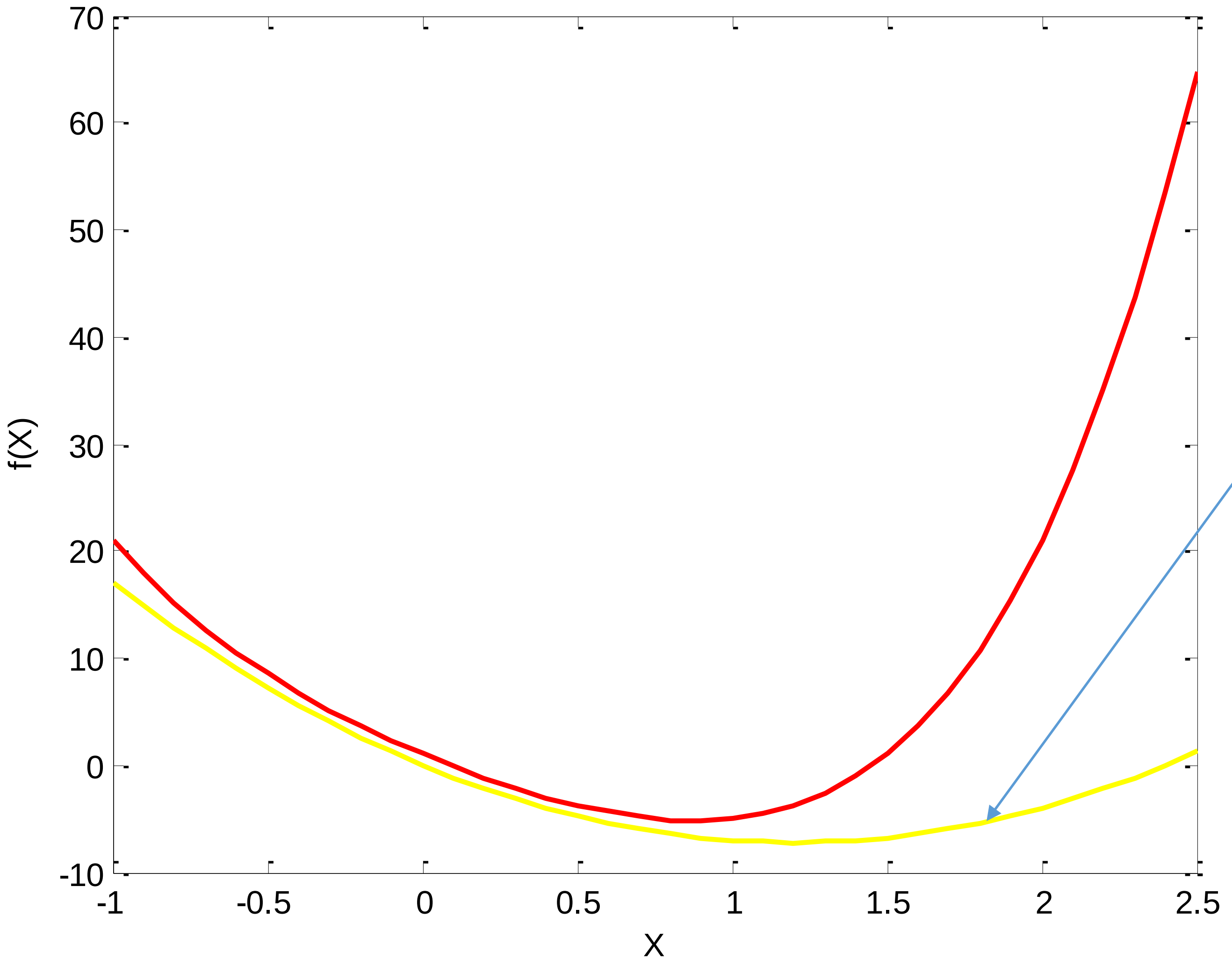


$f(x) = 2x^4 - x^3 + 5x^2 - 12x + 1$

$f^{/}(x) = 8x^3 - 3x^2 + 10x - 12 = 0$

Solving for x

$x^* = 0.8831$ and $f(x^*) = -5.1702$



$f(x) = 2x^4 - x^3 + 5x^2 - 12x + 1$

Quadratic approximation of the function at x_o can be written as

$$f(x) = f(x_o) + f^{/}(x_o)(x - x_o) + 0.5 * f^{//}(x_o)(x - x_o)^2$$

Approximate function for $x_o = 0$

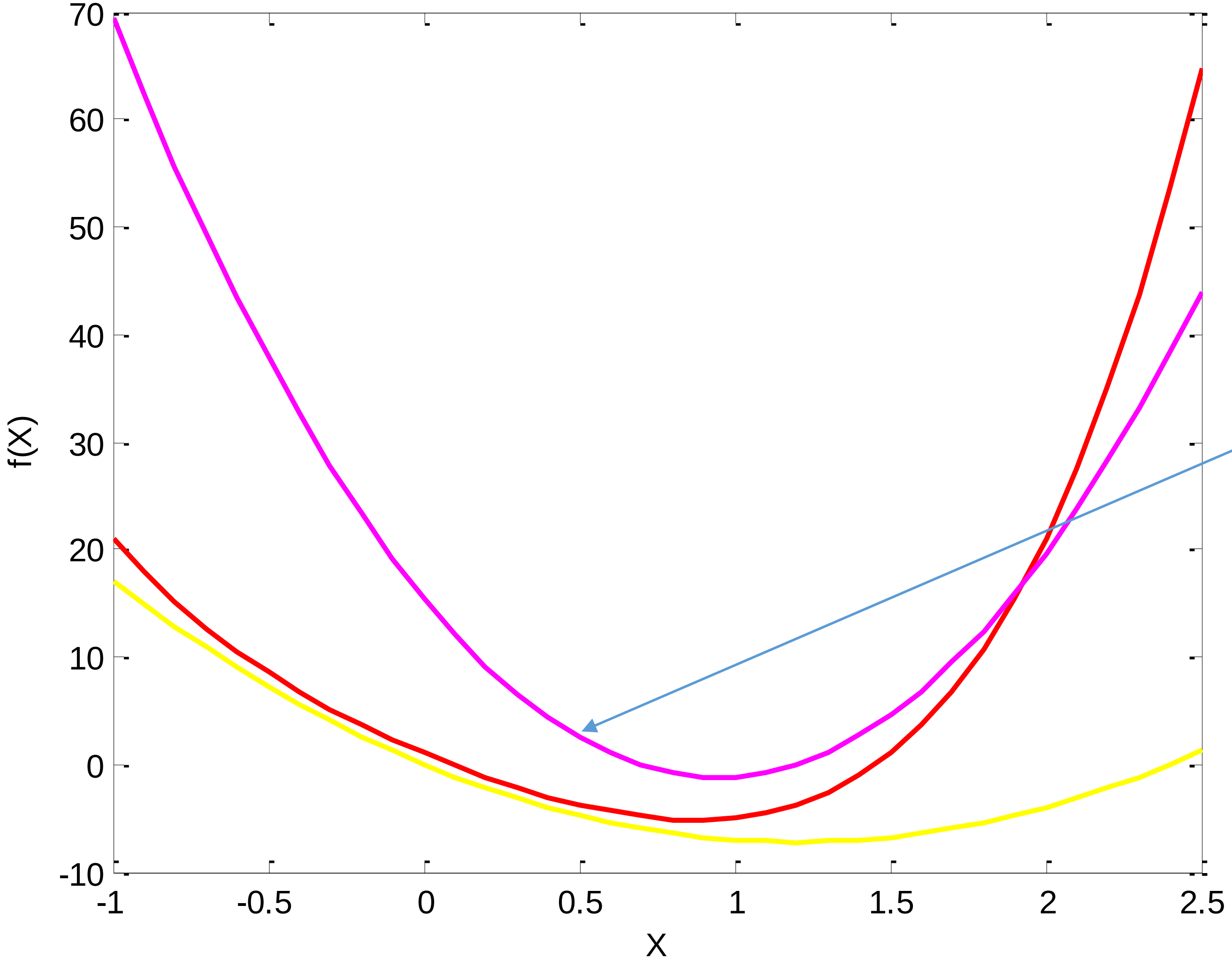
Now we can minimize the function

Minimize $f^{/}(x_o)(x - x_o) + 0.5 * f^{//}(x_o)(x - x_o)^2$

Solution is

$x^* = 1.2$ and $f(x^*) = -3.7808$ and $f^{/}(x^*) = 9.5040$

This is the solution of the approximate function: First trial



$f(x) = 2x^4 - x^3 + 5x^2 - 12x + 1$

Quadratic approximation of the function at x_o can be written as

$$f(x) = f(x_o) + f^{/}(x_o)(x - x_o) + 0.5 * f^{//}(x_o)(x - x_o)^2$$

Approximate function for $x_o = 1.2$

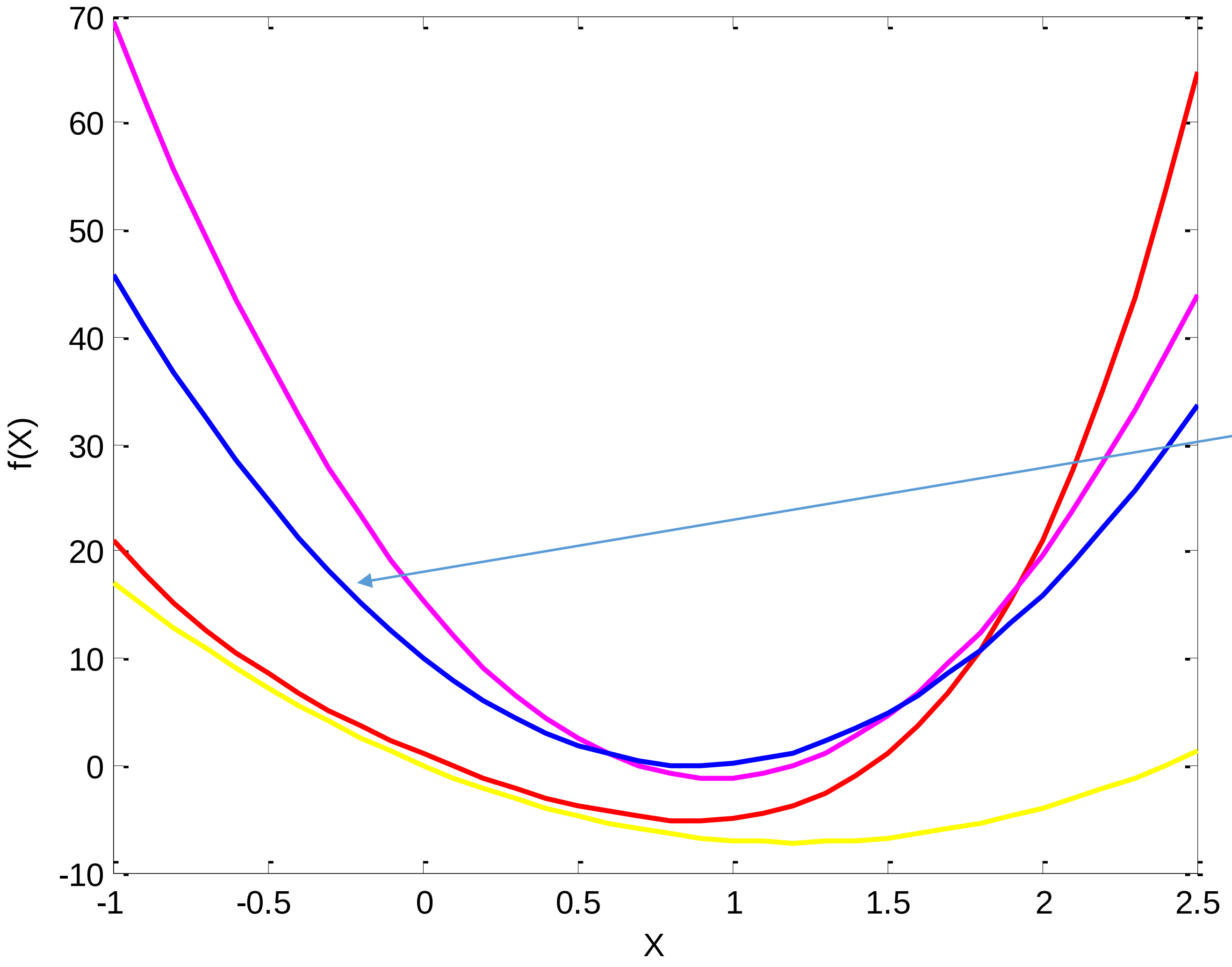
Now we can minimize the function

Minimize $f^{/}(x_o)(x - x_o) + 0.5 * f^{//}(x_o)(x - x_o)^2$

Solution is

$x^* = 0.9456,\ f(x^*) = -5.1229$ and $f^{/}(x^*) = 1.5377$

This is the solution of the approximate function: Second trial



$f(x) = 2x^4 - x^3 + 5x^2 - 12x + 1$

Quadratic approximation of the function at x_o can be written as

$$f(x) = f(x_o) + f^{/}(x_o)(x - x_o) + 0.5 * f^{//}(x_o)(x - x_o)^2$$

Approximate function for $x_o = 0.9456$

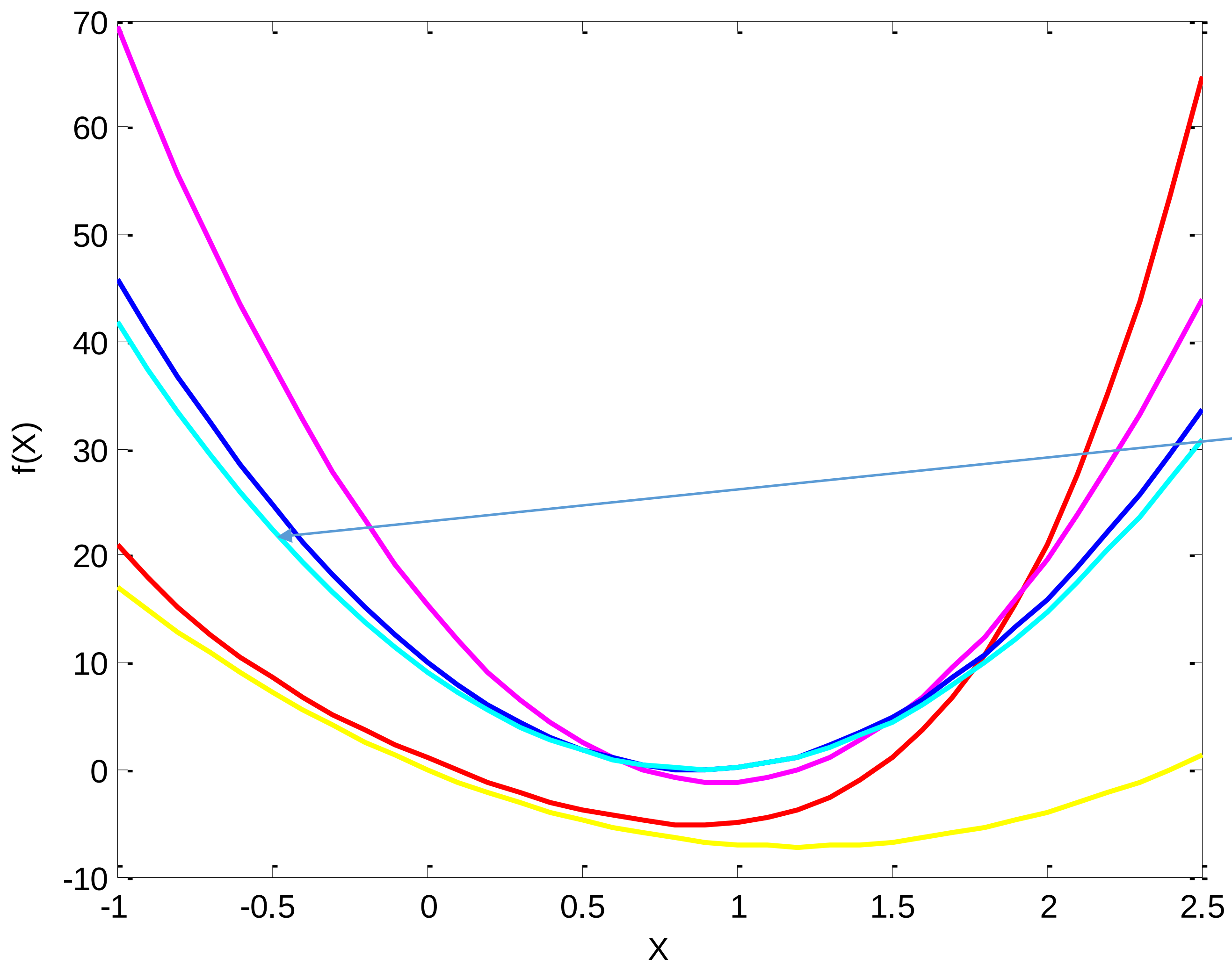
Now we can minimize the function

Minimize $f^{/}(x_o)(x - x_o) + 0.5 * f^{//}(x_o)(x - x_o)^2$

Solution is

$x^* = 0.8864$ and $f(x^*) = -5.1701$ and $f^{/}(x^*) = 0.0785$

This is the solution of the approximate function: Third trial



$f(x) = 2x^4 - x^3 + 5x^2 - 12x + 1$

Quadratic approximation of the function at x_o can be written as

$$f(x) = f(x_o) + f^{/}(x_o)(x - x_o) + 0.5 * f^{//}(x_o)(x - x_o)^2$$

Approximate function for $x_o = 0.8864$

Now we can minimize the function

Minimize $f^{/}(x_o)(x - x_o) + 0.5 * f^{//}(x_o)(x - x_o)^2$

Solution is

$x^* = 0.8831$ and $f(x^*) = -5.1702$ and $f^{/}(x^*) = 0.00099$

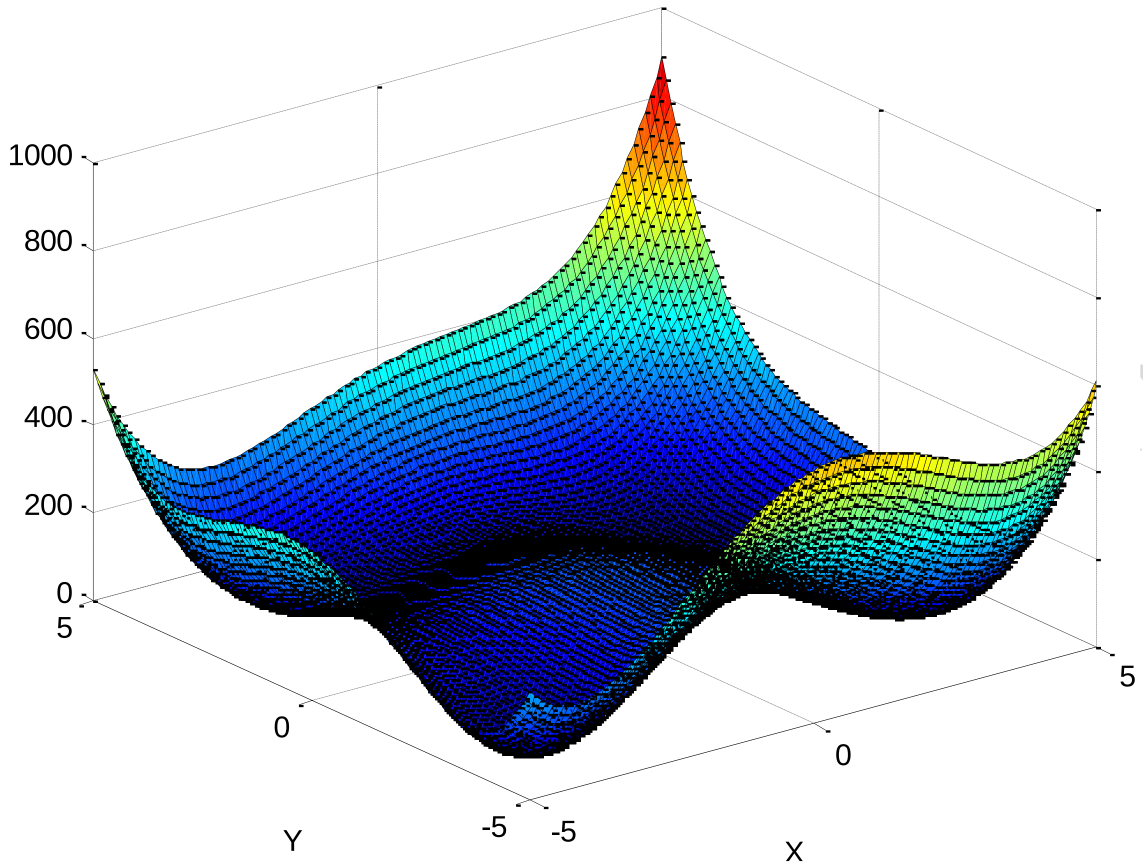
This is the solution of the approximate function: Fourth trial

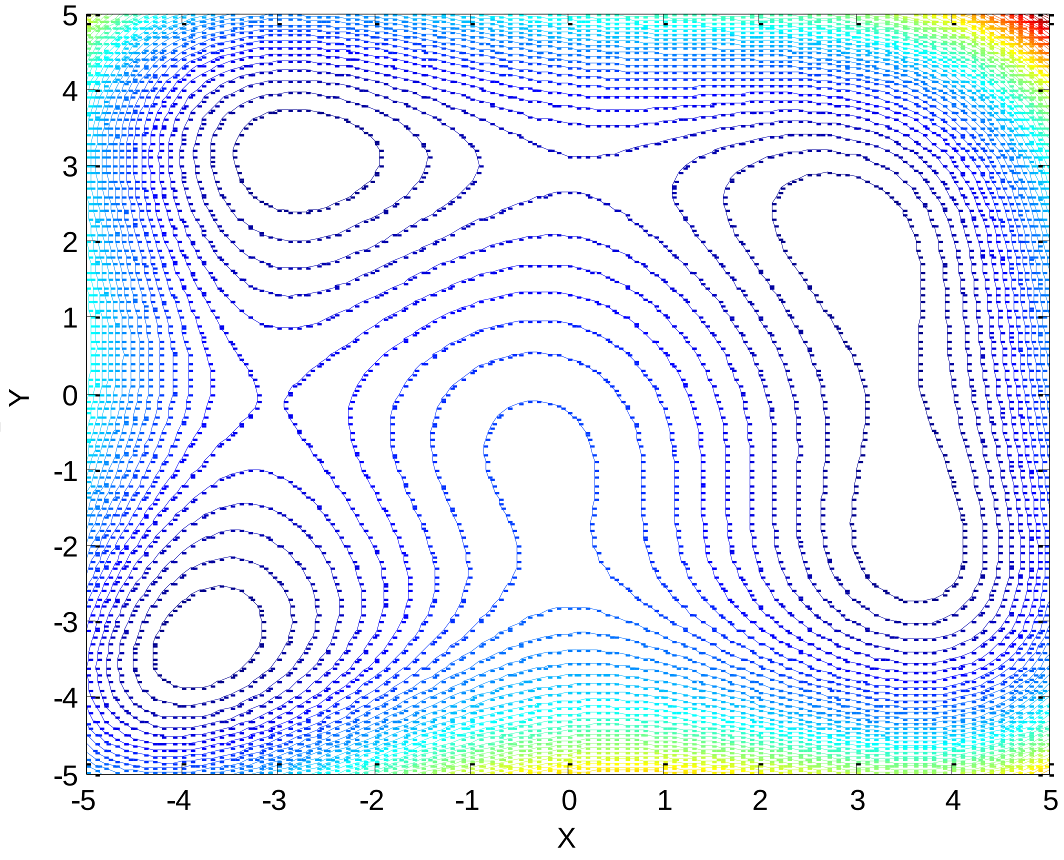
Gradient is negligible

STOP ITERATION

Now take an example of multivariable problem

$$\text{Minimize } f(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 \pm 7)^2$$





Minimize $f(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 \pm 7)^2 \qquad x_o = [2\ 2]^T$

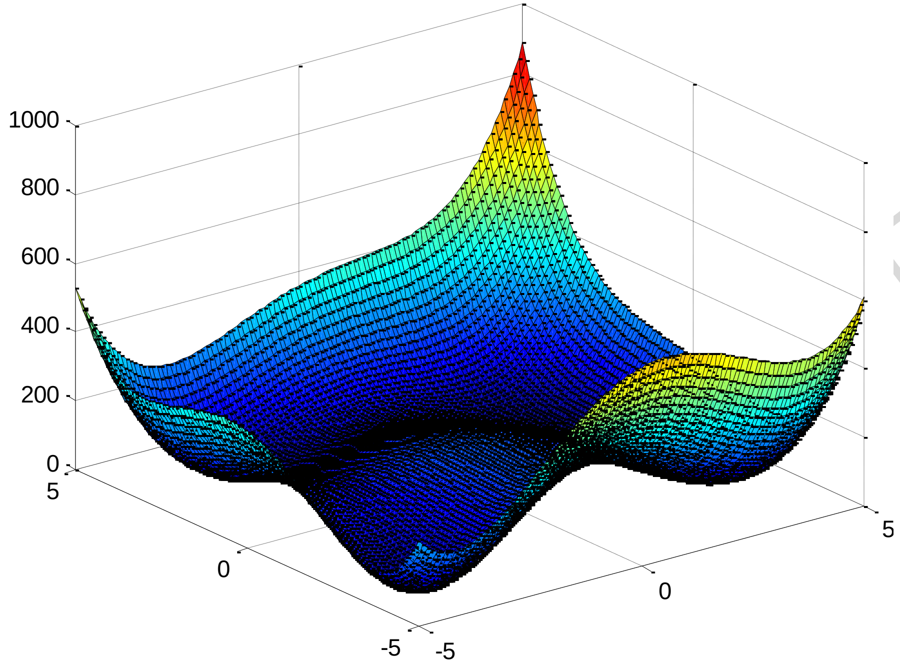
The quadratic approximation of the function at $x_o = [2\ 2]^T$ can be written as

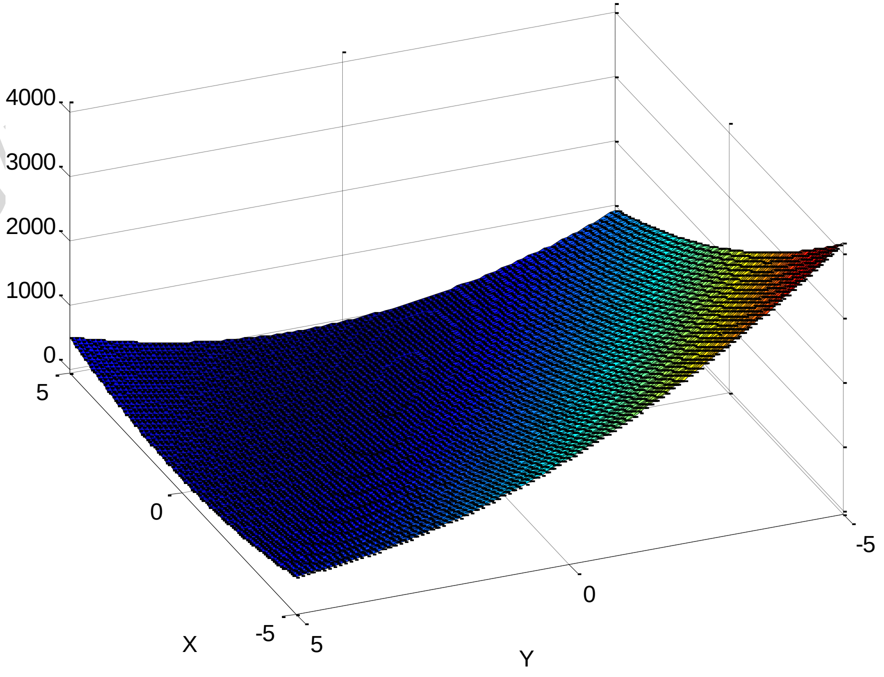
$$f(X) = f(X_o) + (X - X_o)\nabla f(X_o)^T + (X - X_o)H(X_o)(X - X_o)^T$$

For first approximation

Minimize $$f(X) = (X - X_o)\nabla f(X_o)^T + (X - X_o)H(X_o)(X - X_o)^T$$

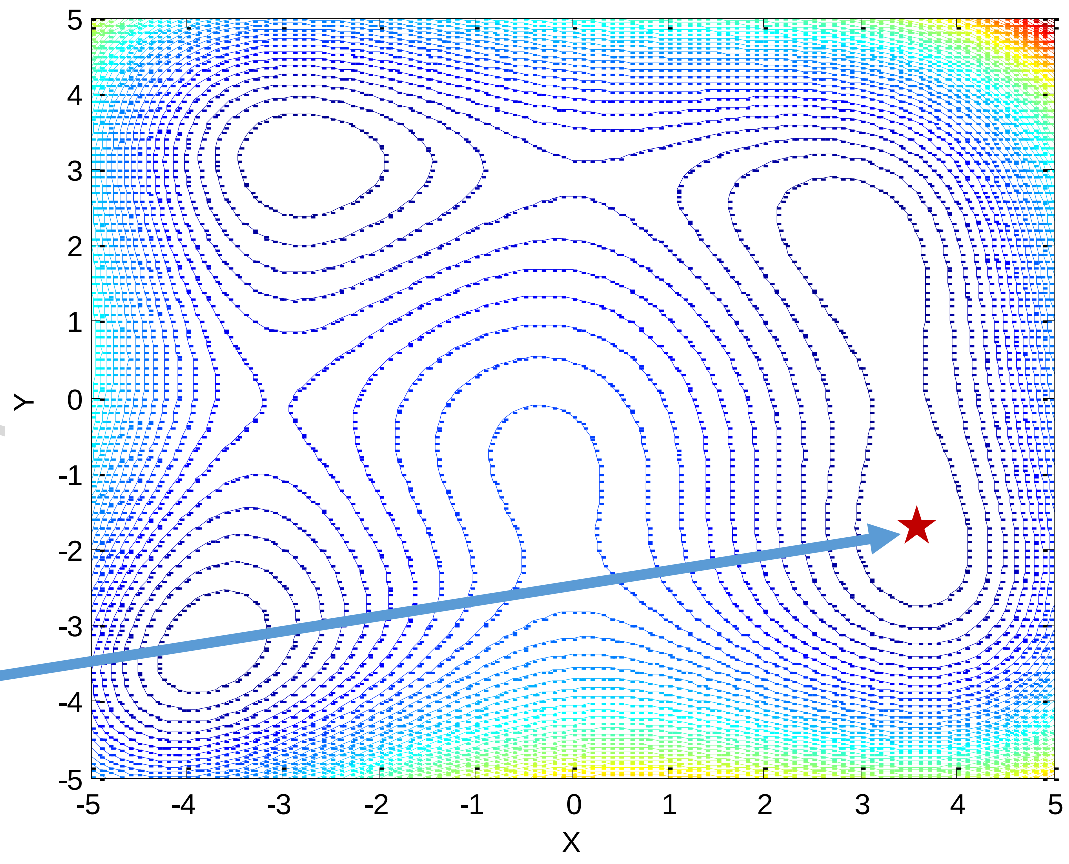
Or, $$f(X) = [x_1 - 2 \quad x_2 - 2]\begin{bmatrix}-42\\-18\end{bmatrix} + [x_1 - 2 \quad x_2 - 2]\begin{bmatrix}14 & 16\\16 & 30\end{bmatrix}\begin{bmatrix}x_1 - 2\\x_2 - 2\end{bmatrix}$$





Solution

Trial	X value		Gradient	
1	7.9268	-0.5610	-42	-18
2	5.7945	-4.4555	1628	99
3	4.4415	-3.1670	457	-296
4	3.7952	-2.3927	113	-83
5	3.6086	-1.9928	20	-22
6	3.5858	-1.8623	1.5811	-4.5637
7	3.5844	-1.8483	0.0457	-0.4106
8	3.5844	-1.8481	-0.0042	-0.0053
9	3.5844	-1.8481	-0.0028	0.0006



Optimal solution

Gradient is almost negligible

THANKS