Linear Problem (LP)

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Linear programming

It is an optimization method applicable for the solution of optimization problems where objective function and the constraints are linear

It was first applied in 1930 by economist, mainly in solving resource allocation problems

During World War II, the US Air force sought more effective procedure for allocation of resources

George B. Dantzig, a member of the US Air Force formulated the general linear problem for solving the resources allocation problems

The devised method is known as Simplex method

Linear programming

It is considered as a revolutionary development that helps in obtaining optimal decision in complex situation

Some of the great contributions are

George B. Dantzig : Devised simplex method
Kuhn and Tucker : Duality theory in LP
Charnes and Cooper: Industrial application of LP
Karmarkar : Karmarkar's method

Nobel prize awarded for contribution related to LP

Nobel prize in economics was awarded in 1975 jointly to L.V. Kantorovich of the former Soviet Union and T.C. Koopmans of USA on the application of LP to the economic problem of resource allocation.

Linear programming

Standard form of Linear Problem (LP)

Minimize $f(x_1, x_2, x_3, \dots, x_n) = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$

Subject to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$x_1, x_2, x_3, \dots, x_n \geq 0$$

Linear programming

Standard form of Linear Problem (LP) in Matrix form

Minimize $f(X) = c^T X$

Subject to

$$aX = b$$

$$X \geq 0$$

Where

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \qquad b = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} \qquad c = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix} \qquad a = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Linear programming

Characteristics of linear problem are

1. The objective function is minimization type
2. All constraints are equality type
3. All the decision variables are non-negative

Linear programming

Characteristics of linear problem are

1. The objective function is minimization type

For maximization problem

Maximize $f(x_1, x_2, x_3, \ldots, x_n) = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$

Equivalent to

Minimize $F = -f(x_1, x_2, x_3, \ldots, x_n) = -c_1x_1 - c_2x_2 - c_3x_3 - \ldots - c_nx_n$

Linear programming

Characteristics of linear problem are

2. All constraints are equality type

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n = b_k$$

If it is less than type

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n \leq b_k$$

It can be converted to

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n + x_{n+1} = b_k$$

Slack variable

If it is greater than type

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n \geq b_k$$

It can be converted to

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n - x_{n+1} = b_k$$

Surplus variable

Linear programming

Characteristics of linear problem are

3. All the decision variables are non-negative

$$x_1, x_2, x_3, \dots, x_n \geq 0$$

Is any variable x_j is unrestricted in sign, it can be expressed as

$$x_j = x_j' - x_j''$$

Where, $x_j', x_j'' \geq 0$

Linear programming

There are m equations and n decision variables

Now see the conditions

If $m > n$, there will be $m - n$ redundant equations which can be eliminated

If $m = n$, there will be an unique solution or there may not be any solution

If $m < n$, a case of undetermined set of linear equations, if they have any solution, there may be innumerable solutions

The problem of linear programming is to find out the best solution that satisfy all the constraints

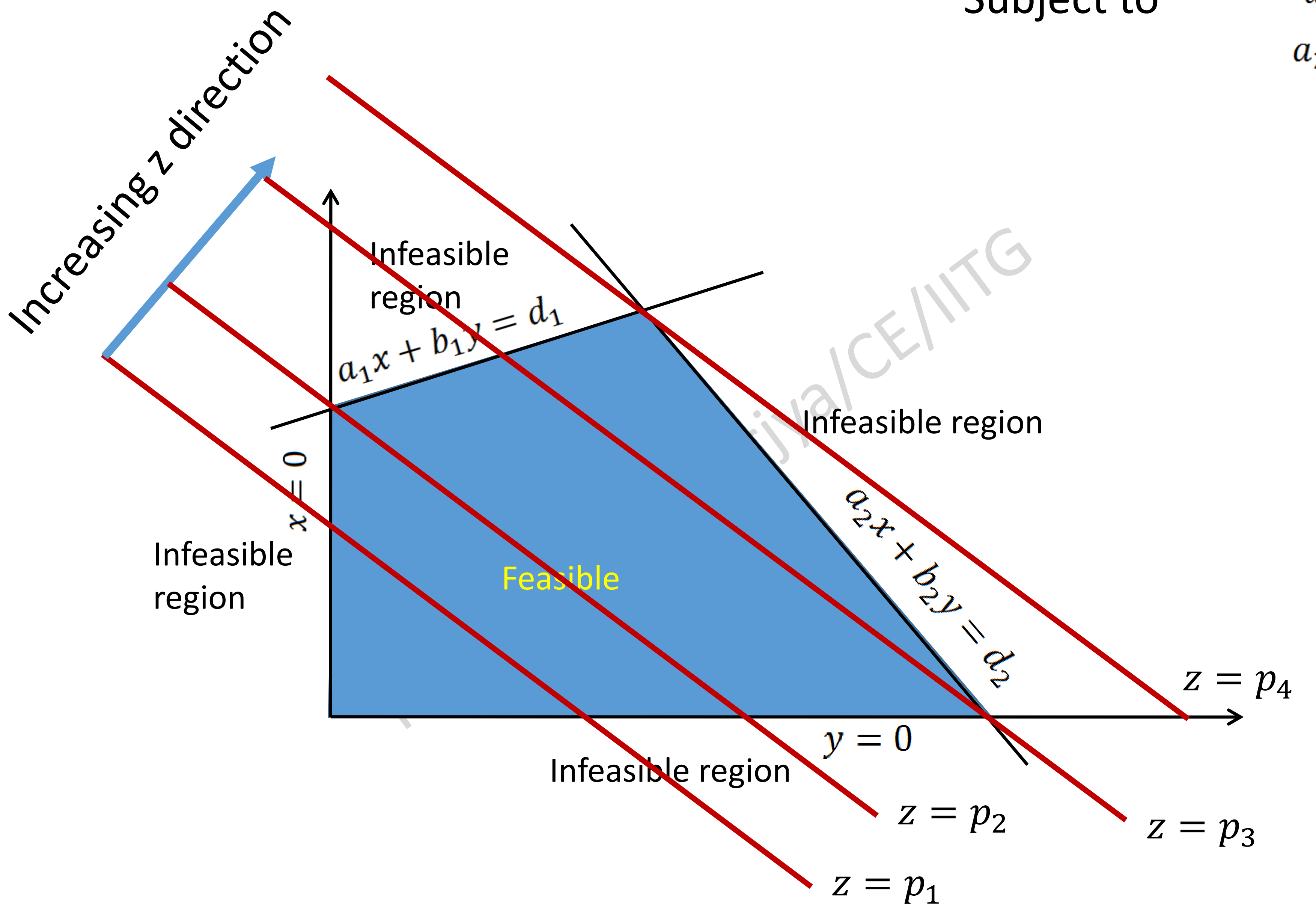
Maximize $z = c_1 x + c_2 y$

Subject to

$$a_1 x + b_1 y \leq d_1$$

$$a_2 x + b_2 y \leq d_2$$

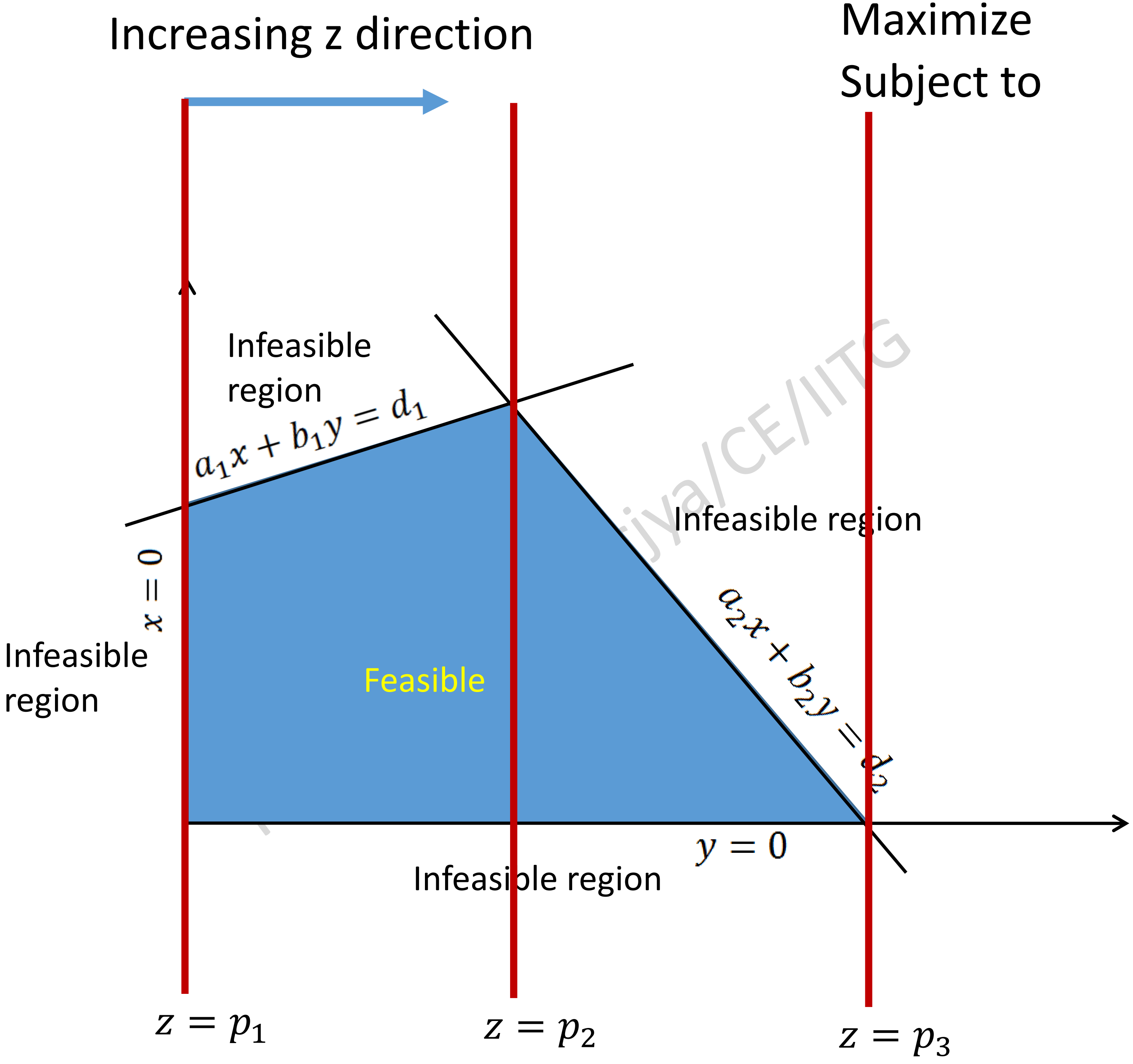
$$x, y \geq 0$$



Maximize $z = c_1 x + c_2 y$

Subject to

$$a_1 x + b_1 y \leq d_1$$
$$a_2 x + b_2 y \leq d_2$$
$$x, y \geq 0$$

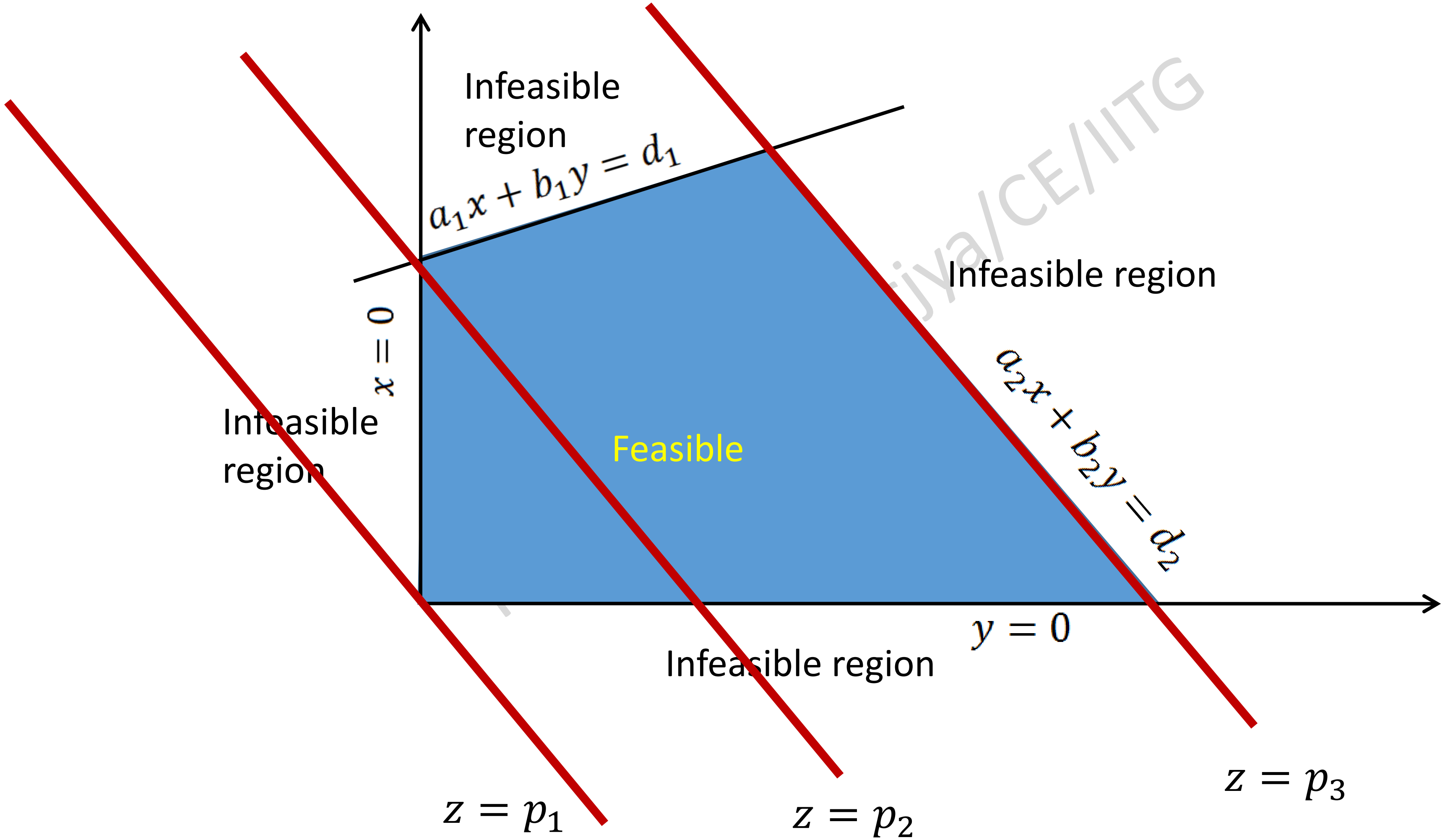


As such solution of the problem will be one of the corners of the search space

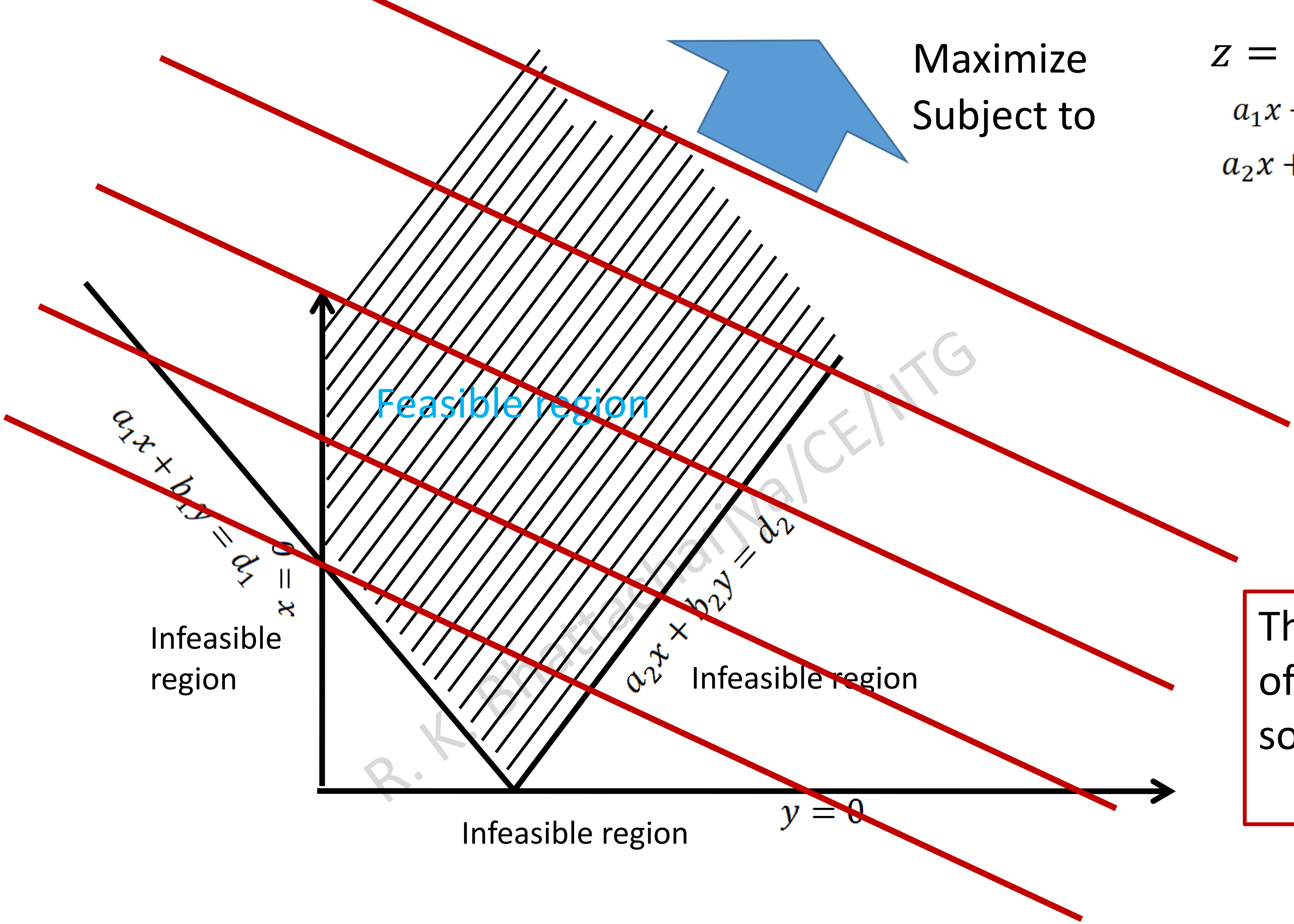
Maximize $z = c_1x + c_2y$

Subject to

$$a_1x + b_1y \le d_1$$
$$a_2x + b_2y \le d_2$$
$$x, y \ge 0$$



This problem has infinite number of solutions



Maximize $z = c_1x + c_2y$

Subject to

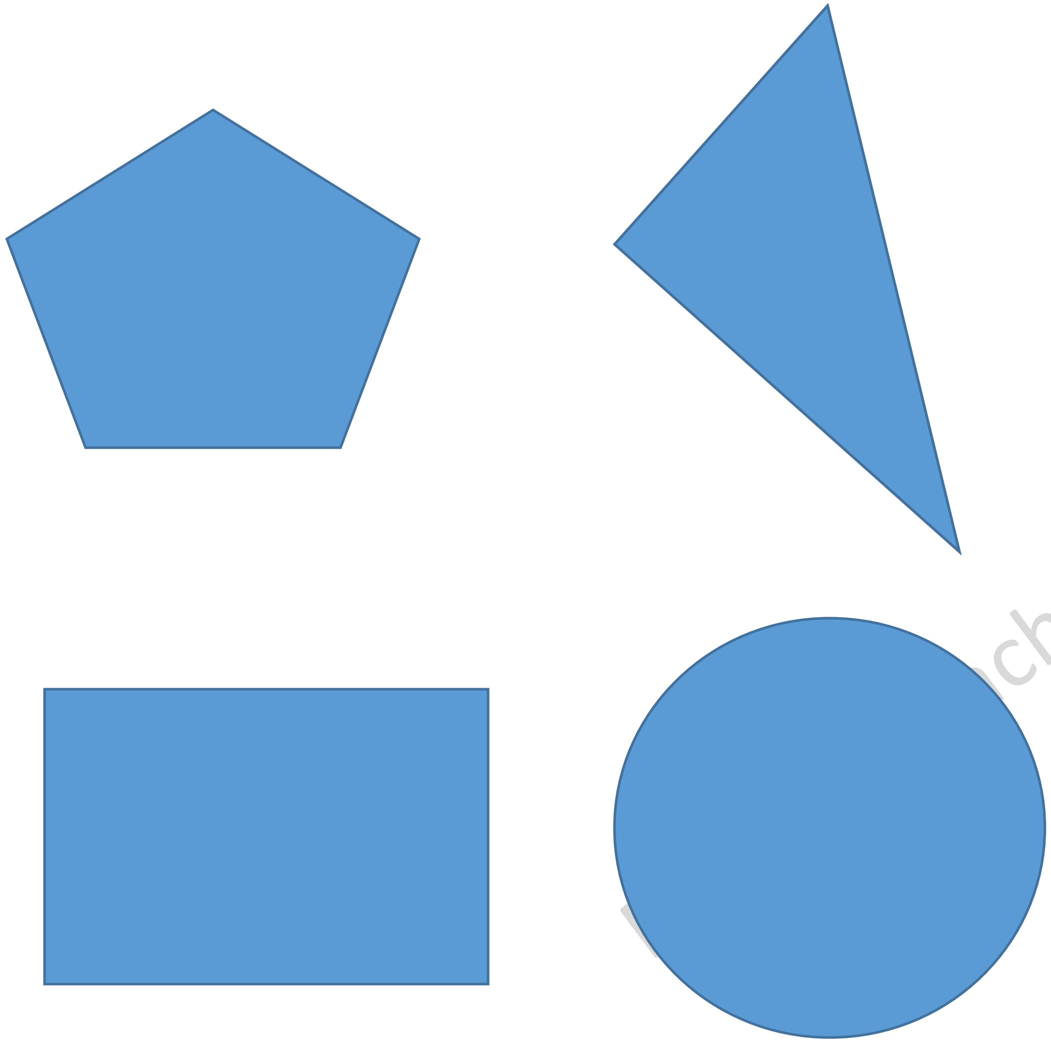
$$a_1x + b_1y \leq d_1$$

$$a_2x + b_2y \leq d_2$$

$$x, y \geq 0$$

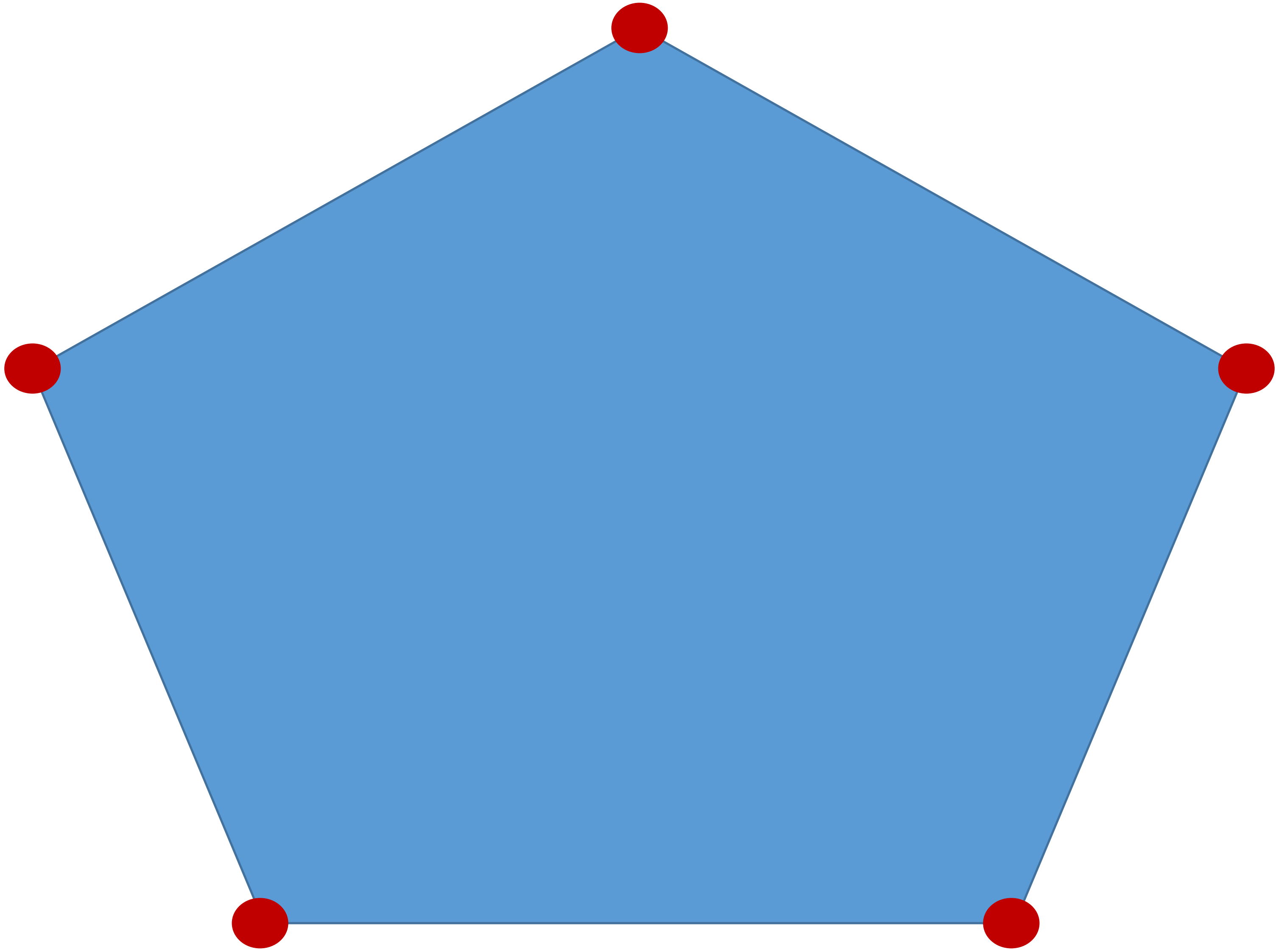
This is a problem of unbounded solution

Search space



Convex Search space

Non Convex Search space



Some definitions

Point of n -Dimensional space

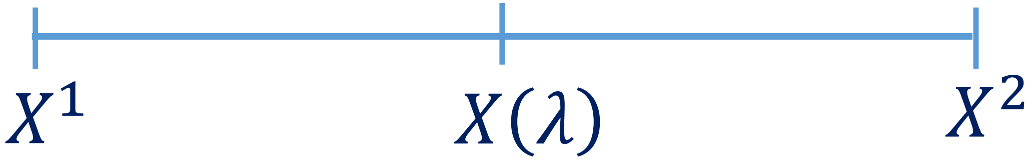
A point X in an n -dimensional space is characterized by an ordered set of n values or coordinates. The coordinate of X are also called the component of X.

Line segment in n-Dimensions (L)

If coordinates of two pints X^1 and X^2 are given, the line segment (L) joining these points is the collection of points $X(\lambda)$ whose coordinates are given by

$$X(\lambda) = \lambda X^1 + (1 - \lambda)X^2$$

Thus $\mathrm{L} = \{X | X = \lambda X^1 + (1 - \lambda)X^2\}$



$$0 \geq \lambda \geq 1$$

Some definitions

Hyperplane

In n -dimensional space, the set of points whose coordinate satisfy a linear equation

$$a_1x_1 + a_2x_1 + a_3x_1 + \cdots + a_nx_n = a^T X = b$$

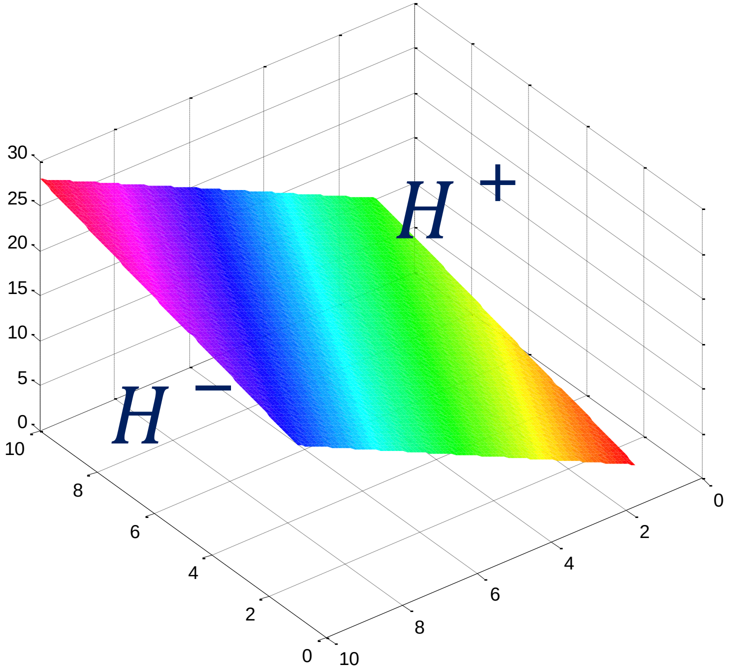
is called a hyperplane

A hyperplane is represented by

$$H(a, b) = \{X | a^T X = b\}$$

A hyperplane has $n - 1$ dimensions in an n-dimensional space

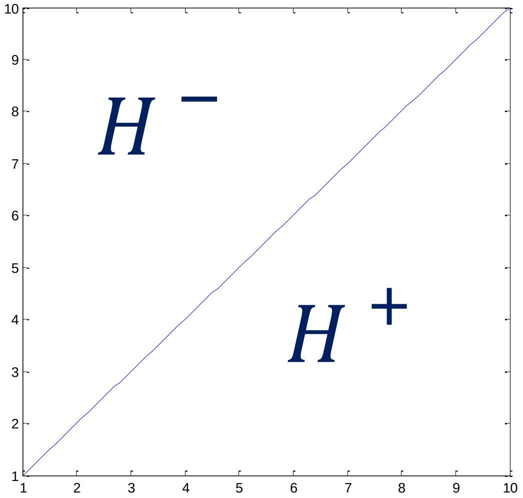
It is a plane in three dimensional space
It is a line in two dimensional space



Plane

$$H^+ = \{X \mid a^T X \geq b\}$$

$$H^- = \{X \mid a^T X \leq b\}$$



Line

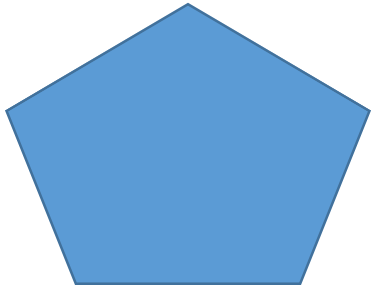
Convex Set

A convex set is a collection of points such that if X^1 and X^2 are any two points in the collection, the line segment joining them is also in the collection, which can be defined as follows

If $X^1, X^2 \in S$, then $X \in S$

Where $X(\lambda) = \lambda X^1 + (1-\lambda)X^2$ $\quad\quad 0 \geq \lambda \geq 1$

Vertex or Extreme point



Feasible solution

In a linear programming problem, any solution that satisfy the conditions

$$aX = b$$

$$X \geq 0$$

is called feasible solution

Basic solution

A basic solution is one in which $n - m$ variables are set equal to zero and solution can be obtained for the m number variables

Basis

The collection of variables not set equal to zero to obtain the basic solution is called the basis.

Basic feasible solution

This is the basic solution that satisfies the non-negativity conditions

Nondegenerate basic feasible solution

This is a basic feasible solution that has got exactly m positive x_i

Optimal solution

A feasible solution that optimized the objective function is called an optimal solution

Solution of system of linear simultaneous equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \longrightarrow \mathrm{E}_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \longrightarrow \mathrm{E}_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \longrightarrow \mathrm{E}_3$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \longrightarrow \mathrm{E}_n$$

Elementary operation

1. Any equation E_r can be replaced by kE_r, where k is a non zero constant
2. Any equation E_r can be replaced by $E_r + kE_s$, where E_s is any other equation

Using these elementary row operation, a particular variable can be eliminated from all but one equation. This operation is known as Pivot operation

Using pivot operation, we can transform the set of equation to the following form

$$
\begin{aligned}
1x_1 + 0x_2 + 0x_3 + \cdots + 0x_n &= b_1' \\
0x_1 + 1x_2 + 0x_3 + \cdots + 0x_n &= b_2' \\
0x_1 + 0x_2 + 1x_3 + \cdots + 0x_n &= b_3' \\
\vdots \qquad\qquad \vdots \qquad\qquad &\quad \vdots \\
0x_1 + 0x_2 + 0x_3 + \cdots + 1x_n &= b_n'
\end{aligned}
$$

Now the solution are

$$x_i = b_i' \qquad i = 1,2,3, \ldots, n$$

General system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_1 + a_{m3}x_1 + \cdots + a_{mn}x_n = b_m$$

And $n > m$

Pivotal variables

Non pivotal variables

Constants

$$1x_1 + 0x_2 + \cdots + 0x_m + a'_{1m+1}x_{m+1} + \cdots + a'_{1n}x_n = b'_1$$
$$0x_1 + 1x_2 + \cdots + 0x_m + a'_{2m+1}x_{m+1} + \cdots + a'_{2n}x_n = b'_2$$
$$0x_1 + 0x_2 + \cdots + 0x_m + a'_{3m+1}x_{m+1} + \cdots + a'_{3n}x_n = b'_3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$0x_1 + 0x_2 + \cdots + 1x_m + a'_{mm+1}x_{m+1} + \cdots + a'_{mn}x_n = b'_m$$

$$1x_1 + 0x_2 + \cdots + 0x_m + a'_{1m+1}x_{m+1} + \cdots + a'_{1n}x_n = b'_1$$
$$0x_1 + 1x_2 + \cdots + 0x_m + a'_{2m+1}x_{m+1} + \cdots + a'_{2n}x_n = b'_2$$
$$0x_1 + 0x_2 + \cdots + 0x_m + a'_{3m+1}x_{m+1} + \cdots + a'_{3n}x_n = b'_3$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$0x_1 + 0x_2 + \cdots + 1x_m + a'_{mm+1}x_{m+1} + \cdots + a'_{mn}x_n = b'_m$$

One solution can be deduced from the system of equations are

$x_i = b'_i$ For $i = 1,2,3, \dots, m$

$x_i = 0$ For $i = m+1, m+2, m+3, \dots, n$

This solution is called basis solution

Basic variable x_i $i = 1,2,3, \dots, m$

Non basic variable x_i $i = m+1, m+2, m+3, \dots, n$

Now let's solve a problem

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1 \qquad R_0$$
$$x_1 + x_2 + x_3 + 3x_4 = 6 \qquad R_1$$
$$x_1 - x_2 + x_3 + 5x_4 = 4 \qquad R_2$$

$$x_1 + \frac{3}{2}x_2 - x_3 - \frac{7}{2}x_4 = \frac{1}{2} \qquad R_{01} = \frac{1}{2}R_0$$

$$0 - \frac{1}{2}x_2 + 2x_3 + \frac{13}{2}x_4 = \frac{11}{2} \qquad R_{11} = R_1 - R_{01}$$

$$0 - \frac{5}{2}x_2 + 2x_3 + \frac{17}{2}x_4 = \frac{7}{2} \qquad R_{21} = R_2 - R_{01}$$

$$x_1 + 0 + 5x_3 + 16x_4 = 17 \qquad R_{02} = R_{01} - \frac{3}{2}R_{12}$$

$$0 + x_2 - 4x_3 - 13x_4 = -11 \qquad R_{12} = -2R_{11}$$

$$0 + 0 - \boxed{8x_3} - 24x_4 = -24 \qquad R_{22} = R_{21} + \frac{5}{2}R_{12}$$

$$x_1 + 0 + 0 + x_4 = 2 \qquad R_{02} = R_{02} - 5R_{22}$$

$$0 + x_2 + 0 - x_4 = 1 \qquad R_{13} = R_{12} + 4R_{23}$$

$$0 + 0 + x_3 + 3x_4 = 3 \qquad R_{23} = -\frac{1}{8}R_{22}$$

Solution of the problem is

$$x_1 = 2 - x_4$$

$$x_2 = 1 + x_4$$

$$x_3 = 3 - 3x_4$$

The solution obtained by setting independent variables equal to zero is called basic solution.

$x_1 = 2 \quad x_2 = 1 \quad x_3 = 3$

$$
\begin{aligned}
2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1\\
x_1 + x_2 + x_3 + 3x_4 &= 6\\
x_1 - x_2 + x_3 + 5x_4 &= 4
\end{aligned}
$$

$$x_1 = 2, x_2 = 1, x_3 = 3, x_4 = 0$$

$$
\begin{aligned}
2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1\\
x_1 + x_2 + x_3 + 3x_4 &= 6\\
x_1 - x_2 + x_3 + 5x_4 &= 4
\end{aligned}
$$

$$x_1 = 1, x_2 = 2, x_3 = 0, x_4 = 1$$

$$
\begin{aligned}
2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1\\
x_1 + x_2 + x_3 + 3x_4 &= 6\\
x_1 - x_2 + x_3 + 5x_4 &= 4
\end{aligned}
$$

$$x_1 = 3, x_2 = 0, x_3 = 6, x_4 = -1$$

$$
\begin{aligned}
2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1\\
x_1 + x_2 + x_3 + 3x_4 &= 6\\
x_1 - x_2 + x_3 + 5x_4 &= 4
\end{aligned}
$$

$$x_1 = 0, x_2 = 3, x_3 = -3, x_4 = 2$$

How many combinations?

$$\binom{n}{m} = \frac{n!}{(n-m)!\,m!}$$

The problem we have just solved has 4 combinations

Now consider a problem of 10 variables and 8 equations, we will have 45 different combinations

If a problem of 15 variables and 10 equations, we will have 3003 different combinations

As such, it is not possible to find solutions for all the combinations

Moreover, many combinations, we may get infeasible solutions

As such we need some set of rules to switch from one feasible solution another feasible solution

Now before discussing any method, let's try to solve a problem

Minimize $-x_1 - 2x_2 - x_3$

Subject to

$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_i \geq 0 \qquad i = 1,2,3$$

$$\begin{aligned}
2x_1 + x_2 - x_3 + x_4 \quad &= 2 \\
2x_1 - x_2 + 5x_3 \quad + x_5 \quad &= 6 \\
4x_1 + x_2 + x_3 \quad + x_6 &= 6 \\
-x_1 - 2x_2 - x_3 \quad - f &= 0
\end{aligned}$$

The initial basic solution is $x_4 = 2$ $\quad x_5 = 6$ $\quad x_6 = 6$ Basic variable

$x_1 = x_2 = x_3 = 0$ Non basic variable

$f = 0$

Now look at the objective function

$$-x_1 - 2x_2 - x_3 \qquad -f = 0$$

Is it an optimal solution?

Can we improve the objective function value by making one non basic variable as basic?

For this problem, all the coefficients of the objective function is negative, as such making one of them as basic variable, we can improve (reduce) the objective value.

However, making x_2 as basic variable we will have maximum advantage

So, select the variable with minimum negative coefficient

In our problem, x_2 is the new entering variable (basic variable)

Now, next question is which one will be pivoting element

$$2x_1 + x_2 - x_3 + x_4 = 2$$
$$2x_1 - x_2 + 5x_3 + x_5 = 6$$
$$4x_1 + x_2 + x_3 + x_6 = 6$$
$$-x_1 - 2x_2 - x_3 - f = 0$$

$$2x_1 + x_2 - x_3 + x_4 = 2$$
$$4x_1 + 0x_2 + 4x_3 + x_4 + x_5 = 8$$
$$2x_1 + 0x_2 + 2x_3 - x_4 + x_6 = 4$$
$$3x_1 + 0x_2 - 3x_3 + x_4 - f = 4$$

The initial basic solution is $x_2 = 2$ $\quad x_5 = 8$ $\quad x_6 = 4$ Basic variable

$x_1 = x_3 = x_4 = 0$ Non basic variable

$f = -4$

$$2x_1 + x_2 - x_3 + x_4 = 2 \qquad 4x_1 + 0x_2 + 4x_3 + x_4 + x_5 = 8$$

$$2x_1 - x_2 + 5x_3 + x_5 = 6 \qquad -2x_1 + x_2 - 5x_3 - x_5 = -6$$

$$4x_1 + x_2 + x_3 + x_6 = 6 \qquad 6x_1 + 0x_2 + 6x_3 + x_5 + x_6 = 12$$

$$-x_1 - 2x_2 - x_3 - f = 0 \qquad -5x_1 + 0x_2 - 11x_3 - 2x_5 - f = -12$$

The initial basic solution is $x_2 = -6 \quad x_4 = 8 \quad x_6 = 12$ Basic variable

$x_1 = x_3 = x_5 = 0$ Non basic variable

$f = -12$

$$2x_1 + x_2 - x_3 + x_4 = 2$$
$$2x_1 - x_2 + 5x_3 + x_5 = 6$$
$$4x_1 + x_2 + x_3 + x_6 = 6$$
$$-x_1 - 2x_2 - x_3 - f = 0$$

$$-2x_1 + 0x_2 - 2x_3 + x_4 - x_6 = -4$$
$$6x_1 + 0x_2 + 6x_3 + x_5 + x_6 = 12$$
$$4x_1 + x_2 + x_3 + x_6 = 6$$
$$7x_1 + 0x_2 + x_3 + 2x_6 - f = 12$$

The initial basic solution is $x_2 = 6 \quad x_4 = -4 \quad x_5 = 12$ Basic variable

$x_1 = x_3 = x_6 = 0$ Non basic variable

$f = +12$

$$2x_1 + x_2 - x_3 + x_4 = 2$$
$$2x_1 - x_2 + 5x_3 + x_5 = 6$$
$$4x_1 + x_2 + x_3 + x_6 = 6$$
$$-x_1 - 2x_2 - x_3 - f = 0$$

$x_2 = 2 \quad x_5 = 8 \quad x_6 = 4$

$x_1 = x_3 = x_4 = 0$

$f = -4$

$x_2 = -6 \quad x_4 = 8 \quad x_6 = 12$

$x_1 = x_3 = x_5 = 0$

$f = -12$

Infeasible solution

$x_2 = 6 \quad x_4 = -4 \quad x_5 = 12$

$x_1 = x_3 = x_6 = 0$

$f = +12$

Infeasible solution

Now what is the rule, how to select the pivoting element?

What is the maximum value of x_2 without making x_2 negative?

$$2x_1 + x_2 - x_3 + x_4 = 2 \qquad x_2 = 2/1$$

$$2x_1 - x_2 + 5x_3 + x_5 = 6$$

$$4x_1 + x_2 + x_3 + x_6 = 6 \qquad x_2 = 6/1$$

Select the minimum one to avoid infeasible solution

Thus the general rule is

1. Calculate the ratio $\frac{b_i}{a_{is}}$ (For $a_{is} \geq 0$)
2. Pivoting element is $x_s^* = \underset{a_{is} \geq 0}{minimum} \left(\frac{b_i}{a_{is}}\right)$

$$2x_1 + x_2 - x_3 + x_4 = 2$$

$$2x_1 - x_2 + 5x_3 + x_5 = 6$$

$$4x_1 + x_2 + x_3 + x_6 = 6$$

$$-x_1 - 2x_2 - x_3 - f = 0$$

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x4	2	1	-1	1	0	0	0	2	2
x5	2	-1	5	0	1	0	0	6	
x6	4	1	1	0	0	1	0	6	6
f	-1	-2	-1	0	0	0	-1	0	

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x2	2	1	-1	1	0	0	0	2	
x5	4	0	4	1	1	0	0	8	2
x6	2	0	2	-1	0	1	0	4	2
f	3	0	-3	2	0	0	-1	4	

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x2	3	1	0	1.25	0.25	0	0	4	
x3	1	0	1	0.25	0.25	0	0	2	
x6	0	0	0	-1.5	-0.5	1	0	0	
f	6	0	0	2.75	0.75	0	-1	10	

All c_j are positive, so no improvement is possible

Basic Variable	x1	x2	x3	x4	x5	x6	f	bi	bi/aij
x2	2	1	-1	1	0	0	0	2	
x5	4	0	4	1	1	0	0	8	2
x6	2	0	2	-1	0	1	0	4	2
f	3	0	-3	2	0	0	-1	4	

Basic Variable	x1	x2	x3	x4	x5	x6	f	bi	bi/aij
x2	3	1	0	0.5	0	0.5	0	4	
x5	0	0	0	3	1	-2	0	0	
x3	1	0	1	-0.5	0	0.5	0	2	
f	6	0	0	0.5	0	1.5	-1	10	

Obtained the same solution

Thanks