Conclude that

$$R^n = W^{(n)} \tag{10.12}$$

for all $n \in \mathbb{N}$.

(c) Conclude that

$$R^+ = \bigcup_{i=1}^{|A|} R^i$$

where R^+ is the positive length walk relation determined by R on the set A.

Problem 10.12.
We can represent a relation S between two sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ as an $n \times m$ matrix M_S of zeroes and ones, with the elements of M_S defined by the rule

$$M_S(i, j) = 1 \quad \text{IFF} \quad a_i \; S \; b_j.$$

If we represent relations as matrices this way, then we can compute the composition of two relations R and S by a “boolean” matrix multiplication $\otimes$ of their matrices. Boolean matrix multiplication is the same as matrix multiplication except that addition is replaced by OR, multiplication is replaced by AND, and 0 and 1 are used as the Boolean values **False** and **True**. Namely, suppose $R : B \to C$ is a binary relation with $C = \{c_1, \ldots, c_p\}$. So M_R is an $m \times p$ matrix. Then $M_S \otimes M_R$ is an $n \times p$ matrix defined by the rule:

$$[M_S \otimes M_R](i, j) ::= \text{OR}_{k=1}^{m}[M_S(i, k) \text{ AND } M_R(k, j)]. \tag{10.13}$$

Prove that the matrix representation $M_{R \circ S}$ of $R \circ S$ equals $M_S \otimes M_R$ (note the reversal of R and S).

Problem 10.13.
Chickens are rather aggressive birds that tend to establish dominance over other chickens by pecking them—hence the term “pecking order.” So for any two chickens in a farmyard, either the first pecks the second, or the second pecks the first, but not both. We say that chicken u *virtually pecks* chicken v if either:

- Chicken u pecks chicken v, or
- Chicken u pecks some other chicken w who in turn pecks chicken v.

A chicken that virtually pecks every other chicken is called a *king chicken*.

We can model this situation with a *chicken digraph* whose vertices are chickens, with an edge from chicken u to chicken v precisely when u pecks v. In the graph in Figure 10.11, three of the four chickens are kings. Chicken c is not a king in this example since it does not peck chicken b and it does not peck any chicken that pecks chicken b. Chicken a *is* a king since it pecks chicken d, who in turn pecks chickens b and c.

In general, a *tournament digraph* is a digraph with exactly one edge between each pair of distinct vertices.

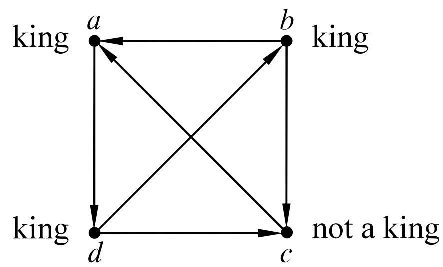

Figure 10.11 A 4-chicken tournament in which chickens a, b and d are kings.

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(a) Define a 10-chicken tournament graph with a king chicken that has outdegree 1.

(b) Describe a 5-chicken tournament graph in which every player is a king.

(c) Prove

Theorem (King Chicken Theorem)**.** *Any chicken with maximum out-degree in a tournament is a king.*

The King Chicken Theorem means that if the player with the most victories is defeated by another player x, then at least he/she defeats some third player that defeats x. In this sense, the player with the most victories has some sort of bragging rights over every other player. Unfortunately, as Figure 10.11 illustrates, there can be many other players with such bragging rights, even some with fewer victories.

(a) Explain how to model the delegate selection problem as a bipartite matching problem. (This is a *modeling problem*; we aren't looking for a description of an algorithm to solve the problem.)

(b) The VP's records show that no student is a member of more than 9 clubs. The VP also knows that to be eligible for support from the Dean's office, a club must have at least 13 members. That's enough for her to guarantee there is a proper delegate selection. Explain. (If only the VP had taken an *Algorithms* class, she could even have found a delegate selection without much effort.)

Problem 12.13.
A simple graph is called *regular* when every vertex has the same degree. Call a graph *balanced* when it is regular and is also a bipartite graph with the same number of left and right vertices.

Prove that if G is a balanced graph, then the edges of G can be partitioned into blocks such that each block is a perfect matching.

For example, if G is a balanced graph with $2k$ vertices each of degree j, then the edges of G can be partitioned into j blocks, where each block consists of k edges, each of which is a perfect matching.

Exam Problems

Problem 12.14.
Overworked and over-caffeinated, the Teaching Assistant's (TA's) decide to oust the lecturer and teach their own recitations. They will run a recitation session at 4 different times in the same room. There are exactly 20 chairs to which a student can be assigned in each recitation. Each student has provided the TA's with a list of the recitation sessions her schedule allows and each student's schedule conflicts with at most two sessions. The TA's must assign each student to a chair during recitation at a time she can attend, if such an assignment is possible.

(a) Describe how to model this situation as a matching problem. Be sure to specify what the vertices/edges should be and briefly describe how a matching would determine seat assignments for each student in a recitation that does not conflict with his schedule. (This is a *modeling problem*; we aren't looking for a description of an algorithm to solve the problem.)

(b) Suppose there are 41 students. Given the information provided above, is a matching guaranteed? Briefly explain.

A	arrows	B
		Q
P XOR Q		
		$\overline{P}$ OR $\overline{Q}$
P AND Q		
		$\overline{P}$ OR $\overline{Q}$ OR $(\overline{P}$ AND $\overline{Q})$
NOT$(P$ AND $Q)$		
		P

(c) The diagram in part (b) defines a bipartite graph G with $L(G) = A$, $R(G) = B$ and an edge between F and G iff $F \ \widehat{R}\ G$. Exhibit a subset S of A such that both S and $A - S$ are nonempty, and the set N(S) of neighbors of S is the same size as S, that is, $|\text{N}(S)| = |S|$.

(d) Let G be an arbitrary, finite, bipartite graph. For any subset $S \subseteq L(G)$, let $\overline{S} ::= L(G) - S$, and likewise for any $M \subseteq R(G)$, let $\overline{M} ::= R(G) - M$. Suppose S is a subset of $L(G)$ such that $|\text{N}(S)| = |S|$, and both S and $\overline{S}$ are nonempty. **Circle the formula** that correctly completes the following statement:

There is a matching from $L(G)$ to $R(G)$ if and only if there is both a matching from S to its neighbors, N(S), and also a matching from $\overline{S}$ to

$\text{N}(\overline{S})$ $\quad$ $\overline{\text{N}(S)}$ $\quad$ $N^{-1}(\text{N}(S))$ $\quad$ $N^{-1}(\text{N}(\overline{S}))$ $\quad$ $\text{N}(\overline{S}) - \overline{\text{N}(S)}$ $\quad$ $\text{N}(S) - \text{N}(\overline{S})$

Hint: The proof of Hall's Bottleneck Theorem.

Problem 12.17. **(a)** Show that there is no matching for the bipartite graph G in Figure 12.25 that covers $L(G)$.

(b) The bipartite graph H in Figure 12.26 has an easily verified property that implies it has a matching that covers $L(H)$. What is the property?

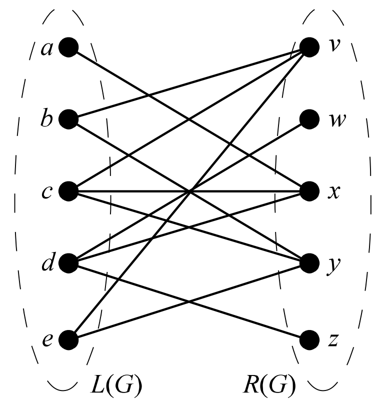

Figure 12.25 Bipartite graph G.

Homework Problems

Problem 12.18.

A *Latin square* is $n \times n$ array whose entries are the number $1, \ldots, n$. These entries satisfy two constraints: every row contains all n integers in some order, and also every column contains all n integers in some order. Latin squares come up frequently in the design of scientific experiments for reasons illustrated by a little story in a footnote[10]

[10] At Guinness brewery in the eary 1900's, W. S. Gosset (a chemist) and E. S. Beavan (a "maltster") were trying to improve the barley used to make the brew. The brewery used different varieties of barley according to price and availability, and their agricultural consultants suggested a different fertilizer mix and best planting month for each variety.

Somewhat sceptical about paying high prices for customized fertilizer, Gosset and Beavan planned a season long test of the influence of fertilizer and planting month on barley yields. For as many months as there were varieties of barley, they would plant one sample of each variety using a different one of the fertilizers. So every month, they would have all the barley varieties planted and all the fertilizers used, which would give them a way to judge the overall quality of that planting month. But they also wanted to judge the fertilizers, so they wanted each fertilizer to be used on each variety during the course of the season. Now they had a little mathematical problem, which we can abstract as follows.

Suppose there are n barley varieties and an equal number of recommended fertilizers. Form an $n \times n$ array with a column for each fertilizer and a row for each planting month. We want to fill in the entries of this array with the integers $1, \ldots, n$ numbering the barley varieties, so that every row contains all n integers in some order (so every month each variety is planted and each fertilizer is used), and also every column contains all n integers (so each fertilizer is used on all the varieties over the course of the growing season).

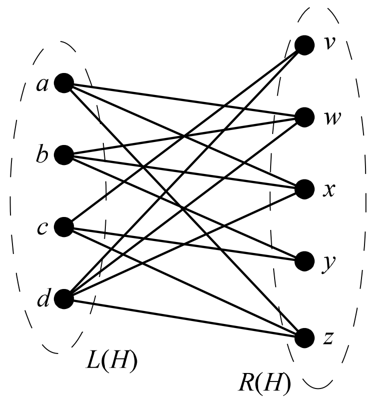

Figure 12.26 Bipartite Graph H.

For example, here is a 4×4 Latin square:

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

(a) Here are three rows of what could be part of a 5×5 Latin square:

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

Fill in the last two rows to extend this “Latin rectangle” to a complete Latin square.

(b) Show that filling in the next row of an $n \times n$ Latin rectangle is equivalent to finding a matching in some $2n$-vertex bipartite graph.

(c) Prove that a matching must exist in this bipartite graph and, consequently, a Latin rectangle can always be extended to a Latin square.

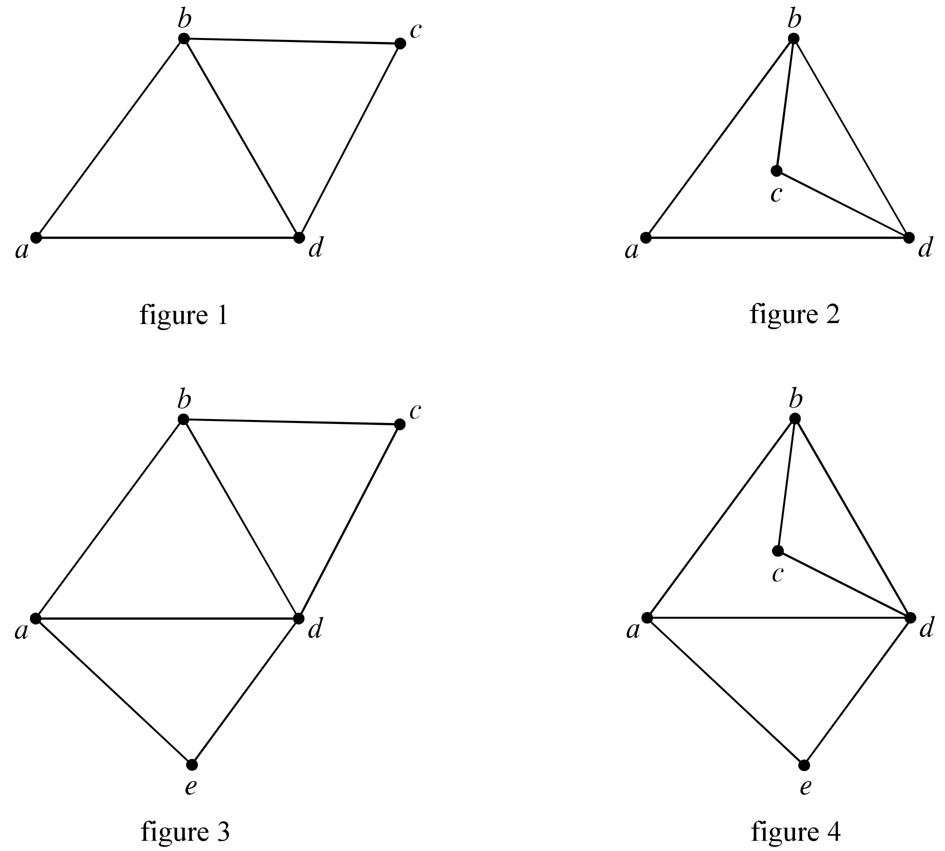

Figure 13.18

(b) In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.

Homework Problems

Problem 13.8.
A simple graph is *triangle-free* when it has no cycle of length three.

(a) Prove for any connected triangle-free planar graph with $v > 2$ vertices and e edges,

$$e \leq 2v - 4. \tag{13.9}$$

(b) Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

(c) Prove that any connected triangle-free planar graph is 4-colorable.

Problem 13.9. **(a)** Prove

Lemma (Switch Edges)**.** *Suppose that, starting from some embeddings of planar graphs with disjoint sets of vertices, it is possible by two successive applications of constructor operations to add edges e and then f to obtain a planar embedding $\mathcal{F}$. Then starting from the same embeddings, it is also possible to obtain $\mathcal{F}$ by adding f and then e with two successive applications of constructor operations.*

Hint: There are four cases to analyze, depending on which two constructor operations are applied to add e and then f. Structural induction is not needed.

(b) Prove

Corollary (Permute Edges)**.** *Suppose that, starting from some embeddings of planar graphs with disjoint sets of vertices, it is possible to add a sequence of edges $e_0, e_1, \ldots, e_n$ by successive applications of constructor operations to obtain a planar embedding $\mathcal{F}$. Then starting from the same embeddings, it is also possible to obtain $\mathcal{F}$ by applications of constructor operations that successively add any permutation*[5] *of the edges $e_0, e_1, \ldots, e_n$.*

Hint: By induction on the number of switches of adjacent elements needed to convert the sequence 0,1,...,n into a permutation $\pi(0), \pi(1), \ldots, \pi(n)$.

(c) Prove

Corollary (Delete Edge)**.** *Deleting an edge from a planar graph leaves a planar graph.*

(d) Conclude that any subgraph of a planar graph is planar.

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[5] If $\pi : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ is a bijection, then the sequence $e_{\pi(0)}, e_{\pi(1)}, \ldots, e_{\pi(n)}$ is called a *permutation* of the sequence $e_0, e_1, \ldots, e_n$.