Introduction to Classical Optimization Methods

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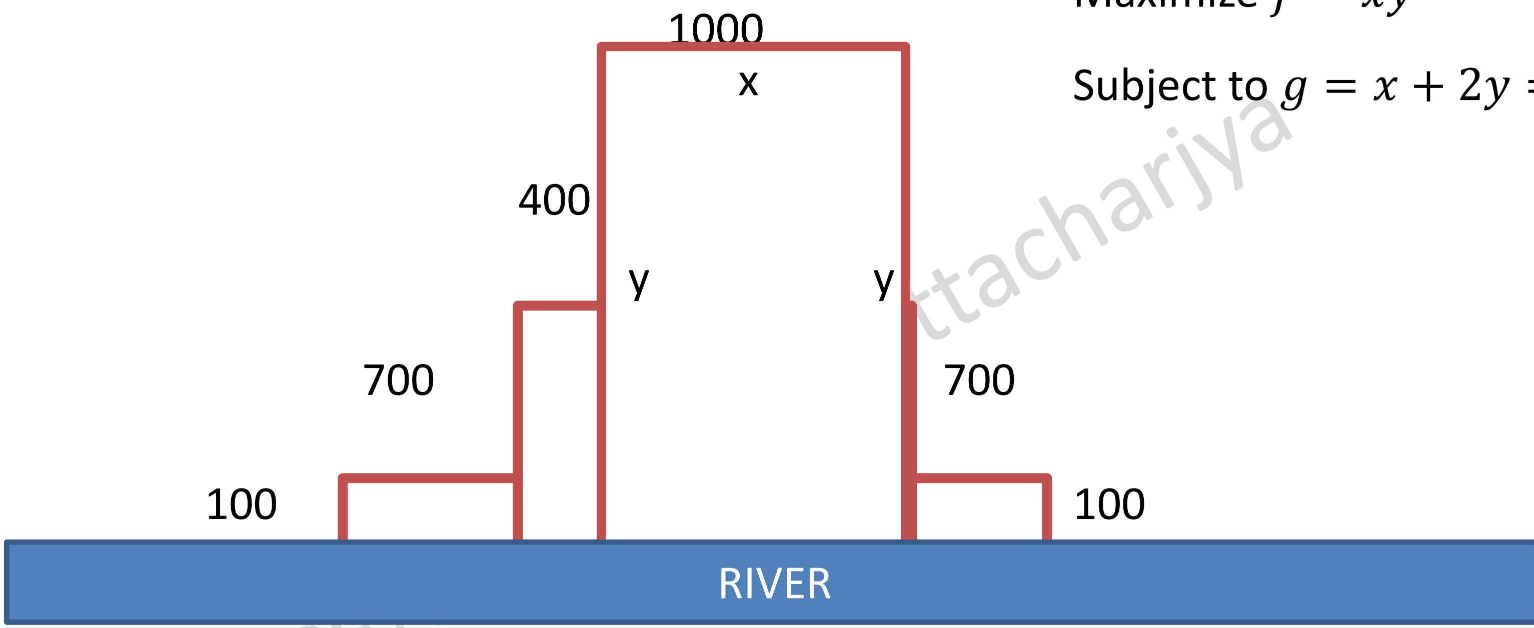
Example

A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

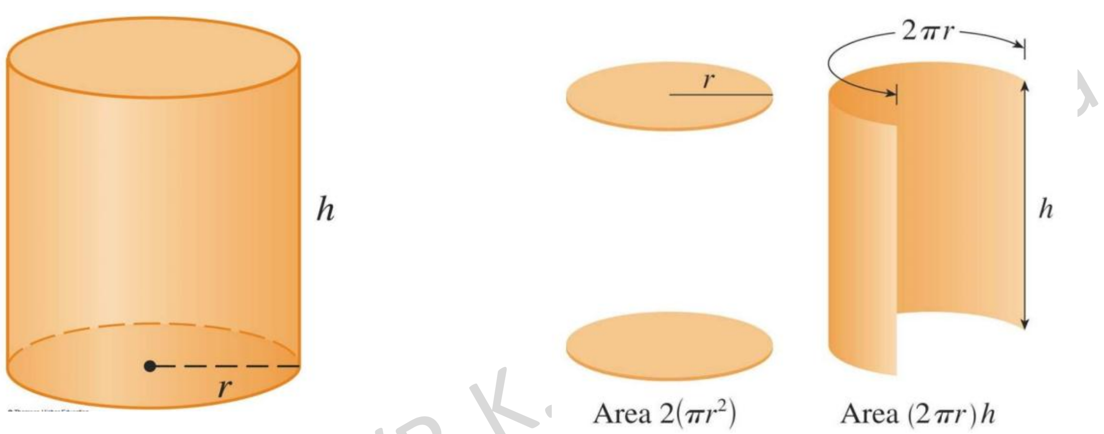
Maximize $f = xy$

Subject to $g = x + 2y = 2400$



Example

A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.



Minimize: $A = 2\pi r^2 + 2\pi rh$

Constraint: $\pi r^2 h = 1500$

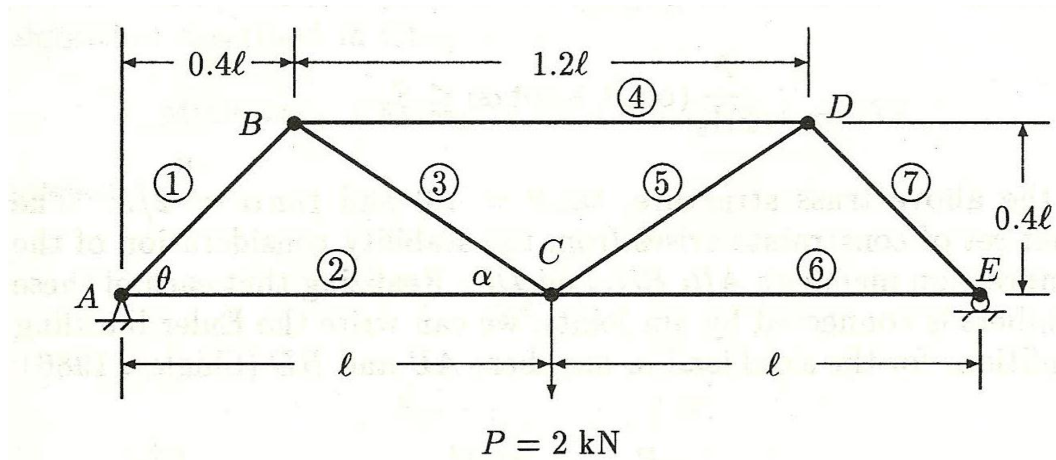
An Example

Objectives

Topology: Optimal connectivity of the structure

Minimum cost of material: optimal cross section of all the members

We will consider the second objective only



The design variables are the cross sectional area of the members, i.e. A1 to A7

Using symmetry of the structure A7=A1, A6=A2, A5=A3

You have only four design variables, i.e., A1 to A4

Optimization formulation

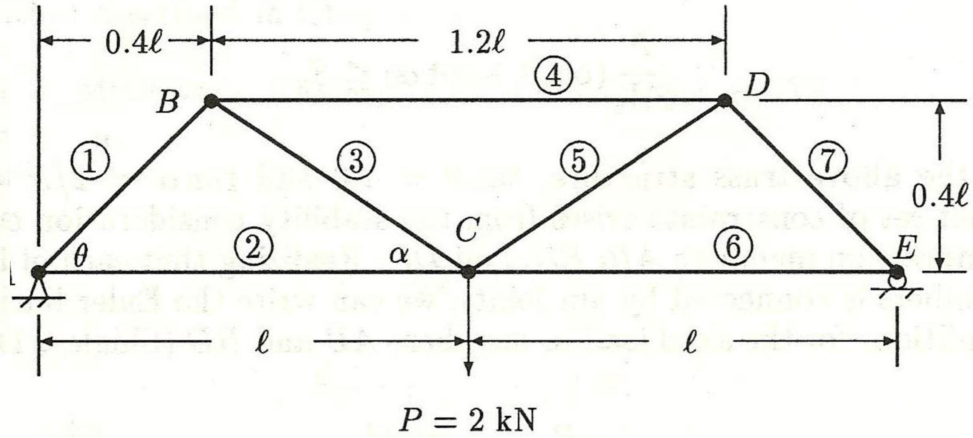
Objective

$$\text{Minimize} \quad 1.132 A_1 \ell + 2 A_2 \ell + 1.789 A_3 \ell + 1.2 A_4 \ell$$

What are the constraints?
One essential constraint is non-negativity of design variables, i.e.
A1, A2, A3, A4 >= 0

Is it complete now?

Member	Force	Member	Force
AB	$-\frac{P}{2}\csc\theta$	BC	$+\frac{P}{2}\csc\alpha$
AC	$+\frac{P}{2}\cot\theta$	BD	$-\frac{P}{2}(\cot\theta + \cot\alpha)$



First set of constraints

$$\frac{P\csc\theta}{2A_1} \le S_{yc},$$

$$\frac{P\cot\theta}{2A_2} \le S_{yt},$$

$$\frac{P\csc\alpha}{2A_3} \le S_{yt},$$

$$\frac{P}{2A_4}(\cot\theta + \cot\alpha) \le S_{yc}$$

Another constraint may be the minimization of deflection at C

$$\frac{P\ell}{E}\left(\frac{0.566}{A_1} + \frac{0.500}{A_2} + \frac{2.236}{A_3} + \frac{2.700}{A_4}\right) \le \delta_{\max}$$

Another constraint is buckling of compression members

$$\frac{P}{2\sin\theta} \le \frac{\pi E A_1^2}{1.281\ell^2}$$

$$\frac{P}{2}(\cot\theta + \cot\alpha) \le \frac{\pi E A_4^2}{5.76\ell^2}$$

$$\text{Minimize} \quad 1.132A_1\ell + 2A_2\ell + 1.789A_3\ell + 1.2A_4\ell$$

subject to

$$S_{yc} - \frac{P}{2A_1 \sin\theta} \geq 0,$$

$$S_{yt} - \frac{P}{2A_2 \cot\theta} \geq 0,$$

$$S_{yt} - \frac{P}{2A_3 \sin\alpha} \geq 0,$$

$$S_{yc} - \frac{P}{2A_4}(\cot\theta + \cot\alpha) \geq 0,$$

$$\frac{\pi E A_1^2}{1.281\ell^2} - \frac{P}{2\sin\theta} \geq 0,$$

$$\frac{\pi E A_4^2}{5.76\ell^2} - \frac{P}{2}(\cot\theta + \cot\alpha) \geq 0,$$

$$\delta_{\max} - \frac{P\ell}{E}\left(\frac{0.566}{A_1} + \frac{0.500}{A_2} + \frac{2.236}{A_3} + \frac{2.700}{A_4}\right) \geq 0,$$

$$10 \times 10^{-6} \leq A_1, A_2, A_3, A_4 \leq 500 \times 10^{-6}.$$

An optimization problem

Minimize $F = (x - p)^2 + (y - q)^2$

Subject to $a_1 x + b_1 y \leq d_1$

$a_2 x + b_2 y \leq d_2$

$x, y \geq 0$

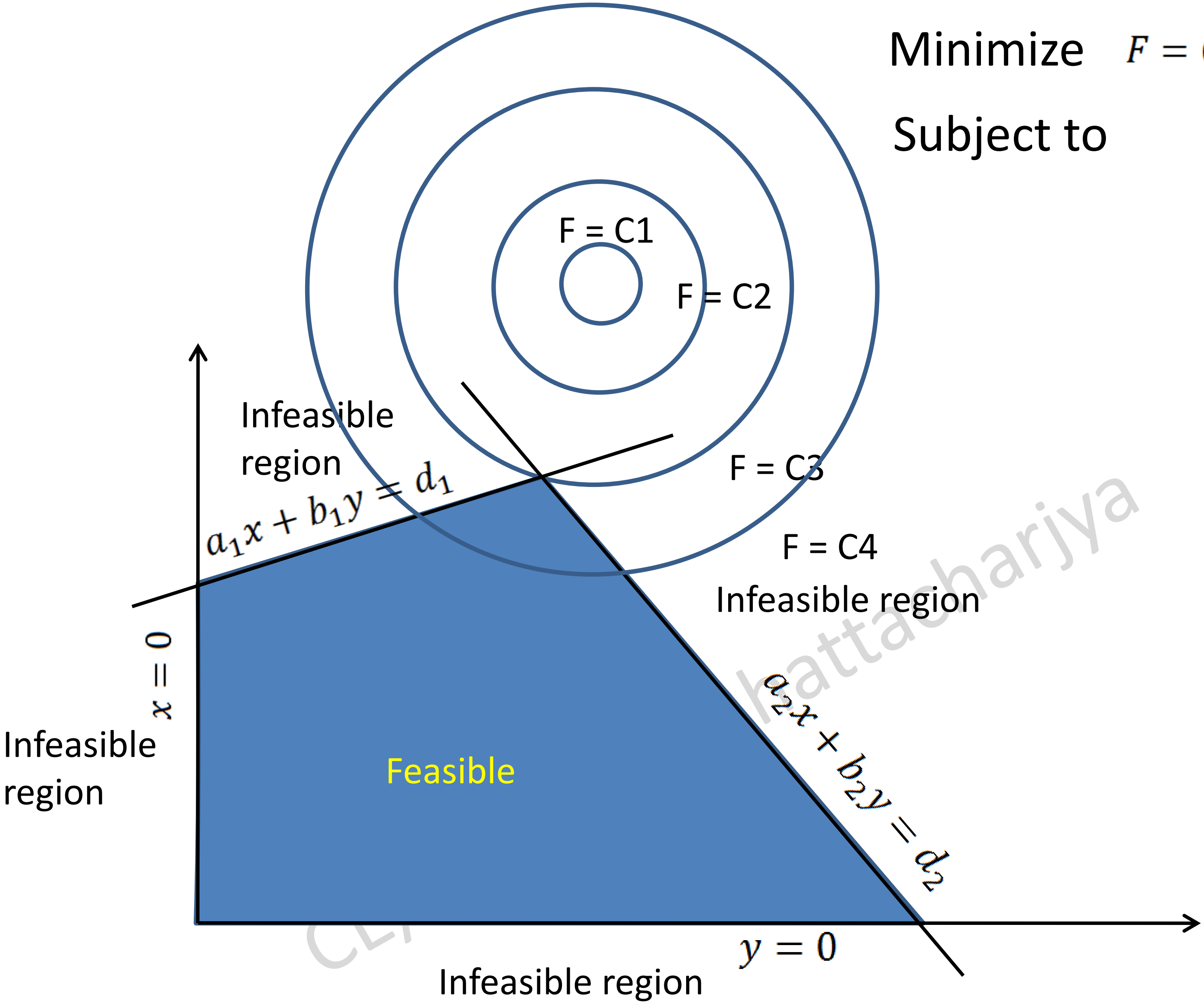
Minimize $F = (x-p)^2 + (y-q)^2$

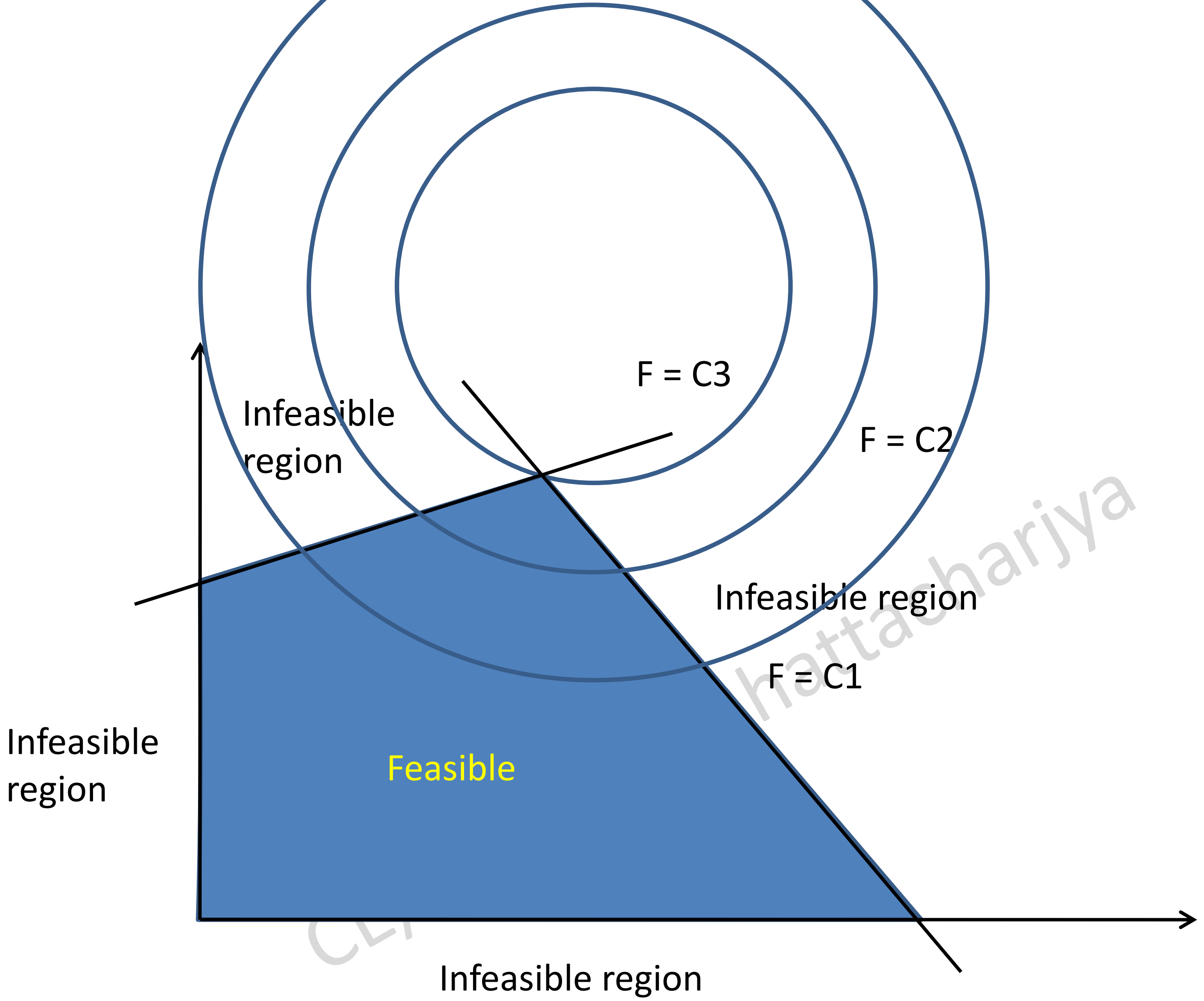
Subject to

$$a_1 x + b_1 y \leq d_1$$

$$a_2 x + b_2 y \leq d_2$$

$$x, y \geq 0$$

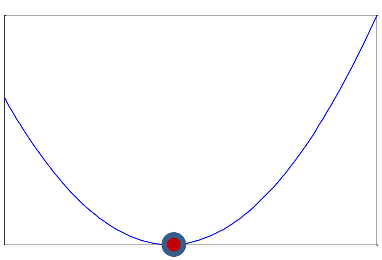



F = C3
Infeasible
region
F = C2
Infeasible region
F = C1
Infeasible
region
Feasible
Infeasible region

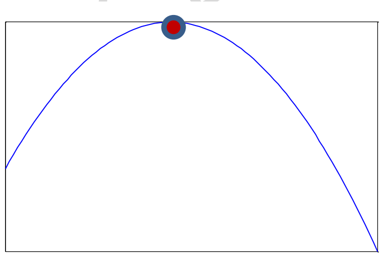
Single variable optimization

Stationary points

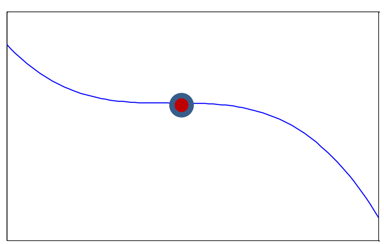
For a continuous and differentiable function *f(x), a stationary point* x^* *is a point at which the* slope of the function is zero, *i.e.* $f'(x) = 0$ *at* $x = x^*$,



Minima



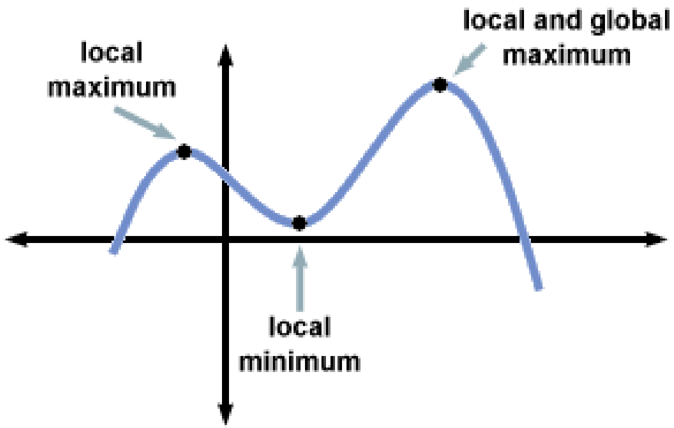
Maxima

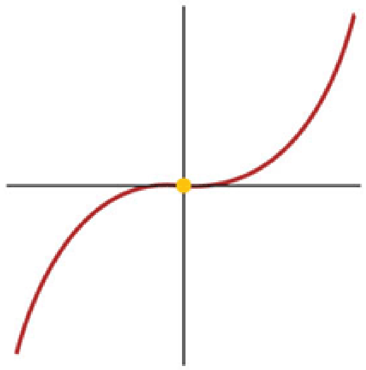


Inflection point

Relative minimum and maximum

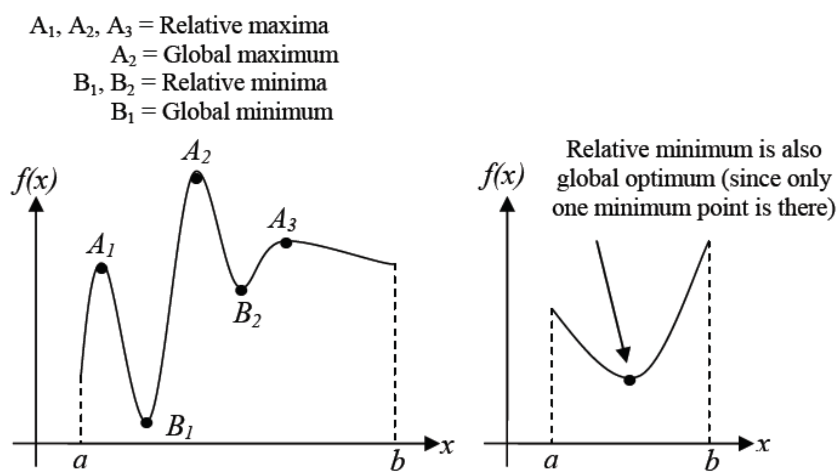
- A function is said to have a *relative or local minimum at* $x = x^*$ *if* $f(x^*) \leq f(x^*+h)$ *for all* sufficiently small positive and negative values of *h, i.e. in the near vicinity of the point x*.*
- Similarly, a point *x* is called a relative or local maximum if* $f(x^*) \geq f(x^*+h)$ for all values of *h sufficiently close to zero.*
- *A point x* is said to be an inflection point if the function value increases locally as x* increases and decreases locally as x* reduces*





Global minimum and maximum

- A function is said to have a *global or absolute minimum at* $x = x^*$ *if* $f(x^*) \leq f(x)$ *for all x in the domain over which f(x) is defined.*
- A function is said to have a *global or absolute maximum at* $x = x^*$ *if* $f(x^*) \geq f(x)$ *for all x in the domain over which f(x) is defined.*



Necessary and sufficient conditions for optimality

Necessary condition

If a function $f(x)$ is defined in the interval $a \leq x \leq b$ and has a relative minimum at $x = x^*$, Where $a \leq x^* \leq b$ and if $f^/(x)$ exists as a finite number at $x = x^*$, then $f^/(x^*) = 0$

Proof

$$f^/(x^*) = \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h}$$

Since x^* is a relative minimum $\quad f(x^*) \leq f(x^* + h)$

For all values of h sufficiently close to zero, hence

$$\frac{f(x^* + h) - f(x^*)}{h} \geq 0 \qquad \text{if } h \geq 0$$

$$\frac{f(x^* + h) - f(x^*)}{h} \leq 0 \qquad \text{if } h \leq 0$$

Thus

$f^{/}(x^{*}) \geq 0$ If h tends to zero through +ve value

$f^{/}(x^{*}) \leq 0$ If h tends to zero through -ve value

The only way to satisfy both the conditions is to have

$$f^{/}(x^{*}) = 0$$

Note:

- This theorem can be proved if x^{*} is a relative maximum
- Derivative must exist at x^{*}
- The theorem does not say what happens if a minimum or maximum occurs at an end point of the interval of the function
- It may be an inflection point also.

Sufficient conditions for optimality

Sufficient condition

Suppose at point *x**, the first derivative is zero and first nonzero higher derivative is denoted by *n,* then

1. *If n is odd, x* is an inflection point*
2. *If n is even, x* is a local optimum*
   1. *If the derivative is positive, x* is a local minimum*
   2. *If the derivative is negative, x* is a local maximum*

Proof

Apply Taylor's series

$$f(x^* + h) = f(x^*) + hf'(x^*) + \frac{h^2}{2!}f''(x^*) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(x^*) + \frac{h^n}{n!}f^n(x^*)$$

Since $f'(x^*) = f''(x^*) = \cdots = f^{n-1}(x^*) = 0$

$$f(x^* + h) - f(x^*) = \frac{h^n}{n!}f^n(x^*)$$

When n is even $\frac{h^n}{n!} \geq 0$

Thus if $f^n(x^*)$ is positive $f(x^* + h) - f(x^*)$ is positive **Hence it is local minimum**

Thus if $f^n(x^*)$ negative $f(x^* + h) - f(x^*)$ is negative **Hence it is local maximum**

When n is odd $\frac{h^n}{n!}$ changes sign with the change in the sign of h.

Hence it is an inflection point

Multivariable optimization without constraints

Minimize $f(X)$ Where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Necessary condition for optimality

If $f(X)$ has an extreme point (maximum or minimum) at $X = X^*$ and if the first partial Derivatives of $f(X)$ exists at X^*, then

$$\frac{\partial f(X^*)}{\partial x_1} = \frac{\partial f(X^*)}{\partial x_2} = \cdots = \frac{\partial f(X^*)}{\partial x_n} = 0$$

Sufficient condition for optimality

The sufficient condition for a stationary point X^* to be an extreme point is that the matrix of second partial derivatives of $f(X)$ evaluated at X^* is

(1) positive definite when X^* is a relative minimum
(2) negative definite when X^* is a relative maximum
(3) Neither positive nor negative definite when X^* is neither a minimum nor a maximum

Proof

Taylor series of two variable function

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y} + \frac{1}{2!}\left(\Delta x^2\frac{\partial^2 f}{\partial x^2} + 2\Delta x\Delta y\frac{\partial^2 f}{\partial x\partial y} + \Delta y^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

$$f(x+\Delta x, y+\Delta y) = f(x,y) + [\Delta x \quad \Delta y]\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{bmatrix} + \frac{1}{2!}[\Delta x \quad \Delta y]\begin{bmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} + \cdots$$

$$f(X^* + h) = f(X^*) + h^T \nabla f(X^*) + \frac{1}{2!} h^T H h + \cdots$$

Since X^* is a stationary point, the necessary condition gives that $\nabla f(X^*) = 0$

Thus

$$f(X^* + h) - f(X^*) = \frac{1}{2!} h^T H h + \cdots$$

Now, X^* will be a minima, if $h^T H h$ is positive

X^* will be a maxima, if $h^T H h$ is negative

$h^T H h$ will be positive if H is a positive definite matrix

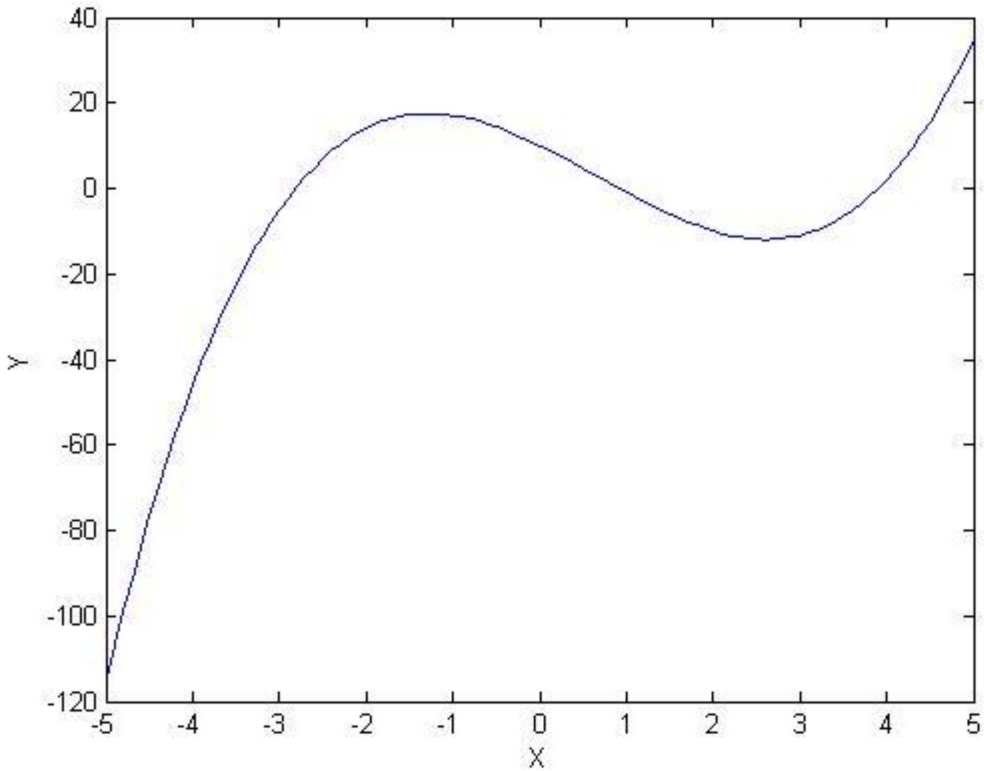
$h^T H h$ will be negative if H is a negative definite matrix

A matrix H will be positive definite if all the eigenvalues are positive, i.e. all the λ values are positive which satisfies the following equation

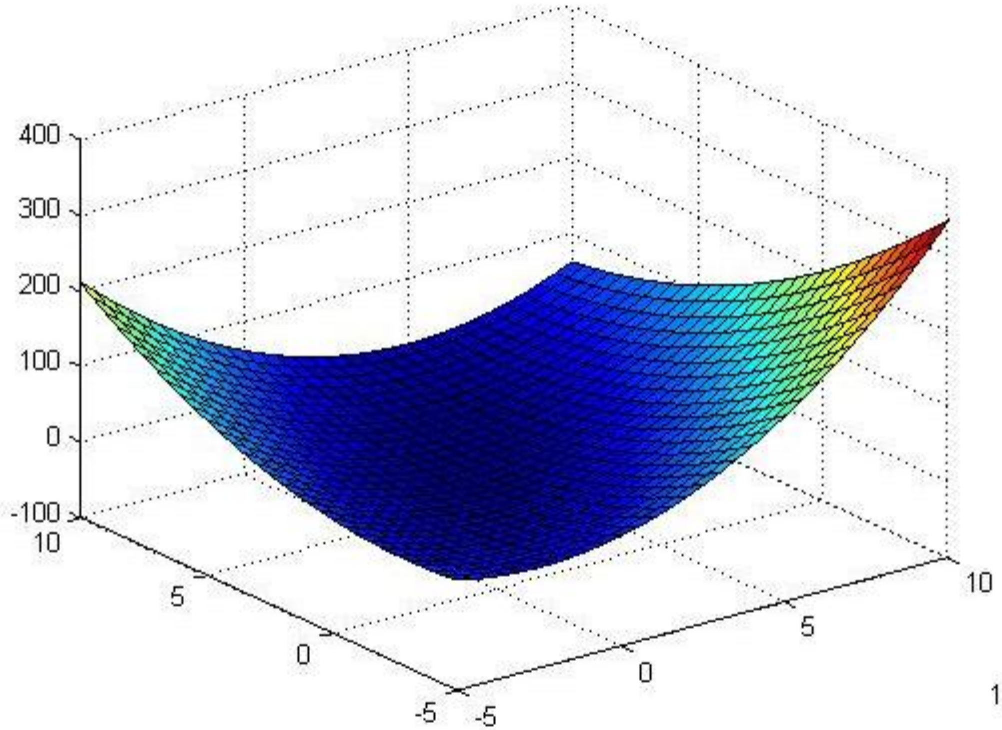
$$|A - \lambda I| = 0$$

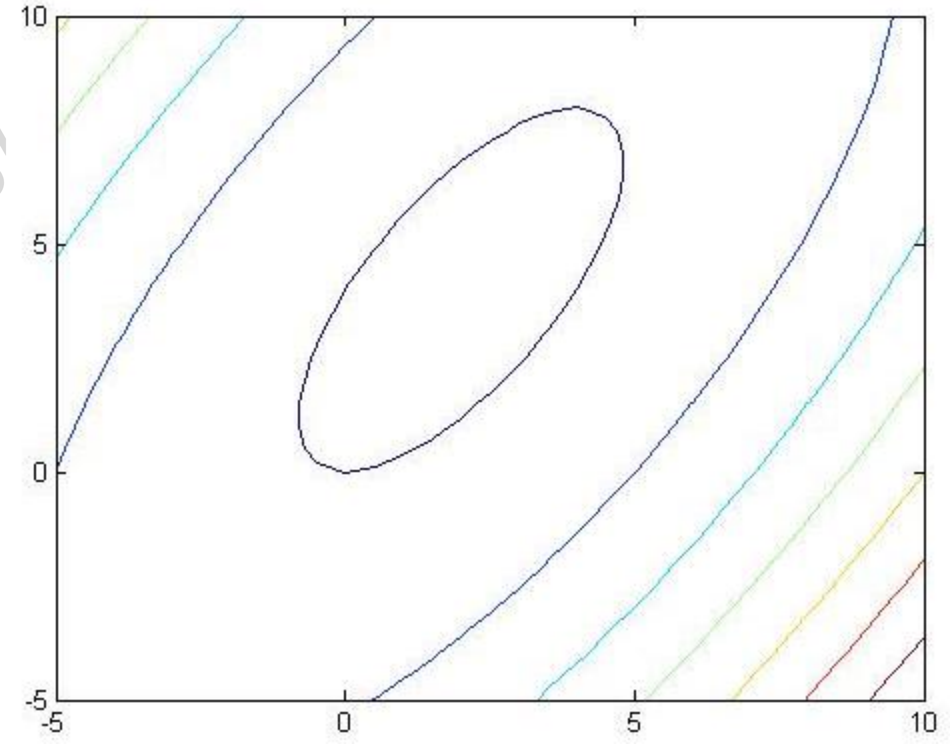
Identify the optimal points of the function given below

$$f(x) = x^3 - 10x - 2x^2 + 10$$



$$x^* = \begin{array}{ll} 2.61 & Minima \\ -1.27 & Maxima \end{array}$$


400
300
200
100
0
-100
10
5
0
-5
-5
0
5
10


10
5
0
-5
-5
0
5
10