The Expressive Power of Linear-time Temporal Logic

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Linear-time Temporal Logic

LTL — convenient specification language

- Atomic propositions, boolean connectives, temporal modalities.

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Formulas are interpreted at positions of a word.

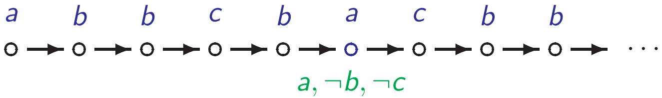
$$w = w_1 w_2 w_3 \ldots \qquad \text{with } w_i \in \Sigma$$

$$w, i \models \varphi \, ?$$

Syntax and Semantics

Atomic propositions: elements of Σ.

$$w, i \models a \iff w_i = a$$



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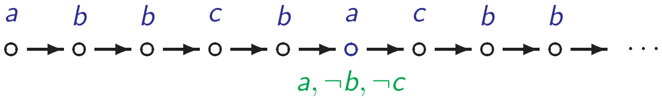
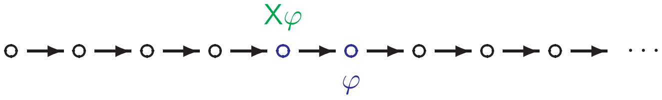


The Next state operator:

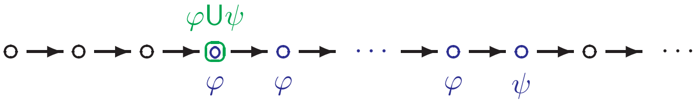
$$w, i \models X\varphi \iff w, i+1 \models \varphi$$



Syntax and Semantics

The Until operator:

$$w, i \models \varphi \mathsf{U} \psi \quad \Longleftrightarrow \quad \exists j \geq i.\ w, j \models \psi \text{ and } \forall i \leq k < j.\ w, k \models \varphi$$



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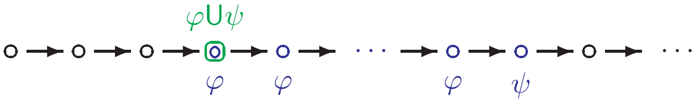


Boolean Connectives:

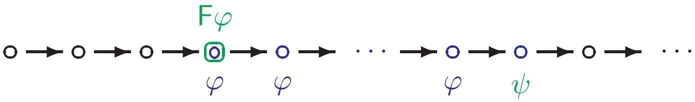
$$\varphi \wedge \psi, \quad \neg\varphi, \quad \ldots$$

with the usual interpretation.

Other Modalities

The Future modality

$$w, i \models \mathsf{F}\varphi \quad \Longleftrightarrow \quad \exists j \geq i.\ w, j \models \varphi$$



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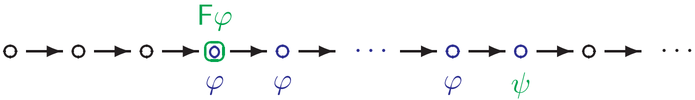
$$F\varphi \quad = \quad \top U \varphi$$



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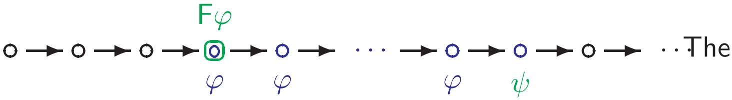
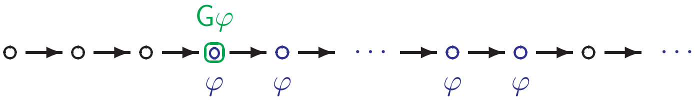


Henceforth modality:

$$w, i \models G\varphi \quad \iff \quad \forall j \geq i.\ w, j \models \varphi$$



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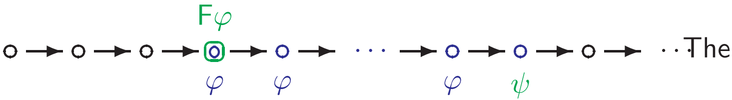
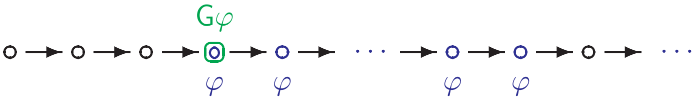


Henceforth modality:

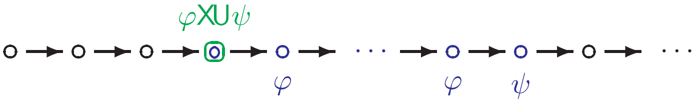
$$\mathsf{G}\varphi \quad = \quad \neg \mathsf{F} \neg \varphi$$



The Universal Modality

The Next-Until modality:

$$w, i \models \varphi \mathsf{XU} \psi \quad \equiv \quad \exists \mathrm{j} > \mathrm{i}.\ \mathrm{w}, \mathrm{j} \models \psi \text{ and } \forall \mathrm{i} < \mathrm{k} \leq \mathrm{j}.\ \mathrm{w}, \mathrm{k} \models \varphi$$



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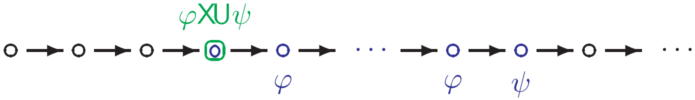

$$\varphi XU \psi \quad = \quad X(\varphi U \psi)$$

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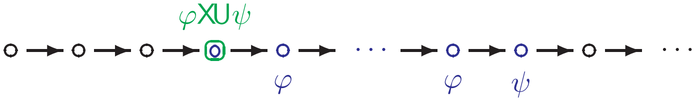

$$\varphi \mathsf{XU} \psi \quad = \quad \mathsf{X}(\varphi \mathsf{U} \psi)$$

Next-Until can express everthing else

$$\begin{array}{rcl} \mathsf{X}\varphi & = & \bot \mathsf{XU} \varphi \\ \varphi \mathsf{U} \psi & = & \psi \vee (\varphi \wedge \varphi \mathsf{XU} \psi) \end{array}$$

LTL definable languages

A word satisfies φ if the initial position satisfies φ

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Formulas define languages. For example,

$$\mathsf{G}(\mathsf{a} \implies \mathsf{Fb})$$

describes words in which there is a b somewhere to the right of every a.

$$b^*(aa^*bb^*)^*$$

Finite/Infinite Words

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$$\mathsf{GX}\top$$

  is satisfied only over infinite words.

$$\mathsf{F}\neg\mathsf{X}\top$$

  is satisfied only by finite words.
- The empty word is not a model.

First-Order Logic of Words

Consider the First-Order formula

$$\varphi \quad = \quad \forall x.\ (a(x) \implies \exists y.\ ((y > x) \wedge b(x))$$

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- The quantifiers have the usual meaning.
- The formula $y > x$ is true if the position y appears somewhere to the right of the position x.

A word w satisfies φ only if for any position (x) with the letter a, there is some position to its right (y) with the letter b.

$$L(\varphi) \quad = \quad b^*(aa^*bb^*)^*$$

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evaluates to true at a position x if and only if it is the first position in the word. Thus

$$\forall x.(\mathrm{First}(x) \implies a(x))$$

identifies all the words that begin with an a.

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\mathcal{T}(a) & = & a(x) \\
\mathcal{T}(X\alpha) & = & \exists y.\ (y = x + 1) \wedge \mathcal{T}(\alpha)[y/x] \\
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- $w, i \models \mathcal{T}(\varphi) \iff w, i \models \varphi$.
- $\mathcal{T}(\varphi)$ uses at the most 3 variables. So, LTL is expressible in FO(3).

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Conclusion: FO seems to be a stronger logic than LTL.

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The rest of this talk and the next would be devoted to proving this result.

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2. Transform aperiodic monoids into equivalent LTL formulas.

    Wilke's technique.

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Model as a word decorated with the variables x and y.

$$\begin{array}{cccccc} b & a & c & a & b & c \\ & x & & & y & \end{array}$$

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Another decorated word:

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ϕ is not satisifed by this word.

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Any formula defines a language of decorated words

Decorated word models

A decorated word is a word over the alphabet $\Sigma \times 2^V$, where V is a set of free variables.

Words corresponding to the decorated words:

$$\begin{array}{cccccc} b & a & c & a & b & c \\ & x & & & y & \end{array}$$

is $(b, \emptyset)(a, \{x\})(c, \emptyset)(a, \emptyset)(b, \{y\})(c, \emptyset)$.

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A V-word is a word $(a_1, U_1)(a_2, U_2)\ldots(a_k, U_k)$ with

1. $U_i \cap U_j = \emptyset$ for $i \neq j$.
2. $\bigcup_{1 \leq i \leq k} U_i = V$.

$L(\varphi)$ is a language of V-words for any V with $\mathrm{free}(\varphi) \subseteq V$.

Stratifying FO formulas

A natural measure of the complexity of a FO formula is its quantifier-depth.

$$
\begin{array}{lcll}
\mathrm{qd}(\varphi) & = & 0 & \text{if } \varphi \text{ is an atomic formula} \\
\mathrm{qd}(\varphi \wedge \psi) & = & \mathsf{Maximum}(\mathrm{qd}(\varphi), \mathrm{qd}(\psi)) & \\
\mathrm{qd}(\neg\varphi) & = & \mathrm{qd}(\varphi) & \\
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Theorem: For any i there are only finitely many formulas of quantifier depth i or less (upto logical equivalence).

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To do this we need an alternative characterization of quantifier-depth.

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Question: When is L definable in FO(k)?
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EF-Games: Set up k-round two player game (between say player 0 and player 1) based on w and w'. Relate winning strategies to distinguishability.

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Let w, w' be two words V-words and let k be an integer. The k round EF-game consists of the two players making k moves. In round i:

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1. Player 0 (who is trying to show that the two words are distinguishable) picks one of the two words and a position p in that word and labels it with x_i.

The Game

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- Pick the first word and position 3.
- No matter how Player 1 responded, pick the first word and position 2.
- If the positions picked by player 1 are not 2 and 4, Player 0 has already won.
- Otherwise, pick the second word and position 3.

EF Games ...

Theorem: (Ehrenfeucht,Fraisse) Player 0 has a winning strategy in the k round game on w, w' if and only if there is a FO(k) formula that distinguishes w and w'.

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The proof is an easy inductive argument.

Note that any distinuishing formula dictates a winning strategy for player 0.

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Example: Consider the words

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Here is a distinguishing formula:

$$\exists x_1.\ (\ b(x_1) \wedge \exists x_2.\ (x_1 < x_2) \wedge \forall x_2 > x_1.\ b(x_2)\ )$$

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Conversely, winning strategies for Player 0 can be turned into distinguishing formulas.

k-equivalence

Two words w and w' are said to be *k-equivalent* if they are indistinguishable by formulas with quantifier depth k or less.

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- $\equiv_k$ is of finite index.
- Let φ be a FO(k) formula. Then $L(\varphi)$ is a (disjoint) union of some of the equivalence classes of $\equiv_k$.

FO definable languages are Regular

Theorem: (Myhill-Nerode) A language L is regular if and only if it is the union of some of the equivalence classes of a right-invariant equivalence relation of finite index.

It suffices to show that $\equiv_k$ is right-invariant.

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Theorem: Every First order definable language of words is regular.

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$$a^s.(a,x).a^t$$

where $s + t = m$ or $s + t + 1 = m$.

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- Suppose $s \leq t$. Player 1 breaks up the other word as

$$a^s.(a,x).a^{t'}$$

  with $s + t' = m$ or $s + t' + 1 = m$.

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The latter asserts that FO definable languages are aperiodic.

Monoids as Language recognizers

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Theorem: For any $X \subseteq M$, $h^{-1}(X)$ is a regular language.

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- Let $x \equiv_L y$ iff $\forall u, v.\ uxv \in L \iff uyv \in L$.

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The Syntactic Monoid of a Regular Language:

- Let $x \equiv_L y$ iff $\forall u, v.\ uxv \in L \iff uyv \in L$.
- $\equiv_L$ is a congruence on Σ^*.
- $SYN(L) = (\Sigma^*/\equiv_L, ., [\epsilon]_{\equiv_L})$ is a finite monoid.

Monoids recognize Regular languages

Let $\eta_L : \Sigma^* \longrightarrow SYN(L)$ be the morphism

$$\eta_L(x) = [x]_{\equiv_L}$$

Then,

$$L = \bigcup_{x \in L} \eta_L^{-1}([x]_{\equiv_L})$$

Theorem: A language is regular if and only if it is recognized by a finite monoid.

Aperiodic Monoids

A Monoid M is said to be aperiodic iff there is an integer N such that

$$a^k = a^{k+1} \quad \text{for all } k \geq N \text{ and } a \in M$$

A language L is aperiodic iff it is recognized by an aperiodic monoid.

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Theorem: $\Sigma^*/\equiv_k$ is an aperiodic monoid. Thus, every FO definable language is aperiodic.

This follows from the fact that $w^m \equiv_k w^{m+1}$ for all $m > 2^k$.

An useful result

If M is an aperiodic monoid and $x, y \in M$ and $x \neq y$ then, $x.y \neq 1$.

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If M is an aperiodic monoid and $x, y \in M$ and $x \neq y$ then, $x.y \neq 1$.

Suppose $x.y = 1$. Then, $x = x.x^N.y^N = x^N.y^N = 1$.

Similarly, $y = 1$, contradicting $x \neq y$.

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Schutzenberger's Theorem: A regular language L is aperiodic if and only if it expressible as a star-free regular expression.