The Expressive Power of Linear-time Temporal Logic

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Summary of Last Lecture

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- FO definable languages are aperiodic. (Via EF Games, Syntacic Monoid)

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How do we put together LTL formulas φ_1 and φ_2 to describe the language $L(\varphi_1).L(\varphi_2)$?

Easy if the decomposition is unambiguous. (eg.) $L_1.c.L_2$ where either L_1 or L_2 is c-free.

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  - L is finite. Easy.
  - L is $\{a^i \mid i \geq N\}$. Easy.

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Observation 1: If φ is a LTL_A formula describing the language L and $A \subseteq \Sigma$ then

$$\varphi \wedge \bigwedge_{a \in \Sigma \setminus A} \mathsf{G}\neg \mathsf{a}$$

is a LTL_Σ formula that describes L.

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It suffices to show that each of these three languages is LTL expressible.

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- L_0 is language over a smaller alphabet A, recognized by M via h.
- So, L_0 is defined by an LTL_A formula φ_0 over A.
- By Observation 1, it is expressible in LTL_Σ.

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- If xcy is in the RHS then $h(xcy) = \alpha.h(c).\beta \in X$. Thus $xcy \in L$.
- Let $w \in L_1$. Therefore, $w = xcy$. Take $\alpha = h(x)$ and $\beta = h(y)$.

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Well, almost! $L_\alpha \cap A^+$ and $L_\beta \cap A^+$ are LTL expressible. We have to deal with ϵ separately

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Otherwise, $\varphi_1 \vee \top\mathrm{U}(\mathrm{c} \wedge \neg\mathrm{X}\top)$ describes the language $A^*.c.L_\beta$.

This case was easy because our modalities walk only to the right and so cannot "stray" to the left. Dealing with $L_\alpha.c.\Sigma^*$ will need a little more work.

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We cannot use φ_α to describe $L_\alpha.c.\Sigma^*$ since the modalities may walk to the right and cross the c boundary.

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This relativization is defined via structural recursion as follows:

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$\varphi_2 = \varphi'_\alpha$ describes $(L_\alpha \cap A^+).c.\Sigma^*$. If $\epsilon \notin L_\alpha$ then φ_2 also describes $L_\alpha.c.\Sigma^*$. Otherwise, use $\varphi_2 \vee c$.

STATUTORY WARNING

I WILL BE SLOPPY WITH ϵ

FROM NOW ON.

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The first and third components are LTL definable. What about the middle component?

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We use m to denote elements of M when treated as letters and m when they are treated as elements of the monoid M.

The map σ and Language K

The map σ is the obvious one:

$$\sigma c t_1 c t_2 \ldots t_{k-2} c t_{k-1} c \quad = \quad \mathrm{h}(\mathrm{t}_1)\mathrm{h}(\mathrm{t}_2)\ldots\mathrm{h}(\mathrm{t}_{\mathrm{k}-1})$$

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Given the map σ and requirement 2.1, the definition of K is also quite obvious:

$$K \quad = \quad \{\mathrm{m}_1\mathrm{m}_2\ldots\mathrm{m}_\mathrm{k} \mid h(c)m_1h(c)m_2\ldots h(c)m_kh(c) = \beta\}$$

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&= \{ct_1ct_2\ldots ct_kc \mid h(c)h(t_1)h(c)h(t_2)\ldots h(c)h(t_k)h(c) = \beta \\
&= L_\beta \cap \Delta \text{ as required by 2.1}
\end{aligned}$$

Localizing a Monoid at an element

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The Monoid $\mathrm{Loc}_m(M)$: Let M be a monoid and $m \in M$. Then

$$\mathrm{Loc}_m(M) \quad = \quad (mM \cap Mm, \circ, m)$$

where $(xm) \circ (my) \stackrel{\triangle}{=} xmy$.

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- Observe that $xm \circ ym = xm \circ my' = xmy' = xym$. Thus $\circ$ is associative and $m = 1.m$ is the identity w.r.t. $\circ$.
- $xm \circ xm \circ \ldots xm \;=\; x^N m$. Thus, $\mathrm{Loc}_m(M)$ is aperiodic whenever M is aperiodic.

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- Observe that $xm \circ ym = xm \circ my' = xmy' = xym$. Thus $\circ$ is associative and $m = 1.m$ is the identity w.r.t. $\circ$.
- $xm \circ xm \circ \ldots xm \;=\; x^N m$. Thus, $\mathrm{Loc}_m(M)$ is aperiodic whenever M is aperiodic.
- $1 \notin \mathrm{Loc}_m(M)$ if $m \neq 1$. This follows from the fact that $1 \neq m'm$ for any $m, m' \neq 1$.

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K is recognized by a smaller monoid and hence there is an LTL_M formula that describes K

Lifting the formula for K

We show that for any formula φ in LTL_M, there is a formula $\varphi^{\#}$ in LTL_Σ such that

$$w \models \varphi^{\#} \iff \begin{array}{l} w = ct_1ct_2c\ldots t_{k-1}ct_k, \text{ with } t_i \in A^* \\ \text{and } \sigma(ct_1ct_2\ldots t_{k-1}c) \models \varphi \end{array}$$

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The formula $\varphi^\#$ is defined recursively on the structure as follows:

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