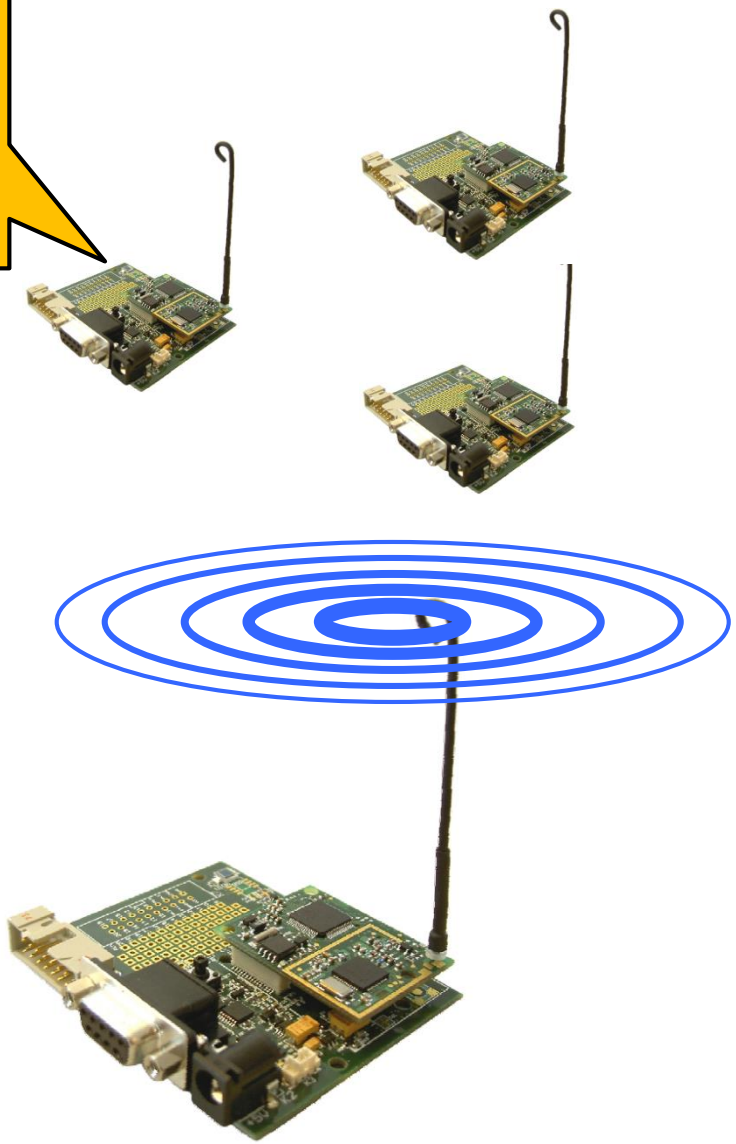
GIAN Course on Distributed Network Algorithms

Distributed Wireless Networks

Thanks to Yvonne-Anne Pignolet from ABB research for basis of slides!

Wireless Networks

Big advantage: no wires! Network setup easy and fast.



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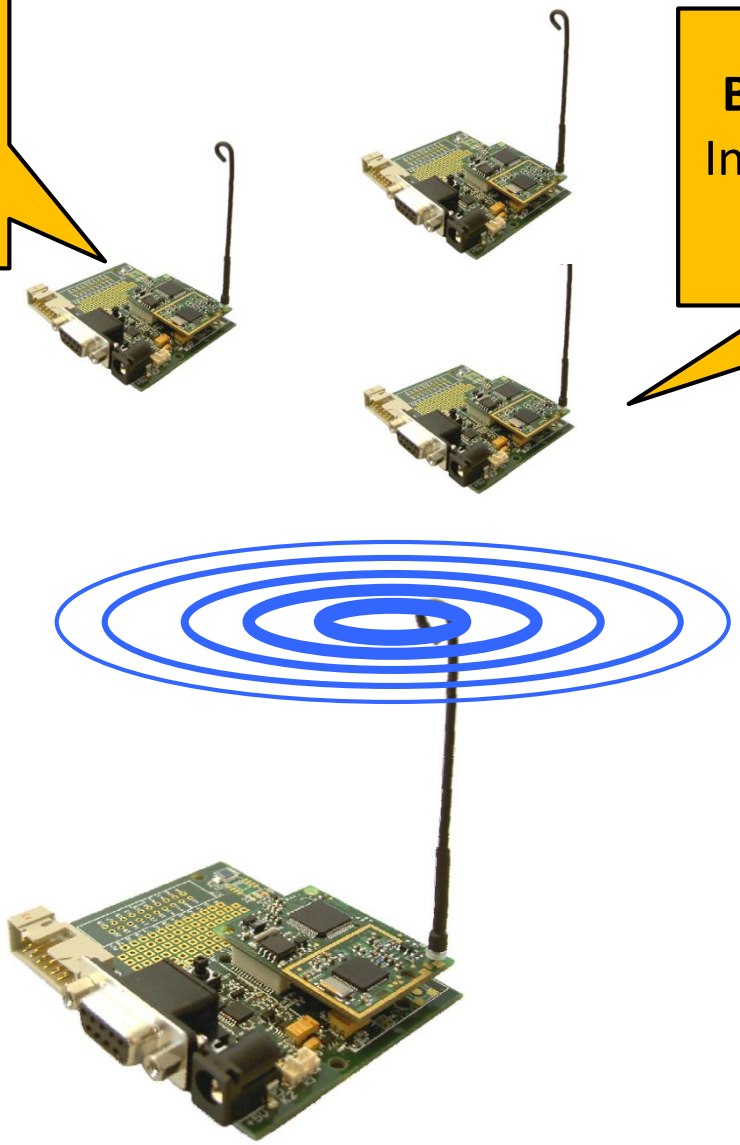

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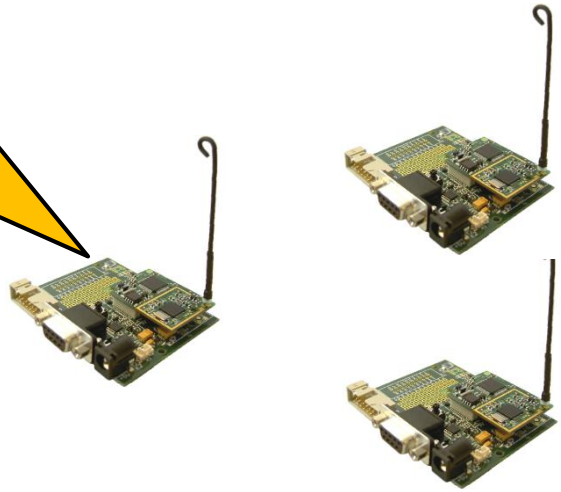
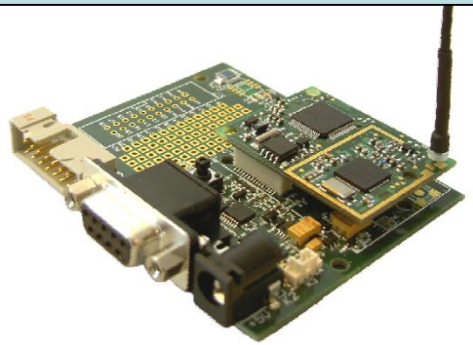


Big challenge: no wires! Interference, attenuation, energy supply.

The big question is:

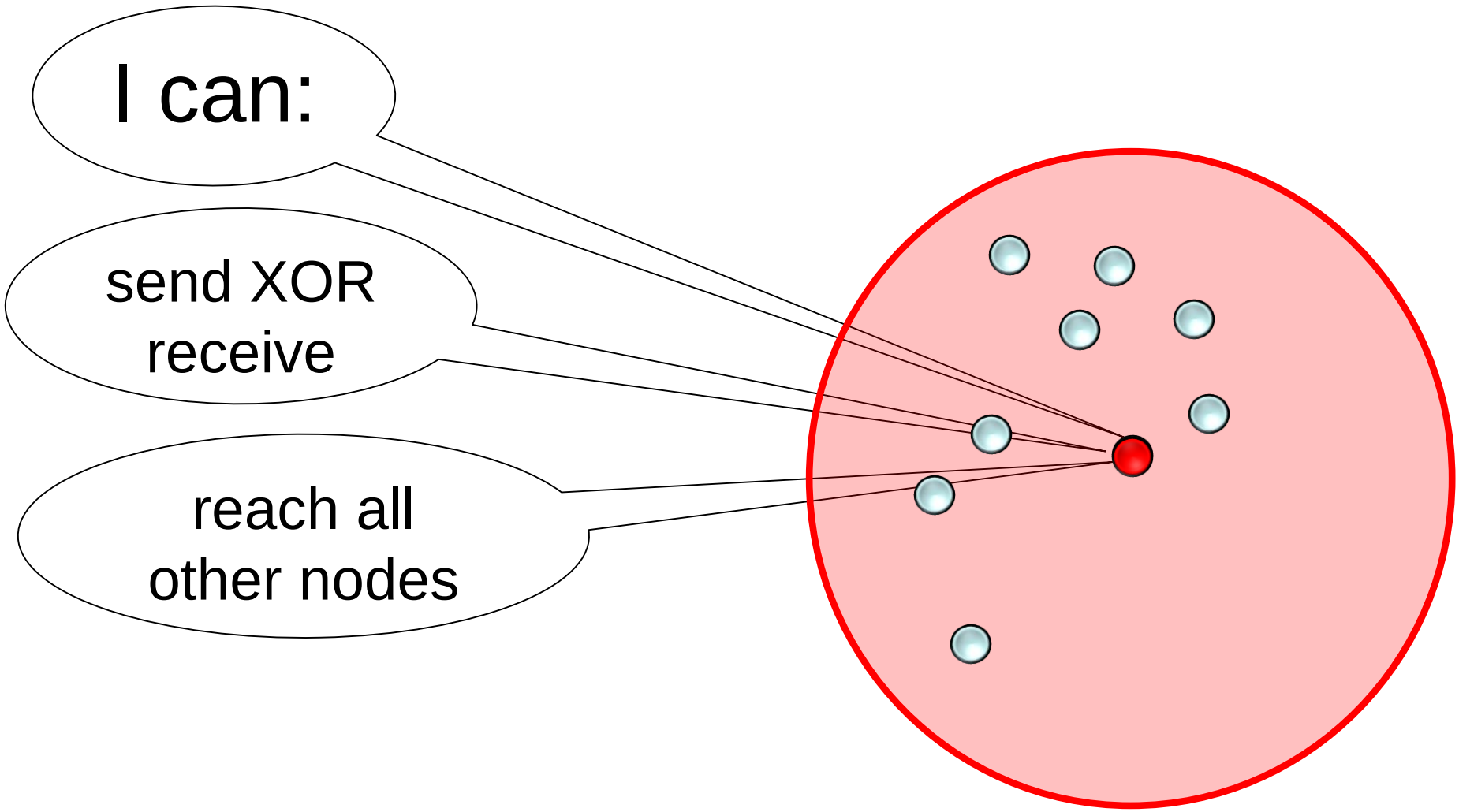
to send or not to send?

Avoid interference!



Radio Network Model

Our model today.



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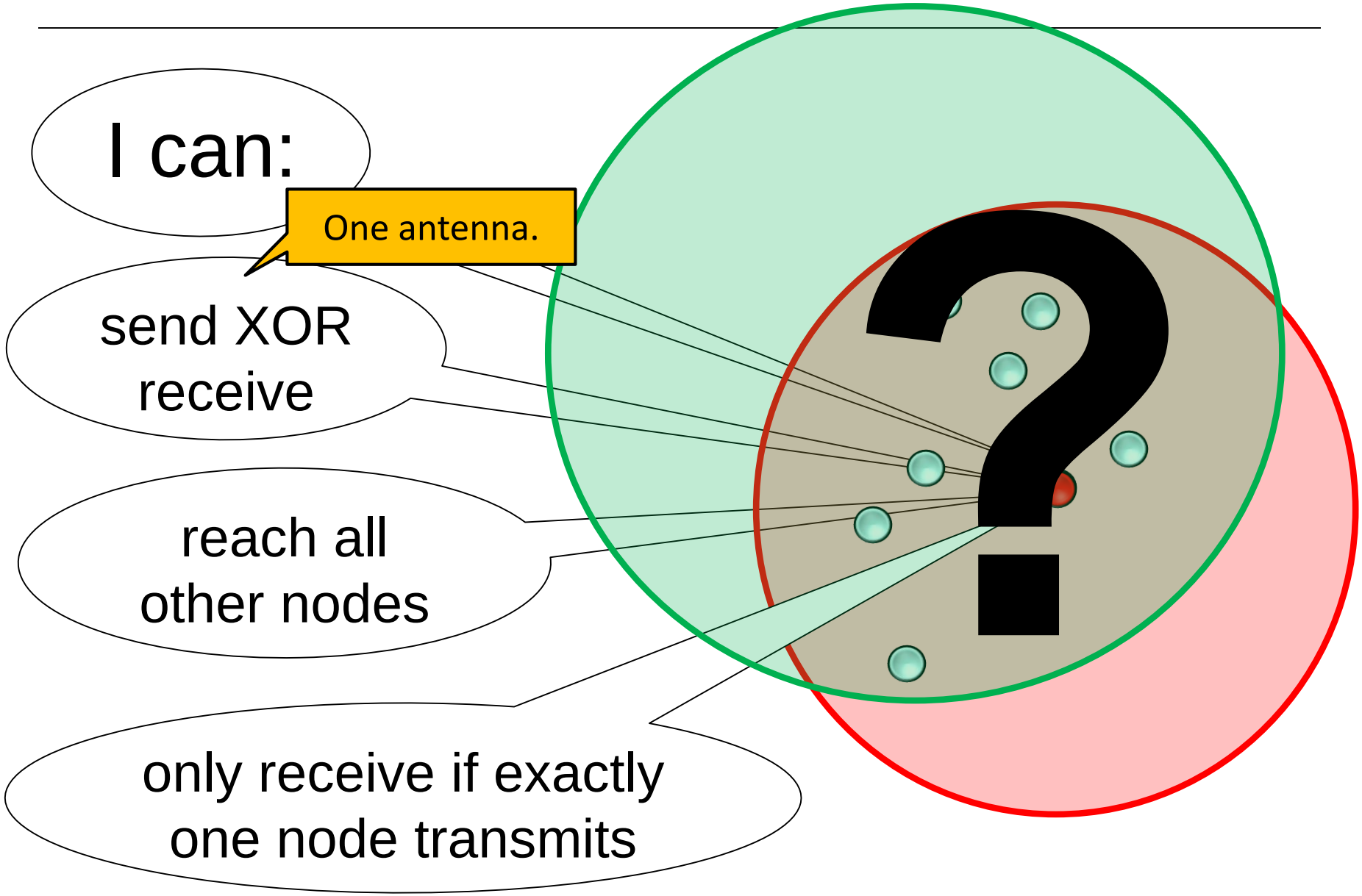


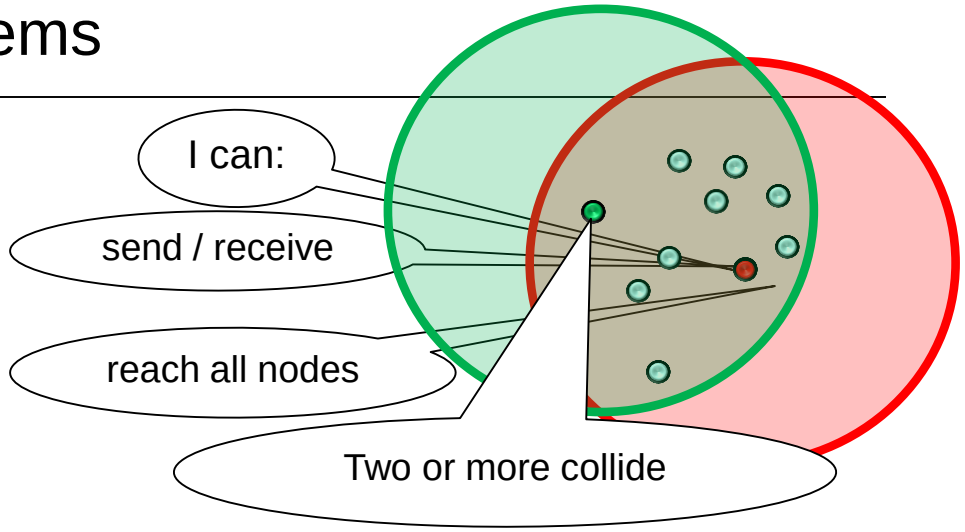
Two Model Variants: With or Without CD

Can I distinguish between busy and idle?

Collision Detection (CD)

In a system with CD, can distinguish busy from idle, i.e., whether one or more nodes send at the same time (interference), from nobody transmitting.

Fundamental Wireless Problems



Leader Election

How long does it take until one node can transmit alone? Leader could also coordinate slots in future...

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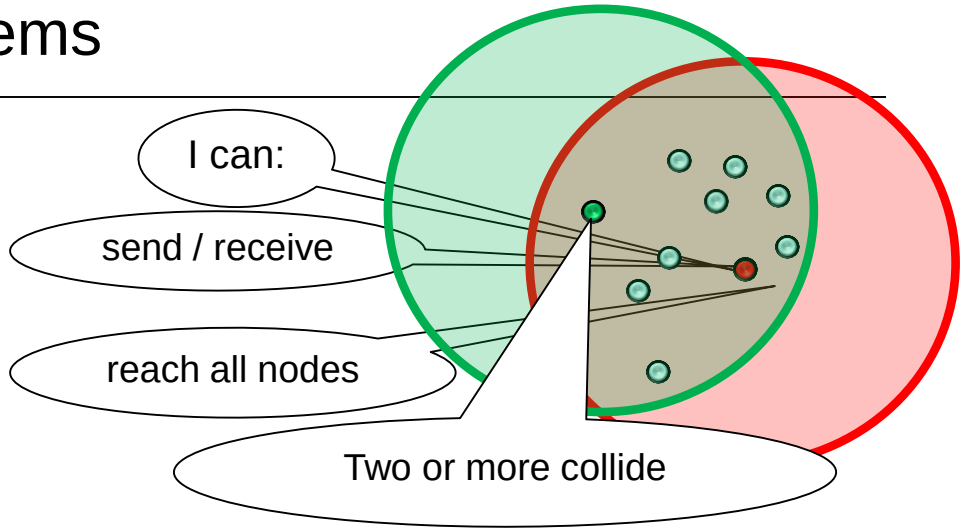


Nice to have: Leader could then also coordinate the future slots efficiently...

Leader Election

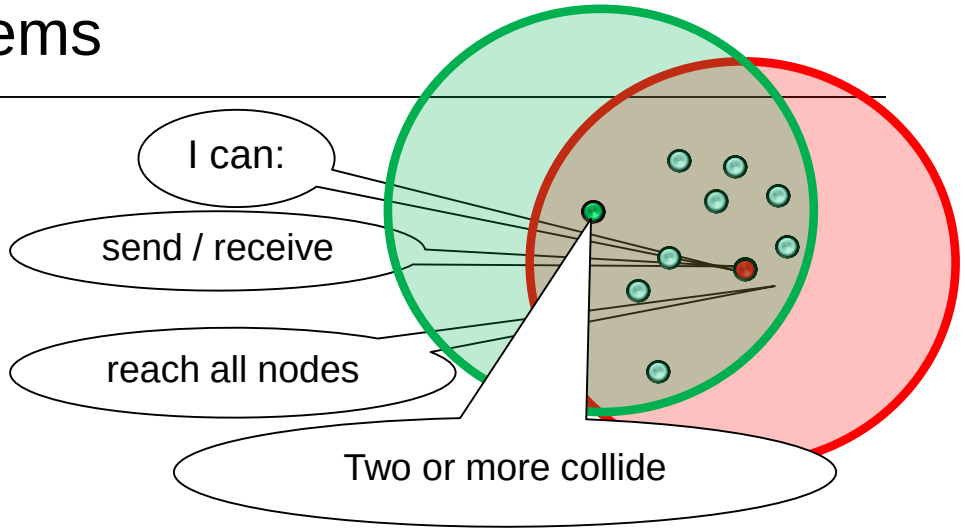
How long does it take until one node can transmit alone?

Initialization

How to assign IDs {1, 2, ….., n}? Slots could be divided accordingly then!

Nice to have: given that, nodes can send one-by-one! Fair and efficient.

Fundamental Wireless Problems



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With and without collision detection, with and without asynchronous wakeup (nodes wakeup up at arbitrary times), ...?

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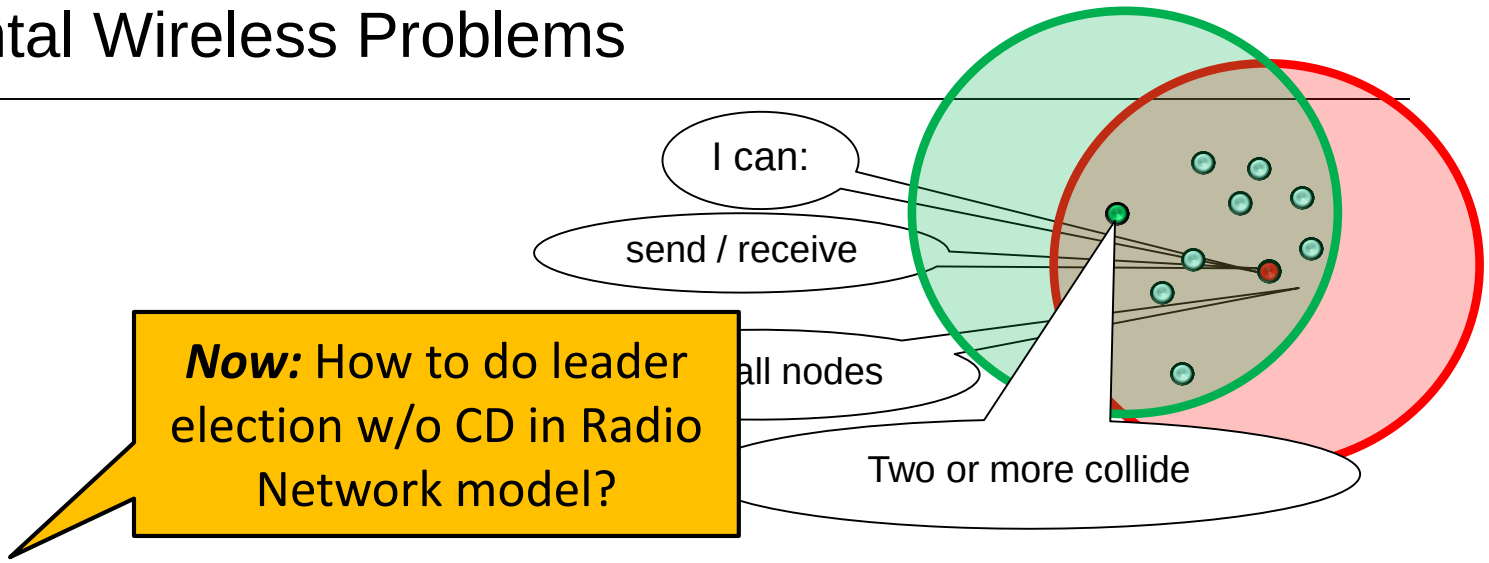

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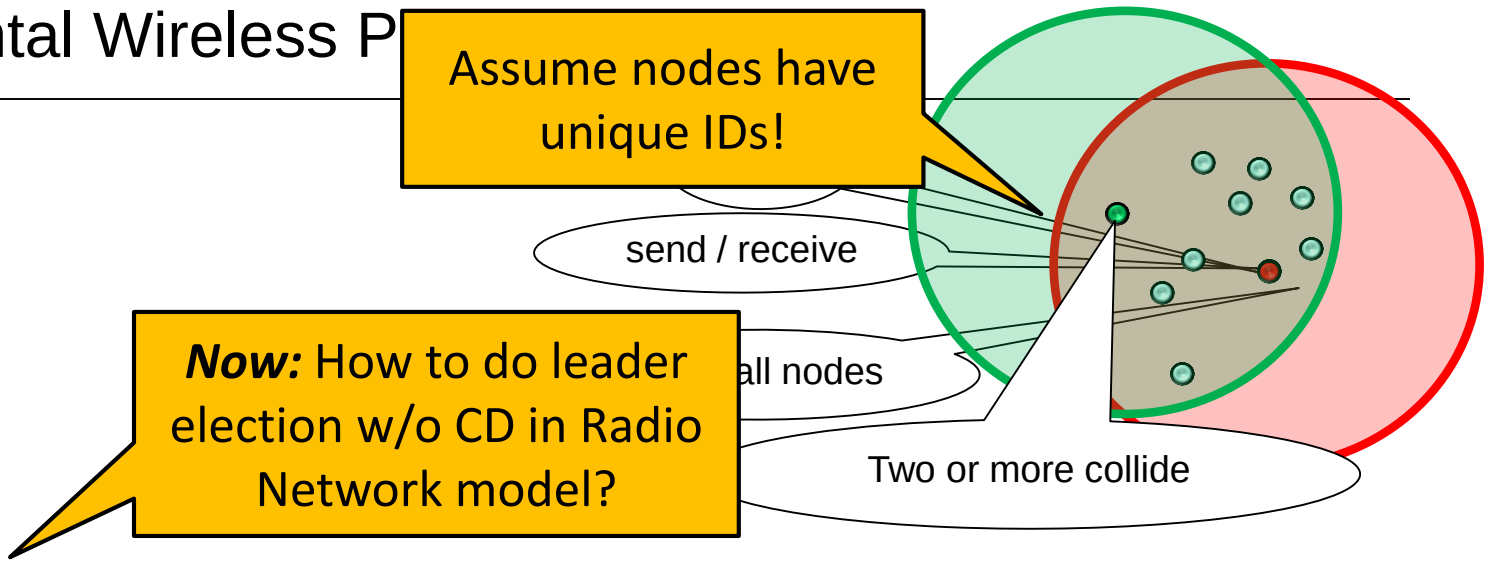

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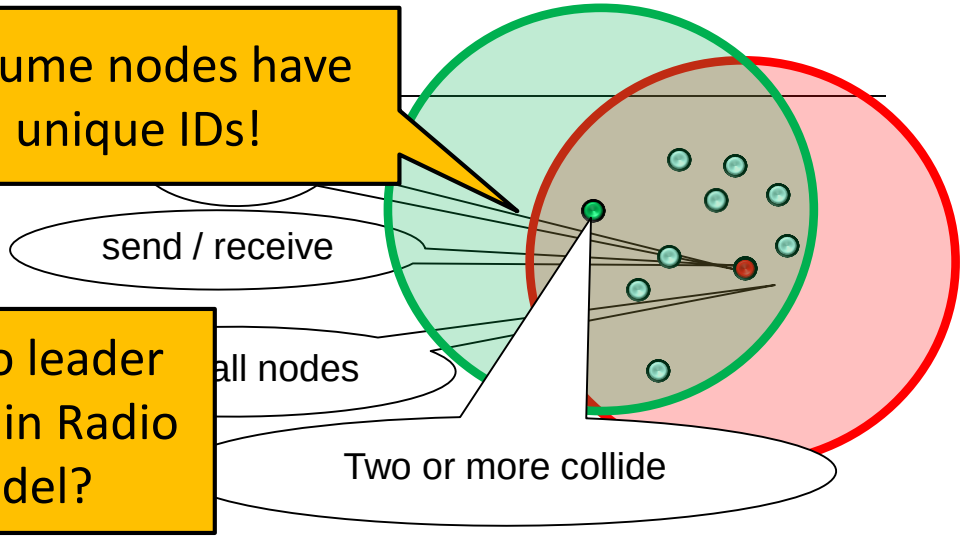
Assume nodes have unique IDs!



Now: How to do leader election w/o CD in Radio Network model?

Leader Election

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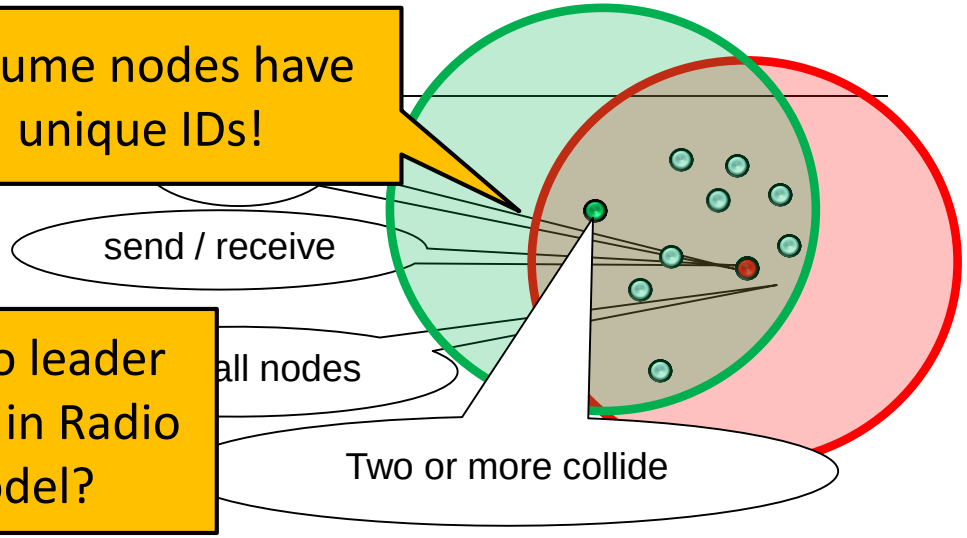

Now: How to do leader election w/o CD in Radio Network model?

Leader Election

How long c
can transm
coordinate

Many problem variants:

Idea: node with ID x sends at time x!
Problem: long time till first node transmits (say IDs are MAC addresses)?

asynchronous wakeup (nodes wakeup up at arbitrary times), ...?

Initialization

How to assign IDs {1, 2, ….., n}?
Slots could be div
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The Classic Solution: Slotted Aloha

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repeat
    transmit with probability 1/n
until one node has transmitted alone

Simple model: n is known (non-uniform)!

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And how does node know that it sent alone??

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Random Variable X

X is the RV denoting the number of nodes transmitting in a given time slot

Slotted Aloha

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Probability that 1st node sends alone plus probability that 2nd node sends alone etc.

Probability that exactly one node sends:

$$Pr[X = 1] = n \cdot \frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \approx \frac{1}{e};$$

So expected time complexity: e

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Nice: constant!

But the ACKs may collide as well!

y: e

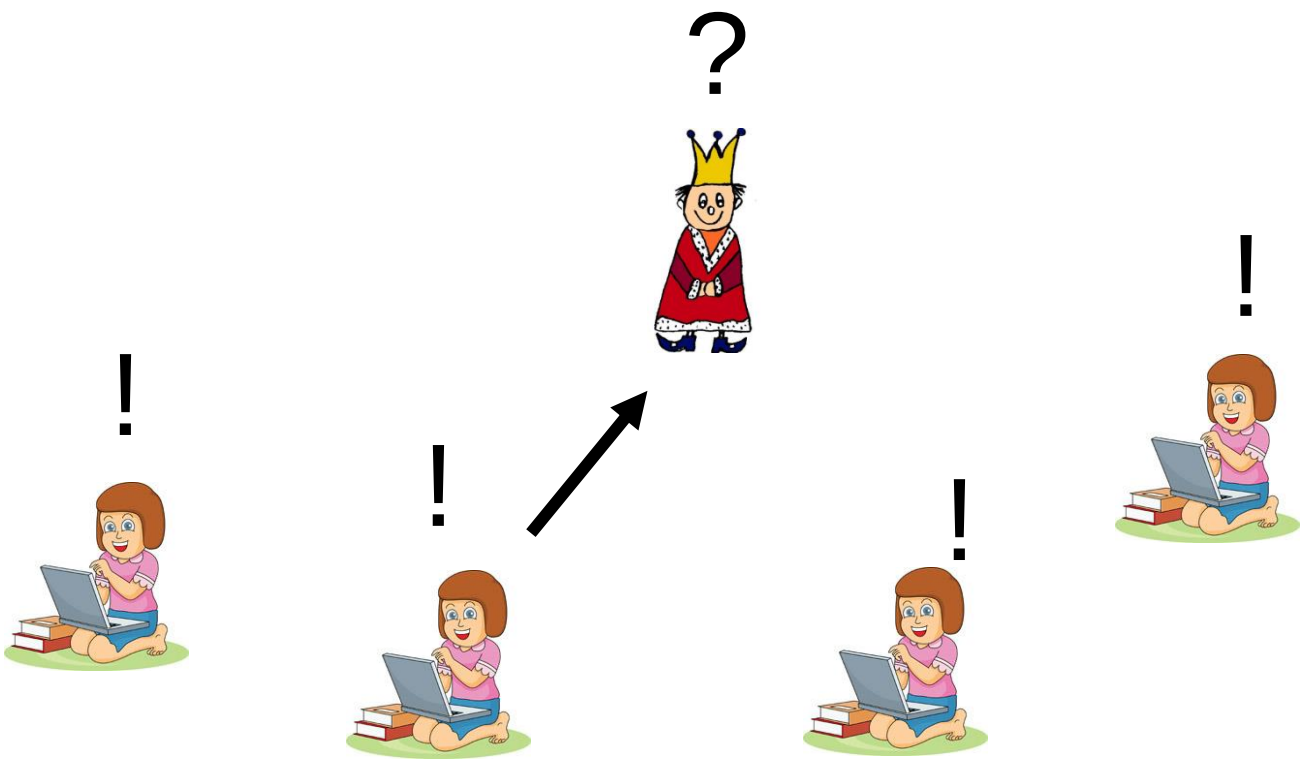
But: The problem is that the leader does not know he was alone and won?!

So we need a leader to let the leader know he is the leader?! ARGH! ☺

The Solution: Distributed ACK

Distributed ACK

After leader successfully sent alone:
all other nodes know the leader now, so the nodes start sending the ID of the leader with 1/n.



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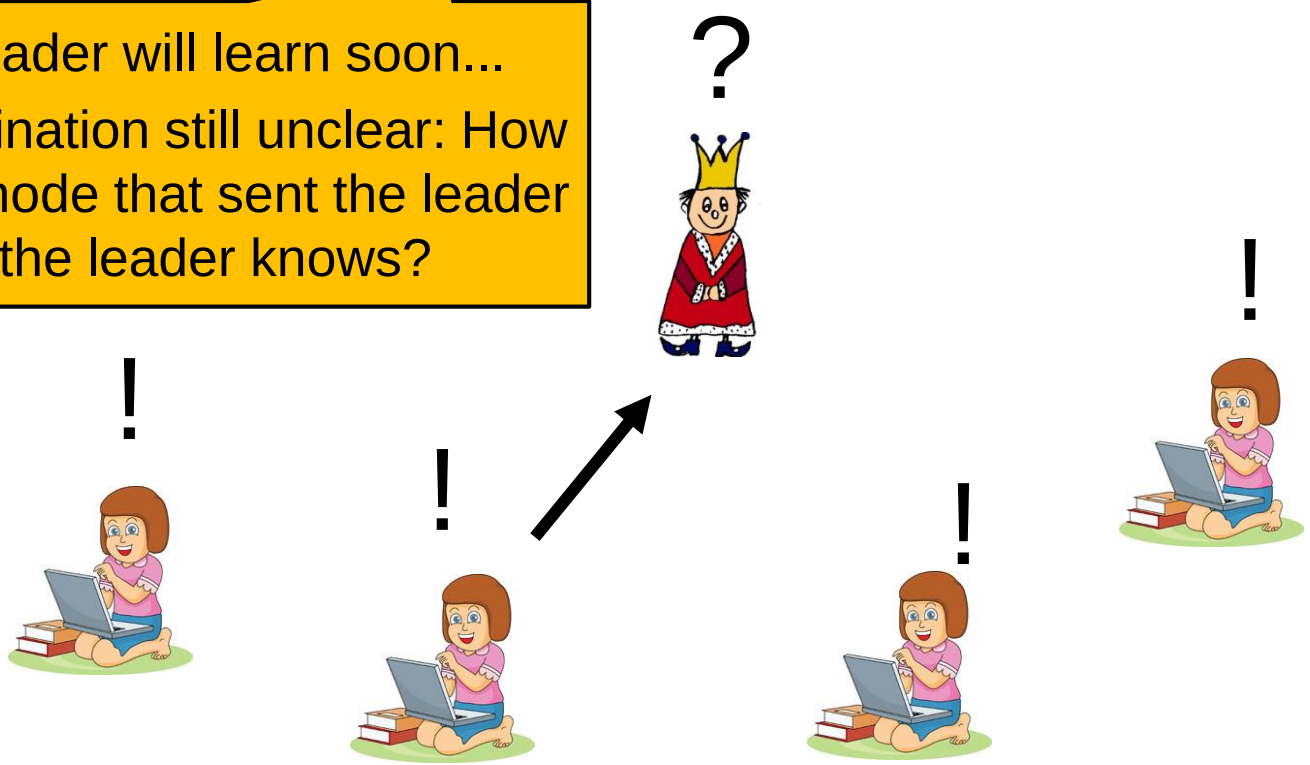

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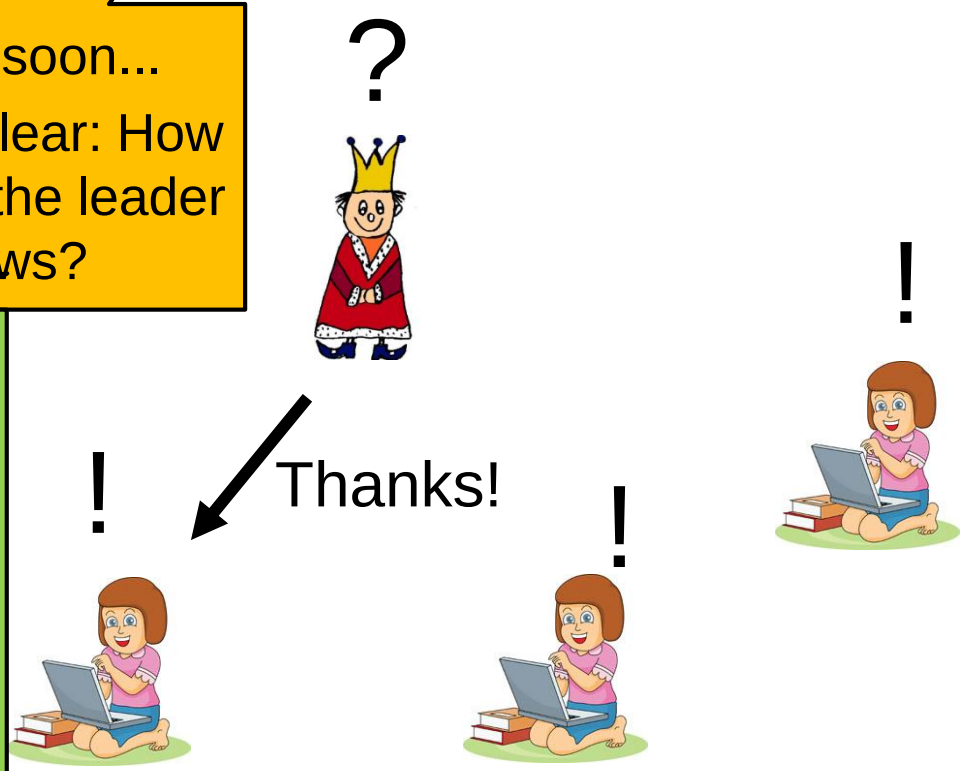
OK now easy: every other node already knows and can shut up, and the leader can send an acknowledgement to this node (e.g., leave next round reserved).



Remark: Synchronous Time Slots Not Needed

Slotted Aloha

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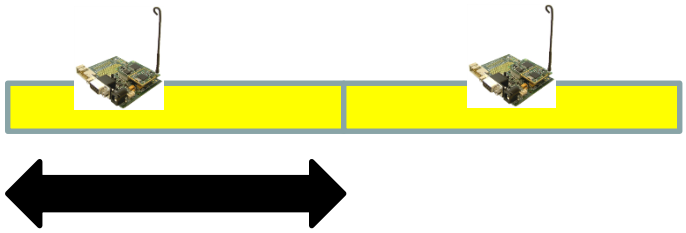
Protocol also works without time slots: just send when needed

«Pure ALOHA» reduces success probability by a factor of two:

Slotted ALOHA



success if nobody else sends here!

Pure ALOHA

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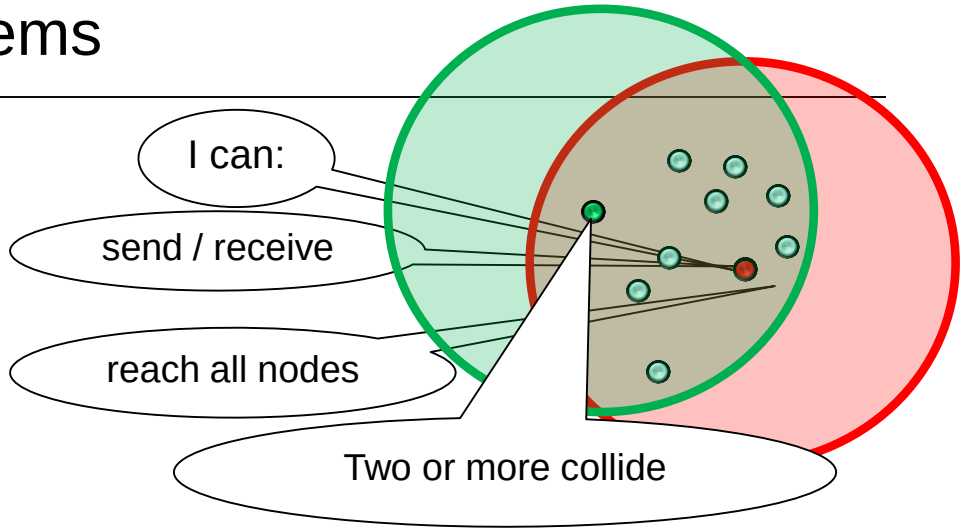
Fundamental Wireless Problems



Leader Election

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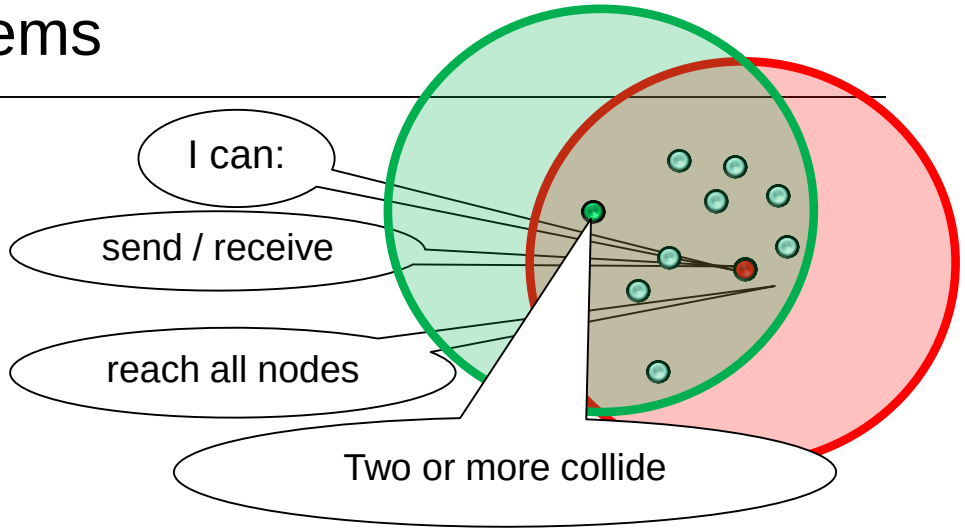

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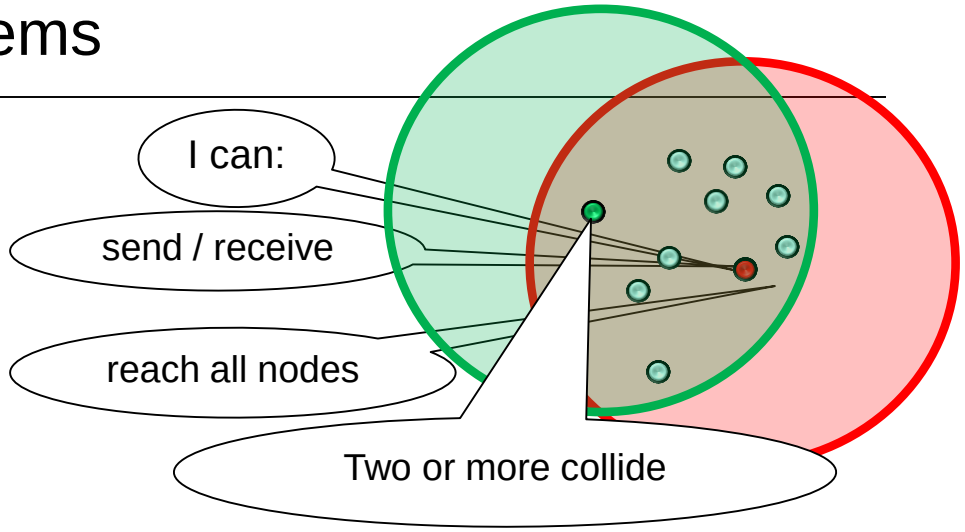

Easy: just do repeated ALOHA / leader election!

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Competition among n, n-1, n-2, n-3, … nodes!

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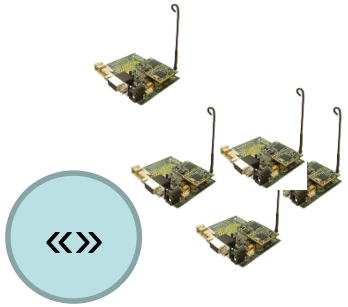
Let‘s do with CD first!

Idea: let nodes determine their ID in a binary manner!



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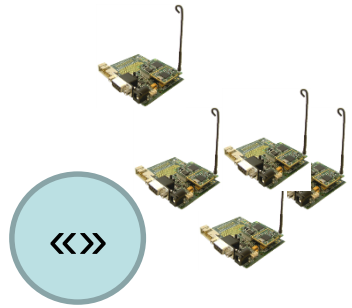

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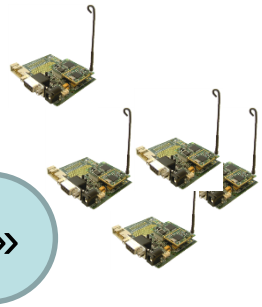

Each node chooses random bit $b_1 \in \{0,1\}$ and sends in slot b_1 and listens in slot $1- b_1$.

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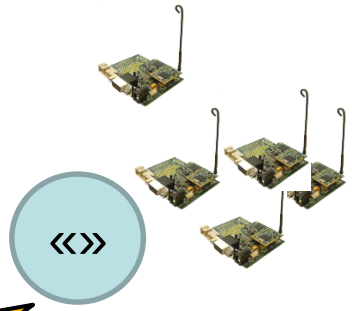

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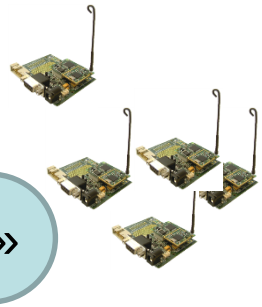

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Requires CD: *idle or busy?*

Slots

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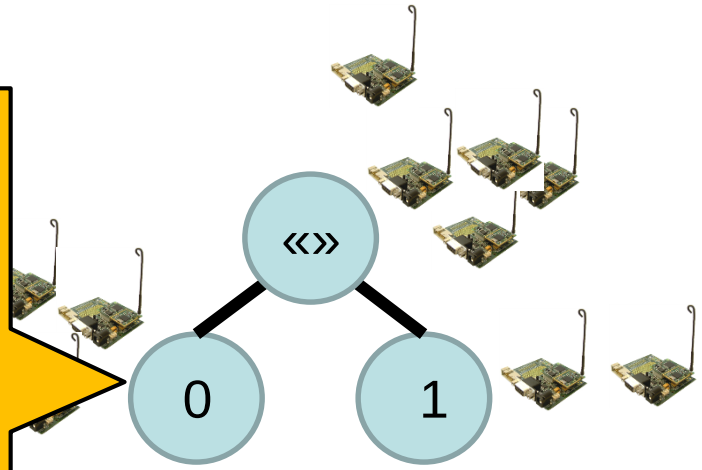
Now repeat: Each node with b_1 chooses next random bit $b_2 \in \{0,1\}$ to append, and sends in slot $b_1\, b_2$ and listens in slot $b_1\,(1- b_2)$.



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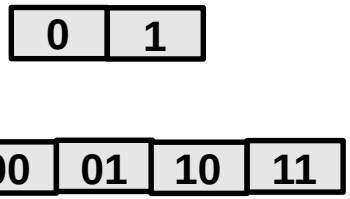

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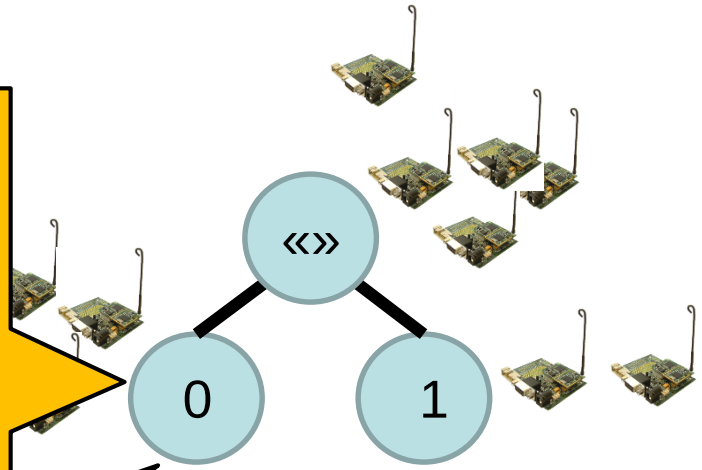
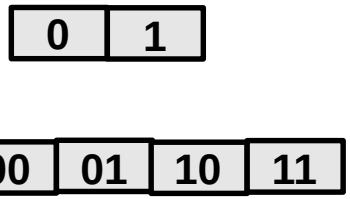


Again: if at least one transmission in both slots: append random bit 0 or 1, otherwise repeat step.

Slots



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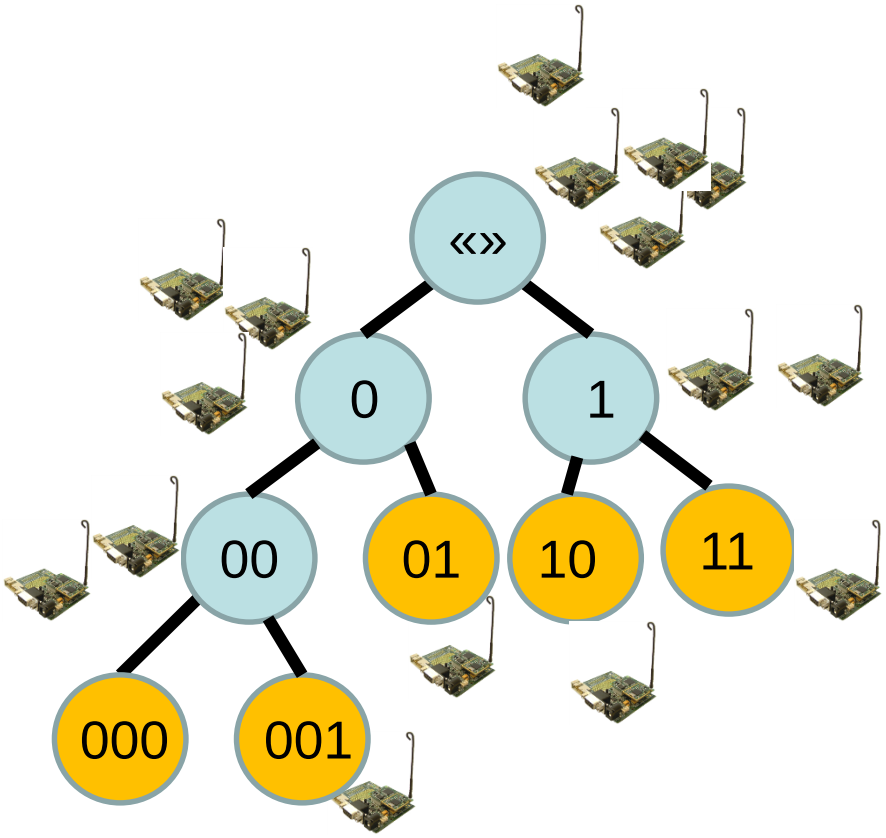

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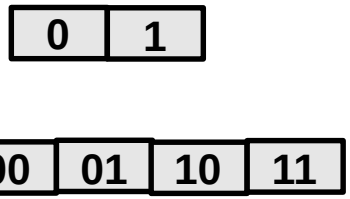

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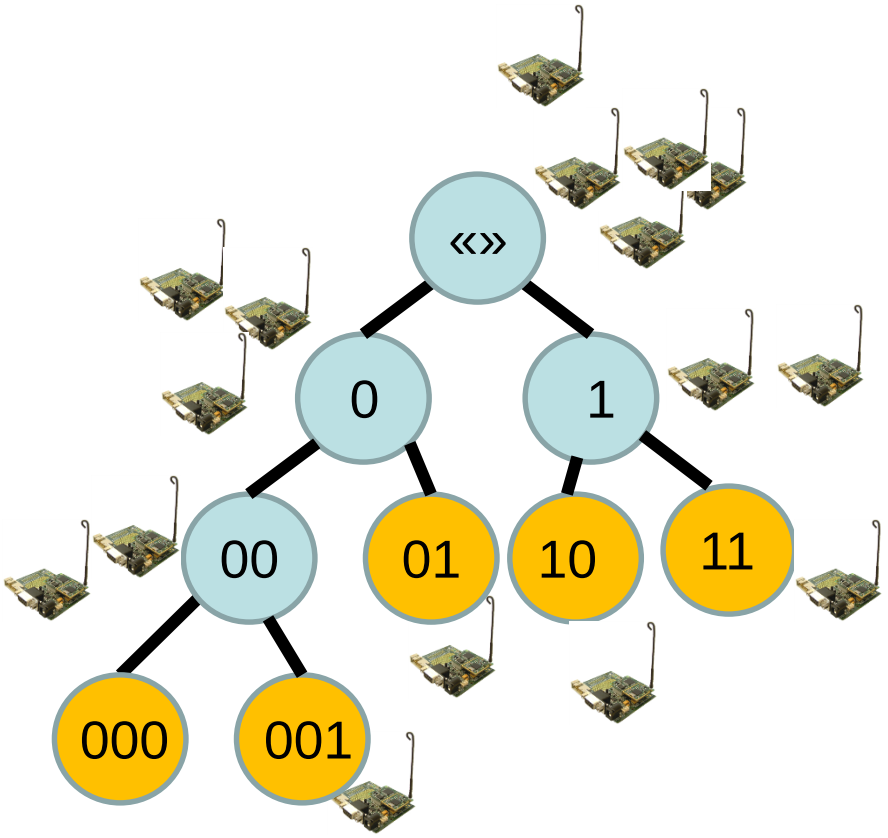

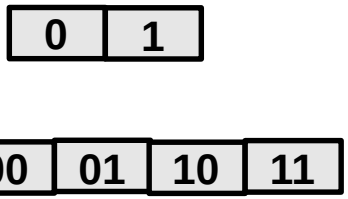

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… repeat until alone!

No need to know n!
But simultaneous start and collision detection.

Summary

Uniform Initialization with CD

For node v:

Main():

1. m = 0 (* already identified nodes *)
2. b_v = «» (* current bitstring of node v *)
3. RandomSplit(b_v)

RandomizedSplit(b)

1. Repeat
2. If b_v = b
3. choose r ∈ {0,1} at random (* next bit in string *)
4. in next two time slots: transmit in r, listen in 1+r mod 2
5. until there was at least one transmission in both slots (two groups, requires collision detection)
6. If b_v = b, append r to my string b_v, i.e., b_v :=b_vr
7. If single node u transmitted in slot r, gets global ID m; m++
8. Else recursively split remaining nodes:
9. RandomizedSplit(b0), RandomizedSplit(b1)

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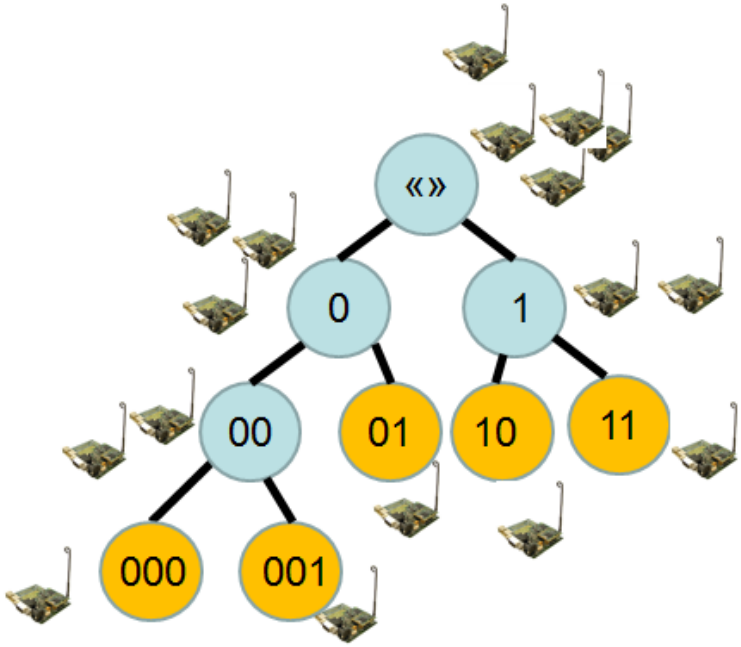
9. RandomizedSplit(b0), RandomizedSplit(b1)

Note: do not transform unique string to ID but use global counter m: only then IDs between 1 and n

Analysis

How many splits needed?



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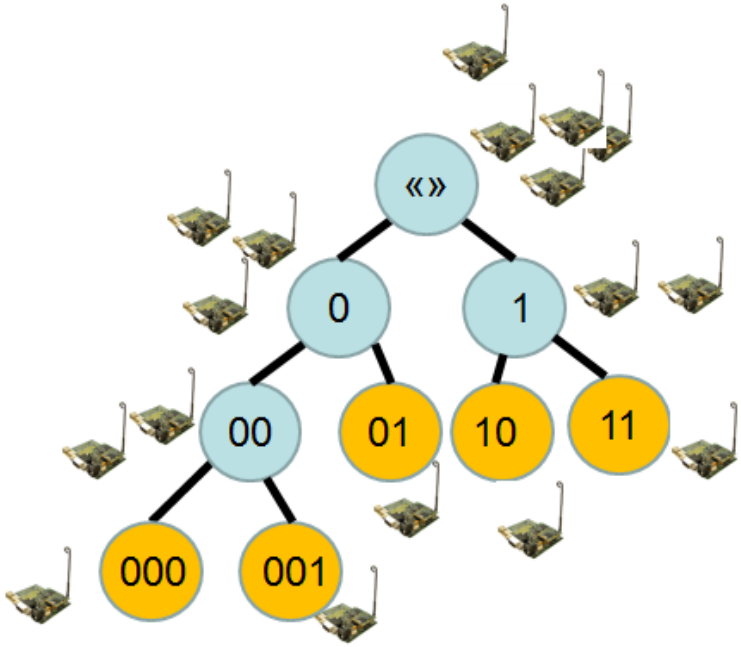
How many splits needed?

n-1: n leaves and n inner nodes in splitting tree.



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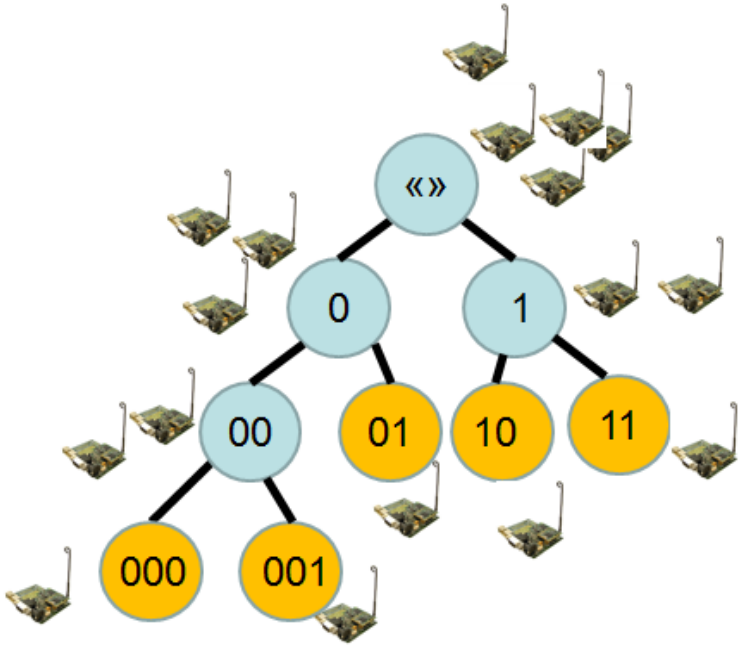

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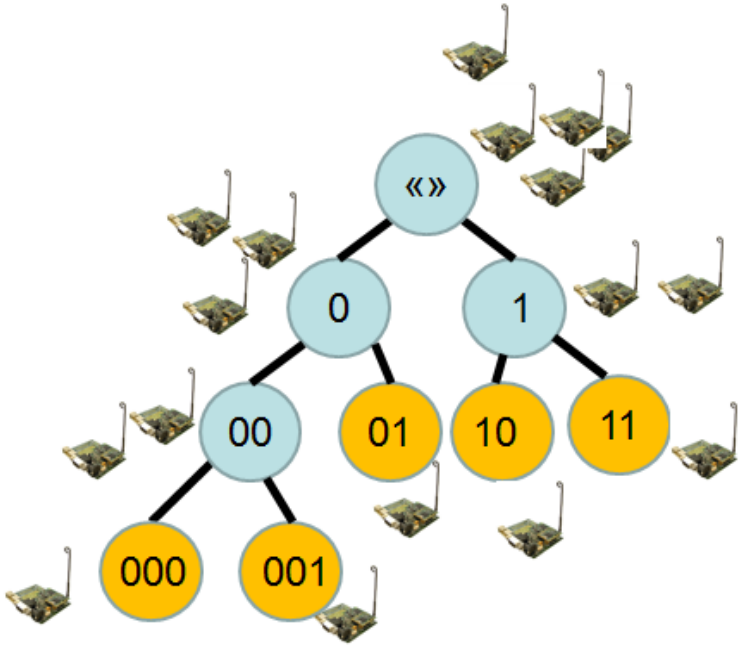

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Let RV X denote the size of the set.

$P[0<X<k] = 1 - P[X=0] - P[X=k] = 1 - 1/2^k - 1/2^k \geq 1/2$

So per split O(1) rounds, so overall runtime O(n).

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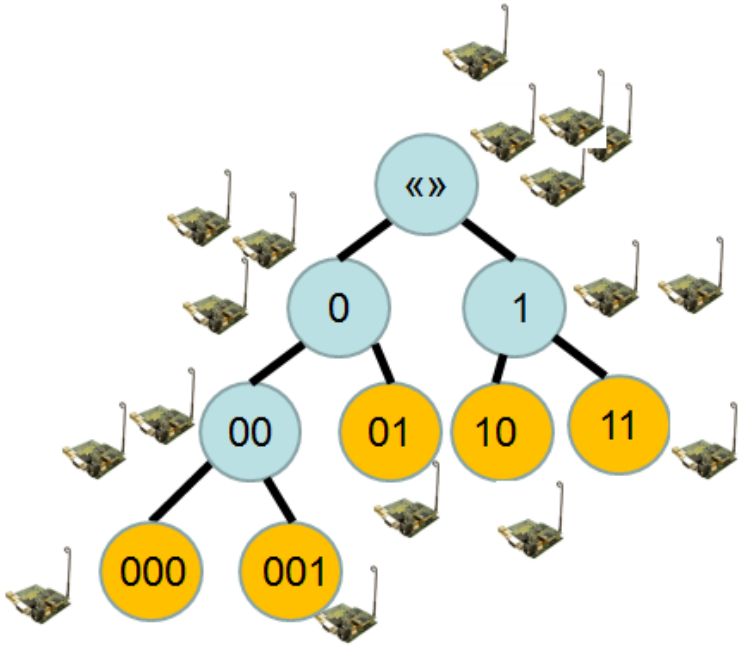


How long until successful (non-empty) split?

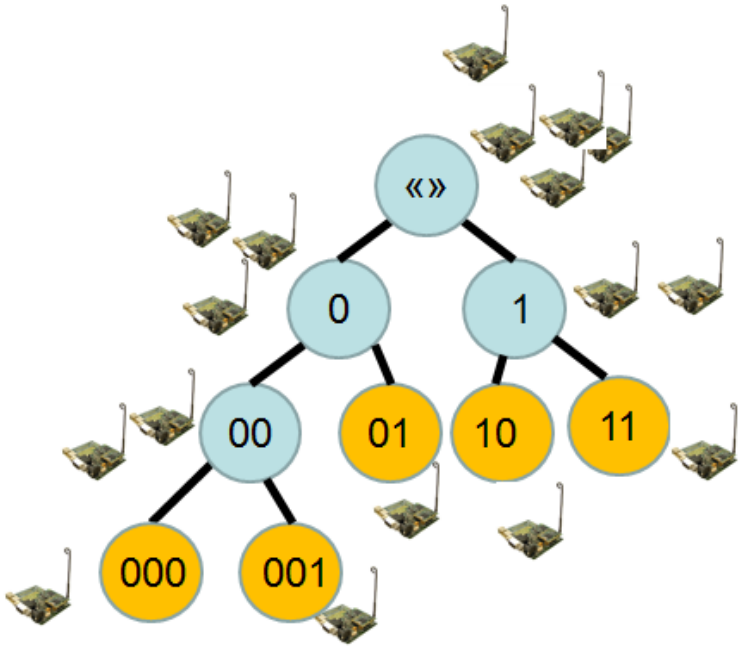
Let RV X denote the size of the set (total k nodes).

$P[0<X<k] = 1 - P[X=0] - P[X=k] = 1 - 1/2^k - 1/2^k \geq 1/2$

So per split O(1) rounds, so overall runtime O(n).

We solved a much more general problem than leader election in the same time as repeated Aloha!

Problems Solved? Really?



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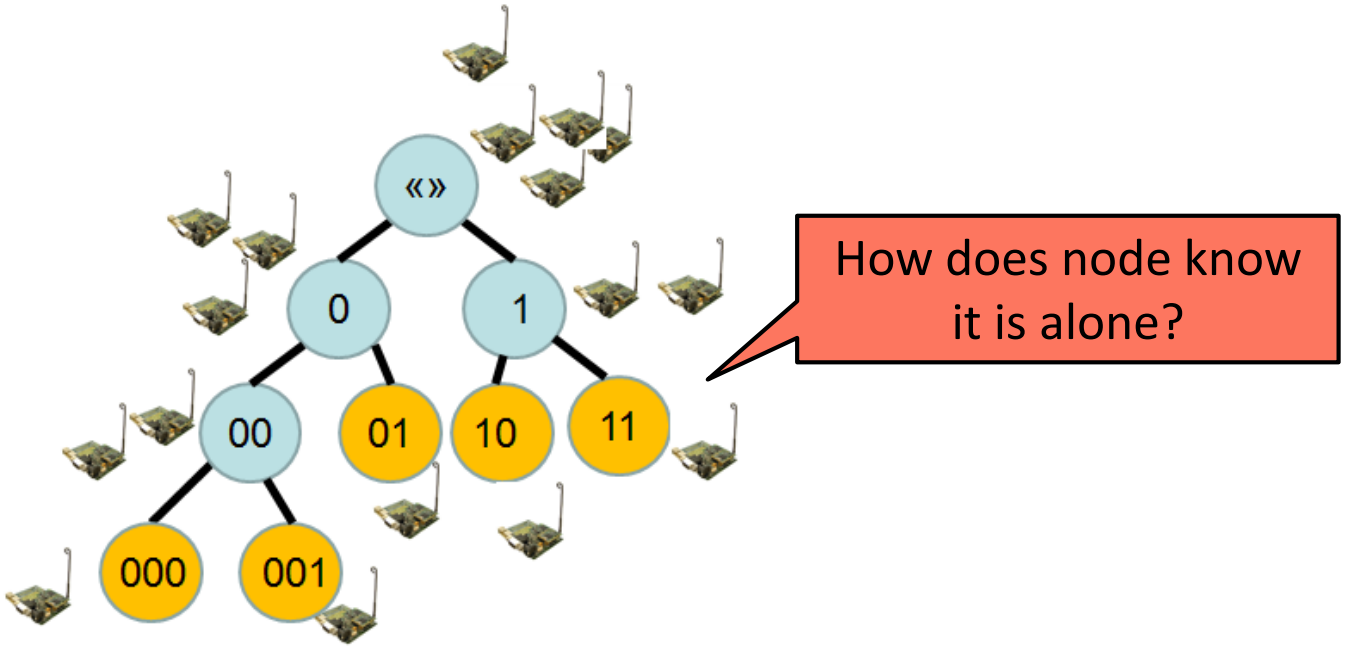

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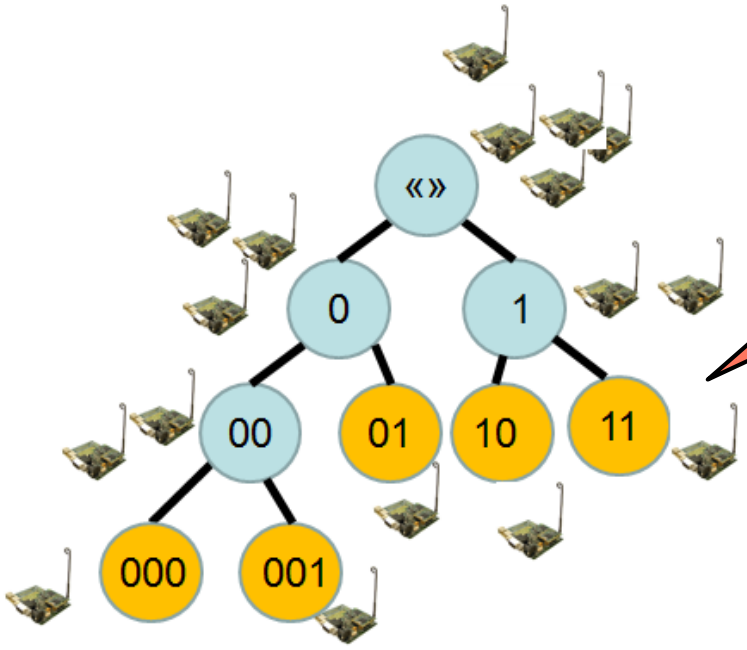

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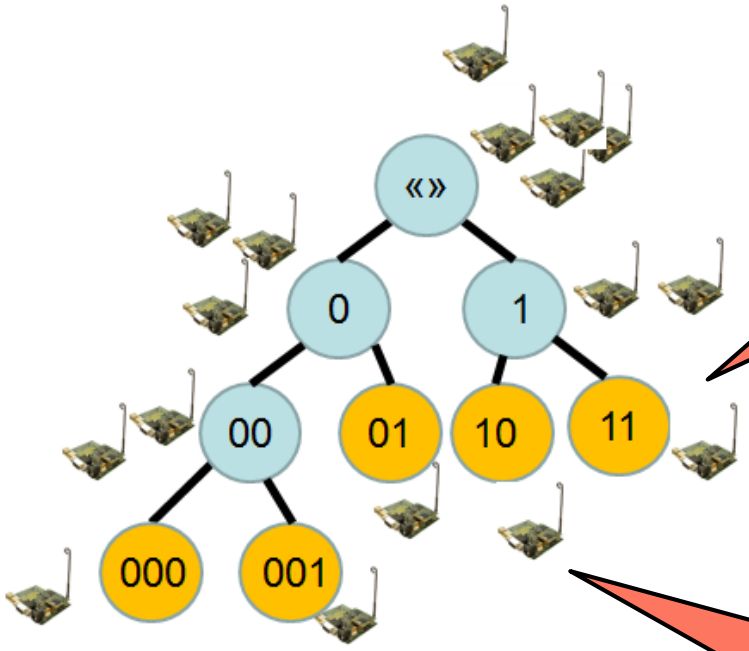

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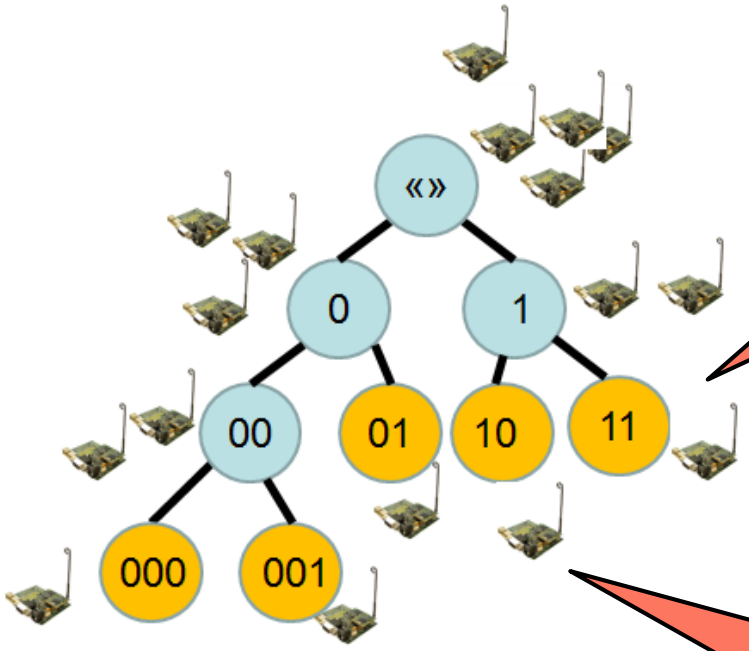

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How to do without collision detection?? Need a mechanism to distinguish between S={} and |S|>0.

There is a neat trick: given a leader, I can emulate CD in a scenario without CD!

Trick: Emulate CD w/o CD

assume: leader

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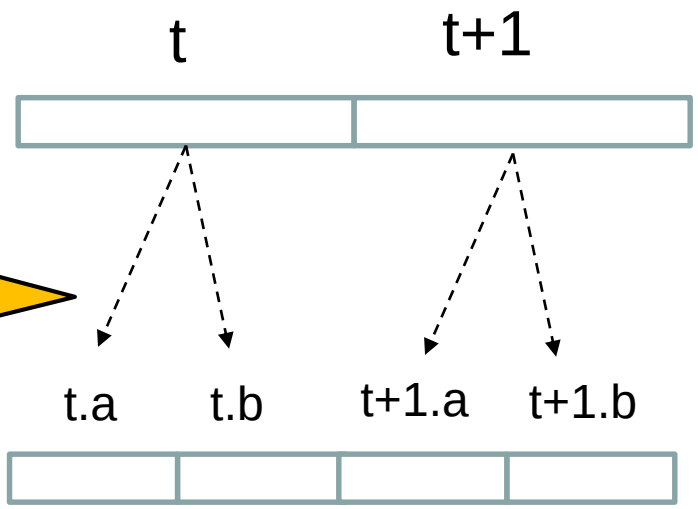
For the emulation, double each time slot (runtime ·2).



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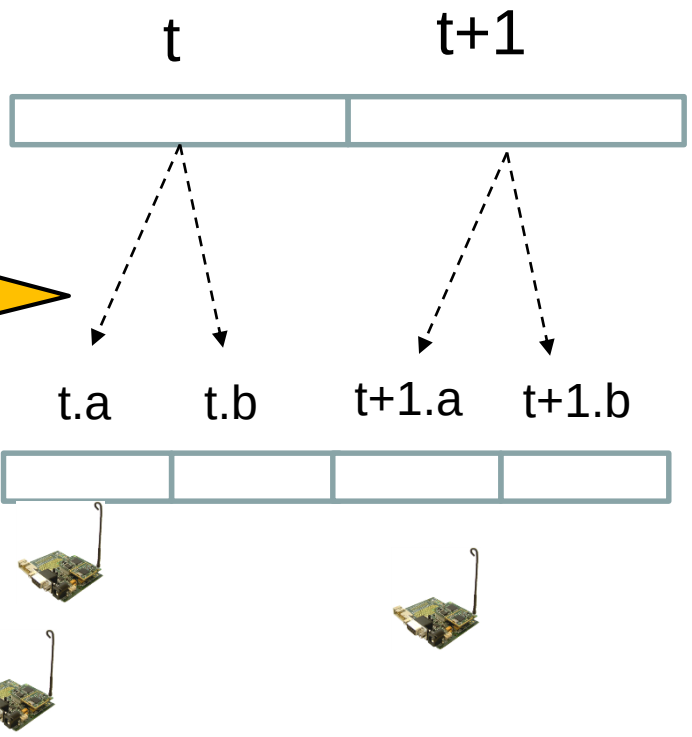

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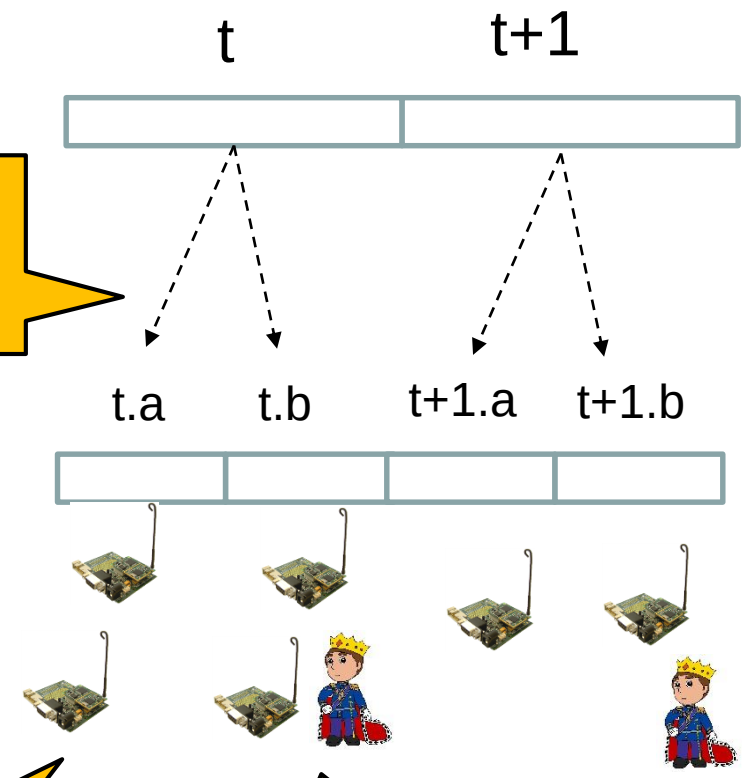

For the emulation, double each time slot (runtime ·2).

assume: leader

In the a-time-slots nodes transmit as usual.

In the b-time-slots nodes transmit as usual but also the leader!

Trick: Emulate CD w/o CD

Result: If idle in A-slot and busy in B-slot:|S|=0.

If busy in both slots, |S|>0. So can distinguish whether sending set S={} or|S|>0!

	nodes in S transmit	nodes in $S \cup \{\ell\}$ transmit
$\|S\| = 0$	✗	✔
$\|S\| = 1, S = \{\ell\}$	✔	✔
$\|S\| = 1, S \neq \{\ell\}$	✔	✗
$\|S\| \geq 2$	✗	✗

assume: leader

In the a-time-slots nodes transmit as usual.

In the b-time-slots nodes transmit as usual but also the leader!

Uniform Initialization (no CD)

1. Elect a leader
2. Divide every slot of the protocol with CD into two slots
   a) In the first slot, the nodes S transmit according to the protocol
   b) In the second slot, the nodes S from a) and the leader transmit
3. Distinguish the cases according to the table

   noise / silence : ✗

   successful transmission: ✔

	nodes in S transmit	nodes in $S \cup \{\ell\}$ transmit
$\|S\| = 0$	✗	✔
$\|S\| = 1, S = \{\ell\}$	✔	✔
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$\|S\| \geq 2$	✗	✗

Leader brings CD to any protocol! E.g., RFID protocols. Overhead: factor 2 runtime.

Result

From our O(n) runtime result w/ CD, by emulation we obtain:

Uniform Init. w/o CD

Given leader election, even without knowing the number of nodes and without collision detection, nodes can be initialized in time O(n).

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But how to elect leader if n is unknown?

Also: how long does it take to elect leader with high probability, not just on expection?

Leader Election With High Probability

With High Probability (whp.)

An event happens with high probability if it occurs with $p \geq 1\text{-}1/n^c$ for some constant c.

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Remember:

Slotted Aloha

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    transmit with probability 1/n
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log(n) many rounds: probability of not having a leader after log n rounds:

$$\left(1-\frac{1}{e}\right)^{c\ln n} = \left(1-\frac{1}{e}\right)^{e\cdot c'\ln n} \leq \frac{1}{e^{\ln n\cdot c'}} = \frac{1}{n^{c'}}.$$

Where do we stand?

Slotted Aloha

Slotted Aloha solves leader election without CD but if n is known in O(log n) rounds, whp.

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Idea: nodes should just start with high sending probability, and decrease it over time till successful (single transmission).

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It is easy to see: if the sum of sending probabilities is constant, then a leader will be elected with constant probability!

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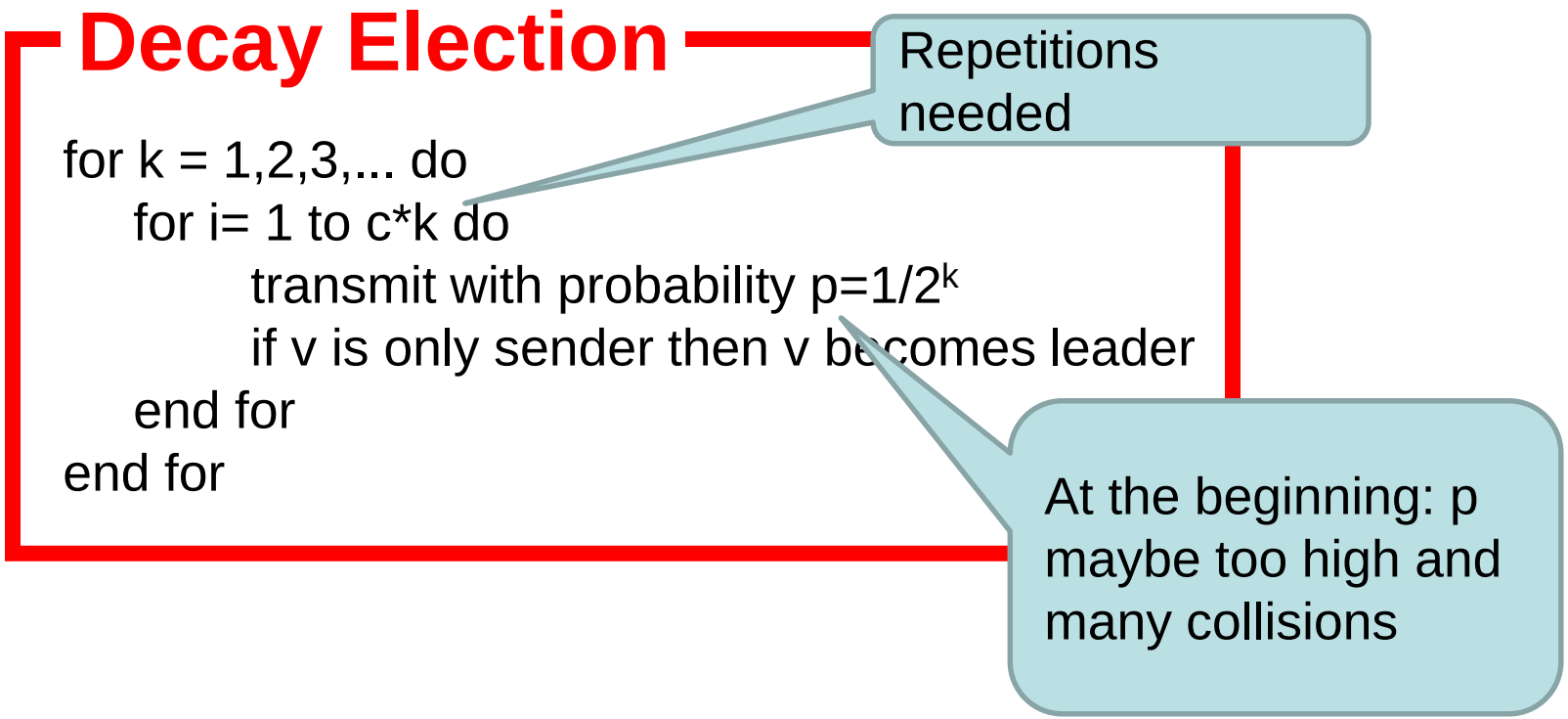
What if n is unknown?

Idea: nodes should just start with high sending probability, and decrease it over time till successful (single transmission).

But how? Tradeoff: **Decrease too slow:** takes long until constant cumulative probability, if **decrease too fast** I may miss the „sweet spot“!

It is easy to see: if the sum of sending probabilities is constant, then a leader will be elected with constant probability!

Leader Election With Unknown n



Leader Election With Unknown n

Decay Election

```
for k = 1,2,3,... do
    for i= 1 to c*k do
        transmit with probability p=1/2^k
        if v is only sender then v becomes leader
    end for
end for
```

Repetitions needed

At the beginning: p maybe too high and many collisions

Analysis: After k ≈ log n rounds, p ≈ 1/n, then we have c*k = c * log n many repetitions with constant cumulative probability! We know that slotted Aloha solves leader election in this situation in log n rounds, whp. So overall log(n)*log(n) rounds, with high probability.

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Decay Election solves uniform leader election without CD in $O(\log^2 n)$ rounds, whp.

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An algorithm based on «knock-out»/tournament:

Transmit or Keep Silent

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    transmit with probability p=1/2
    if at least one node transmitted then
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An algorithm based on «knock-out»/tournament:

Transmit or Keep

repeat

transmit with probability p

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learn via ACK

~ half of the nodes will never transmit again

Fast Uniform Leader Election (with CD)

Transmit or Keep Silent

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Fast Uniform Leader Election

Transmit or Keep S

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Define **successful round**: at most half of the active nodes transmit. I.e., problem divided in half.

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Transmit or Keep Silent

Transmit or Keep Silent solves uniform leader election without CD in O(log n) rounds, whp.

Super Fast Uniform Leader Election (with CD)

Super Fast Uniform Leader Election (with CD)

Idea:

1. Get a very rough estimation of the number of nodes n very fast
2. Get a more accurate estimation of the number of nodes (binary search)
3. Random walk to find constant approximation

At any point: stop if leader found ☺

Guess, guess, walk

Phase 1:

```
1: i := 1
2: repeat
3:    i := 2 · i
4:    transmit with probability 1/2^i
5: until no node transmitted
   {End of Phase 1}
```

Phase 2:

```
6:  l := 2^(i−2)
7:  u := 2^i
8:  while l + 1 < u do
9:     j := ⌈(l+u)/2⌉
10:    transmit with probability 1/2^j
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12:       u := j
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14:       l := j
15:    end if
16: end while
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```

Phase 3:

```
17: k := u
18: repeat
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24:    end if
25: until exactly one node transmitted
```

Phase 1 super-exponential decrease until nobody transmits: $1/2^2$,$1/2^4$, $1/2^8$, $1/2^{16}$, ...

finds raw estimate of $n \approx 2^i$, $i \approx 2^k$, i.e., of 2^{2^k}

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Reduce quickly, so log log n time

Phase 2 … estim… …ter sea… …nary …

Binary search in log interval so also log log n time

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As we will see, also log log n time

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Phase 2:

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Analysis: see lecture notes.

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Backup Slides

Guess Guess Walk

Guess-Guess-Walk elects leader

- with probability at least 1-loglog(n)/log(n)
- in time O(loglog n).

Why?

Guess Guess Walk

Guess-Guess-Walk elects leader
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- in time O(loglog n).

Why?

We will show:

1. In Phase 1+2, we fail in each round (go to next phase too early/late) with probability 1/log(n), so union bound over all rounds: overall we fail with probability loglog(n)/log(n)
2. Phase 1 terminates after O(loglog n) rounds
3. Phase 2 is a binary search on interval of size O(log n), hence terminates in O(loglog n) time: after the phase, our estimate of log n is at most loglog n away from the true value
4. Phase 3 requires O(loglog n) slots to elect a leader with probability 1-1/log(n).

Guess Guess Walk

Guess-Guess-Walk elects leader
- with probability at least 1-loglog(n)/log(n)
- in time O(loglog n).

Why?

We can show that with a small loglog(n) additive deviation from ideal sending probability, we almost always have busy or idle (CD detects)!

$$If\ j > \log n + \log\log n,\ then\ Pr[X > 1] \leq \frac{1}{\log n}.$$

$$If\ j < \log n - \log\log n,\ then\ P[X = 0] \leq \frac{1}{n}.$$

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Lemma 1: For large enough j when sending with probability $1/2^j$, unlikely to have many senders:

If $j > \log n + \log\log n$, then $P[X>1] \leq 1/\log(n)$.

Proof. The nodes transmit with probability $1/2^j < 1/2^{\log n + \log\log n} = \frac{1}{n \log n}$. The expected number of nodes transmitting is $E[X] = \frac{n}{n \log n}$. Using Markov's inequality (see Theorem 13.21) yields $Pr[X > 1] \leq Pr[X > E[X] \cdot \log n] \leq \frac{1}{\log n}$. □

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Lemma 2: For small j, many nodes will send:

If $j < \log n - \log\log n$, then $P[X=0] \leq 1/n$.

Proof. The nodes transmit with probability $1/2^j < 1/2^{\log n - \log\log n} = \frac{\log n}{n}$. Hence, the probability for a silent time slot is $(1 - \frac{\log n}{n})^n = e^{-\log n} = \frac{1}{n}$. □

```
1: i := 1
2: repeat
3:    i := 2 · i
4:    transmit with probability 1/2^i
5: until no node transmitted
   {End of Phase 1}
```

```
6: l := 2^(i−2)
7: u := 2^i
8: while l + 1 < u do
9:    j := ⌈(l+u)/2⌉
10:   transmit with probability 1/2^j
11:   if no node transmitted then
12:      u := j
13:   else
14:      l := j
15:   end if
16: end while
   {End of Phase 2}
```

```
17: k := u
18: repeat
19:    transmit with probability 1/2^k
20:    if no node transmitted then
21:       k := k − 1
22:    else
23:       k := k + 1
24:    end if
25: until exactly one node transmitted
```

Lemma 3: Let v be such that $2^{v-1} < n \leq 2^v$, i.e., $v \approx \log n$. If $k > v+2$, then $P[X>1] \leq 1/4$.

Proof. Markov's inequality yields

$$Pr[X > 1] = Pr\left[X > \frac{2^k}{n}E[X]\right] < Pr[X > \frac{2^k}{2^v}E[X]] < Pr[X > 4E[X]] < \frac{1}{4}.$$

□

```
1: i := 1
2: repeat
3:    i := 2 · i
4:    transmit with probability 1/2^i
5: until no node transmitted
   {End of Phase 1}
```

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17: k := u
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21:       k := k − 1
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23:       k := k + 1
24:    end if
25: until exactly one node transmitted
```

Lemma 4: If k < v-2, then P[X=0] ≤ 1/4.

Proof. A similar analysis is possible to upper bound the probability that a transmission fails if our estimate is too small. We know that $k \leq v-2$ and thus

$$Pr[X=0] = \left(1 - \frac{1}{2^k}\right)^n < e^{-\frac{n}{2^k}} < e^{-\frac{2^{v-1}}{2^k}} < e^{-2} < \frac{1}{4}.$$

□

```
1: i := 1
2: repeat
3:    i := 2 · i
4:    transmit with probability 1/2^i
5: until no node transmitted
   {End of Phase 1}
```

```
6: l := 2^(i−2)
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   {End of Phase 2}
```

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17: k := u
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19:    transmit with probability 1/2^k
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21:       k := k − 1
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23:       k := k + 1
24:    end if
25: until exactly one node transmitted
```

Lemma 5: If v-2 ≤ k ≤ v+2, then the probability that exactly one node transmits is constant.

Proof. The transmission probability is $p = \frac{1}{2^{v \pm \Theta(1)}} = \Theta(1/n)$, and the lemma follows with a slightly adapted version of Theorem 13.1.

□

```
1: i := 1
2: repeat
3:    i := 2 · i
4:    transmit with probability 1/2^i
5: until no node transmitted
   {End of Phase 1}
```

```
6: l := 2^(i−2)
7: u := 2^i
8: while l + 1 < u do
9:    j := ⌈(l+u)/2⌉
10:   transmit with probability 1/2^j
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15:   end if
16: end while
   {End of Phase 2}
```

```
17: k := u
18: repeat
19:    transmit with probability 1/2^k
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24:    end if
25: until exactly one node transmitted
```

Lemma 6: With probability 1-1/log(n), a leader is found in Phase 3 in O(loglog n) time.

Proof. For any k, because of Lemmas 13.13 and 13.14, the random walk of the third phase is biased towards the good area. One can show that in $\mathcal{O}(\log\log n)$ steps one gets $\Omega(\log\log n)$ good transmissions. Let Y denote the number of times exactly one node transmitted. With Lemma 13.15 we obtain $E[Y] = \Omega(\log\log n)$. Now a direct application of a Chernoff bound (see Theorem 13.22) yields that these transmissions elect a leader with probability $1 - \frac{1}{\log n}$. □

Guess Guess Walk

Guess-Guess-Walk elects leader
- with probability at least 1-loglog(n)/log(n)
- in time O(loglog n).

Comments:

- With a more detailed analysis, we can increase the success probability to 1-1/log(n)

Uniform Lower Bound

Any uniform protocol which elects leader with probability at least $1-1/2^t$ must run for at least t rounds.

Why?

Uniform Lower Bound

Any uniform protocol which elects leader with probability at least $1-1/2^t$ must run for at least t rounds.

Why?

Claim already holds for n=2 nodes. (Hence also for n>2: if a network with n>2 nodes could find a leader quicker with higher probability then so could n=2 nodes: one node could simulate the rest of the network.)

For two nodes, they must reach a situation where exactly one transmits. Uniform, so nodes use same p in each round (cannot adapt: same feedback). The corresponding probability is at most

$$P[X=1] = 2 * p * (1-p) \leq 1/2$$

Thus after time t, the election probability is at most $1-1/2^t$.

For t=loglog(n), Guess-Guess-Walk almost tight!

Leader Election with Asynchronous Wakeup?

Recall:

Aynchronous Wakeup: node start executing algorithm at different times

Assume uniform and anonymous: nodes do not know n, do not have an identifier, execute the same code.

Observations:

- At some point the nodes must transmit.
- Look at first time slot where some nodes transmit. They will do so with probability p, independent of n. (Uniform)

Strategy for adversary to make runtime high?

Wakeup Lower Bound

Any uniform protocol has time complexity $\Omega(n/\log n)$ for leader election if nodes wake up arbitrarily.

Analysis.

Leader Election with Asynchronous Wakeup?

Wakeup Lower Bound

Any uniform protocol has time complexity Ω(n/log n) for leader election if nodes wake up arbitrarily.

Analysis.

- Assume first transmission at time t, with probability p, independent of n
- Adversary wakes up w = c/p ln(n) nodes in each time slot, for some constant c.
- First batch of w nodes will transmit with probability p (uniform = independent of n)
- Probability that exactly one of them transmits in first time slot (event E_1)

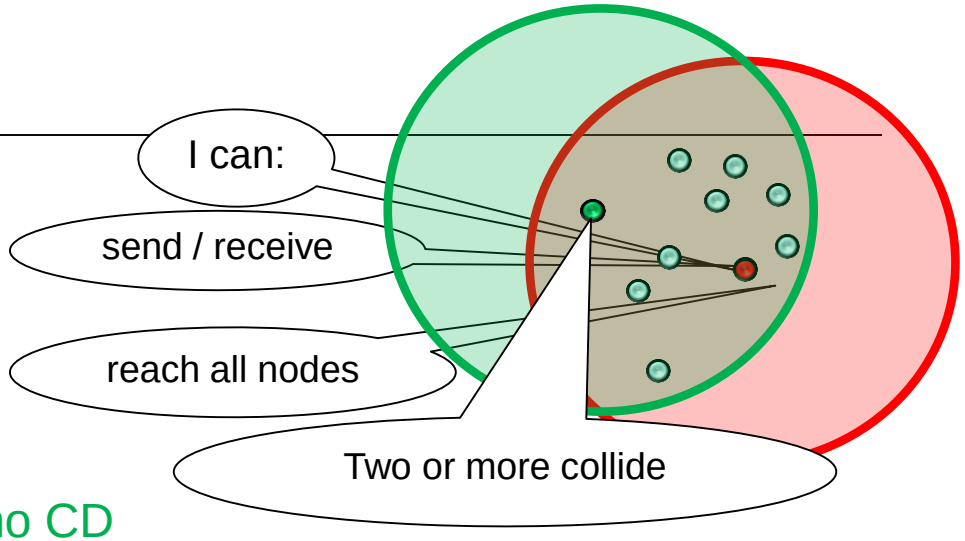
$$P[E_1] = w * p * (1-p)^{w-1} < 1/n^c$$

- ... for w = n/log(n) it is polynomially small (whp. unsuccessful).
- Nodes cannot distinguish between noise and idle: so is the same for every batch of nodes that wakes up! (No CD: do not know whether too many or too few nodes.)
- So we have n/w many time slots. Probabilty that none of those works out

$$P[E] = (1-P[E_1])^{n/w} > 1-1/n^c$$

for w = n/log(n), so does not work out whp.

Summary



Leader Election

How long does it take until one node can transmit alone?

- e in expectation, knowing n
- O(log n) whp, (without) knowing n, no CD
- O(log log n) without knowing n, with CD, with probability 1-loglog n/log n
- 1-1/log n election probability lower bound for log log n

Initialization

How to assign IDs {1, 2, ….., n}?

- O(n) with RandomizedSplit

Asynchronous Wakeup

How long for leader election if nodes wakeup up at arbitrary times?

- Ω(n/log n) without IDs and without knowing n

End of Lecture