Solving parity games

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Formal Methods Update 2006, IIT Guwahati
4 July 2006

Outline

- Parity games
- An efficient algorithm for solving parity games [Jurdziński]
- Solving parity games through strategy improvement [Jurdziński and Vöge]

Parity games

- Two players, 0 and 1

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  - Max-parity: largest colour that occurs infinitely often in the play is even
  - Min-parity: smallest colour that occurs infinitely often in the play is even

Memoryless determinacy for parity games

Theorem

The set of positions of a parity game can be partitioned as W_0, from where player 0 wins with a memoryless strategy, and W_1, from where player 1 wins with a memoryless strategy.

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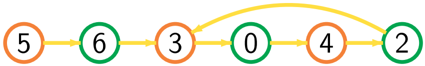
- Complexity is $\mathbf{O(mn^d)}$
  - $\mathbf{m}$ edges, $\mathbf{n}$ states, largest colour $\mathbf{d}$
- Can we identify $\mathbf{W_0}$ and $\mathbf{W_1}$ more efficiently?

Solitaire games

- Observation If both players play by memoryless strategy, each infinite play is a finite prefix followed by a simple loop



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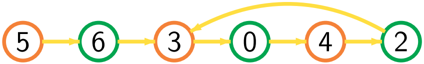

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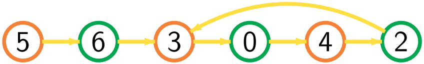

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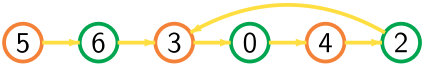

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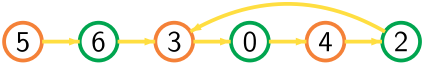

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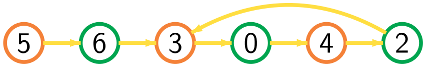

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Lemma

f_0 closed on X wins from all states in X iff all simple cycles in the game restricted to X are even.

Parity progress measures

- For a game with $\mathbf{d}$ colours, assign a $\mathbf{d+1}$-tuple $\rho(\mathbf{v}) = (\mathbf{n_0}, \mathbf{n_1}, \ldots, \mathbf{n_d})$ to each position

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- Parity progress measure
  For each edge $v \longrightarrow w$
  - $c(v)$ even $\Rightarrow \rho(v) \geq_{c(v)} \rho(w)$
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If a solitaire game admits a parity progress measure, then every simple cycle in the game is even.

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  Claim 1 Let $\rho(v) = (n_0, n_1, \ldots, n_d)$.
  For odd i, $n_i \leq |V_i + 1|$.

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  Claim 1 Let $\rho(v) = (n_0, n_1, \ldots, n_d)$.
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- **Claim 2** $\rho(v)$ is a parity progress measure.

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- Range of ρ is M where $M = \{0\} \times \{0, \ldots, |V_1|\} \times \{0\} \times \cdots \times \{0, \ldots, |V_d|\}$

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- A trivial game progress measure assigns $\top$ everywhere.
- Let $\|\rho\| = \{\mathbf{v} \mid \rho(\mathbf{v}) \neq \top\}$
- Our aim is to find ρ such that $\|\rho\|$ is maximized.

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- Given ρ, define the strategy $\mathbf{f}_0^{\rho}$ that chooses for each position $\mathbf{v}$ the successor $\mathbf{w}$ with minimum $\rho(\mathbf{w})$

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- Player 0 has a memoryless winning strategy $\mathbf{f}_0$ with winning set $\mathbf{W}_0$. The solitaire game over $\mathbf{W}_0$ defined by $\mathbf{f}_0$ has only even cycles $\Rightarrow$ we can assign a parity progress measure over $\mathbf{W}_0$, which lifts to a game progress measure ρ with $\mathbf{W}_0 = \|\rho\|$.

Computing game progress measures

- Define an operator $\mathbf{Lift}(\rho, \mathbf{v})$ that updates ρ at $\mathbf{v}$

$$\mathbf{Lift}(\rho, \mathbf{v})(\mathbf{u}) =$$

$$\begin{cases} \rho(\mathbf{u}), & \text{if } \mathbf{u} \neq \mathbf{v} \\ \max\{\rho(\mathbf{v}), \min_{\mathbf{v} \longrightarrow \mathbf{w}} \mathbf{Dom}(\rho, \mathbf{v}, \mathbf{w})\}, & \text{if } \mathbf{u} = \mathbf{v} \in \mathbf{V_0} \\ \max\{\rho(\mathbf{v}), \max_{\mathbf{v} \longrightarrow \mathbf{w}} \mathbf{Dom}(\rho, \mathbf{v}, \mathbf{w})\}, & \text{if } \mathbf{u} = \mathbf{v} \in \mathbf{V_1} \end{cases}$$

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- **Lift** tries to raise the measure of each position in $\mathbf{V}_0$ above at least one neighbour and each position in $\mathbf{V}_1$ strictly above all neighbours

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- **Lift**$(\rho, \mathbf{v})$ is monotone for each $\mathbf{v}$

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- $\mathbf{Lift}(\rho, \mathbf{v})$ is monotone for each $\mathbf{v}$
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- Can compute simultaneous fixed point of all $\mathbf{Lift}(\rho, \mathbf{v})$ iteratively

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  Analysis is a bit complicated

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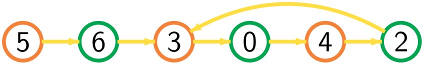
  - All positions have distinct colours—assume positions and colours are both numbered $\{0, 1, \ldots, d\}$ so that position i has colour i

Valuations

A typical play **P** consistent with memoryless (f_0, f_1)



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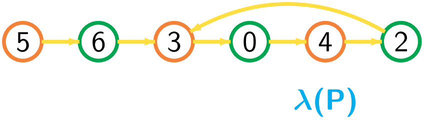

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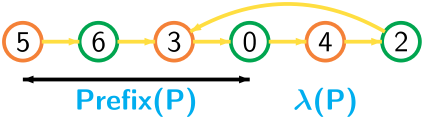

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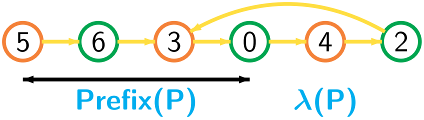

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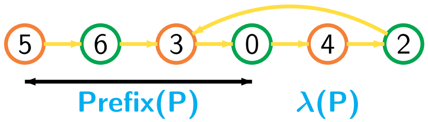

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Valuation

$\Theta : v \mapsto (\lambda(P), \pi(P), \ell(P))$ for some play **P** starting at **v**

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- From any locally progressive valuation Θ, we can extract a pair of strategies (f_0, f_1) that induce Θ
- From any pair of strategies (f_0, f_1) we can derive a locally progressive valuation Θ

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- Order $\Theta(u) = (w, P, \ell)$ and $\Theta(v) = (x, Q, m)$ lexicographically

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- Order on l and m is normal $\leq$

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A valuation Θ is optimal if we have:

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Lemma

If Θ is an optimal valuation for player 0 (player 1), the corresponding strategy is winning for player 0 (player 1).

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  Claim This procedure converges.
- No theoretical bound is known on the complexity of convergence.

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