Problems for Section 1.7

Practice Problems

Problem 1.8.
Prove by cases that

$$\max(r, s) + \min(r, s) = r + s \tag{*}$$

for all real numbers r, s.

Class Problems

Problem 1.9.
If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

Problem 1.10.
Prove by cases that

$$|r + s| \leq |r| + |s| \tag{1}$$

for all real numbers r, s.[9]

Homework Problems

Problem 1.11. **(a)** Suppose that

$$a + b + c = d,$$

where a, b, c, d are nonnegative integers.

Let P be the assertion that d is even. Let W be the assertion that exactly one among a, b, c are even, and let T be the assertion that all three are even.

Prove by cases that

$$P \text{ IFF } [W \text{ OR } T].$$

(b) Now suppose that

$$w^2 + x^2 + y^2 = z^2,$$

[9] The *absolute value* $|r|$ of r equals whichever of r or $-r$ is not negative.

where w, x, y, z are nonnegative integers. Let P be the assertion that z is even, and let R be the assertion that all three of w, x, y are even. Prove by cases that

$$P \text{ IFF } R.$$

Hint: An odd number equals $2m + 1$ for some integer m, so its square equals $4(m^2 + m) + 1$.

Exam Problems

Problem 1.12.
Prove that there is an irrational number a such that $a^{\sqrt{3}}$ is rational.
Hint: Consider $\sqrt[3]{2}^{\sqrt{3}}$ and argue by cases.

Problems for Section 1.8

Practice Problems

Problem 1.13.
Prove that for any $n > 0$, if a^n is even, then a is even.
Hint: Contradiction.

Problem 1.14.
Prove that if $a \cdot b = n$, then either a or b must be $\leq \sqrt{n}$, where a, b, and n are nonnegative real numbers. *Hint:* by contradiction, Section 1.8.

Problem 1.15.
Let n be a nonnegative integer.

(a) Explain why if n^2 is even—that is, a multiple of 2—then n is even.

(b) Explain why if n^2 is a multiple of 3, then n must be a multiple of 3.

Problem 1.16.
Give an example of two distinct positive integers m, n such that n^2 is a multiple of m, but n is not a multiple of m. How about having m be less than n?

Class Problems

Problem 1.17.
How far can you generalize the proof of Theorem 1.8.1 that $\sqrt{2}$ is irrational? For example, how about $\sqrt{3}$?

Problem 1.18.
Prove that $\log_4 6$ is irrational.

Problem 1.19.
Prove by contradiction that $\sqrt{3} + \sqrt{2}$ is irrational.
Hint: $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$

Problem 1.20.
Here is a generalization of Problem 1.17 that you may not have thought of:

Lemma. *Let the coefficients of the polynomial*

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1} + x^m$$

be integers. Then any real root of the polynomial is either integral or irrational.

(a) Explain why the Lemma immediately implies that $\sqrt[m]{k}$ is irrational whenever k is not an mth power of some integer.

(b) Carefully prove the Lemma.

You may find it helpful to appeal to:
Fact. If a prime p is a factor of some power of an integer, then it is a factor of that integer.

You may assume this Fact without writing down its proof, but see if you can explain why it is true.

Exam Problems

Problem 1.21.
Prove that $\log_9 12$ is irrational.

must be a smallest rational $q_0 \in C$. So since $q_0/2 < q_0$, it must be possible to express $q_0/2$ in lowest terms, namely,

$$\frac{q_0}{2} = \frac{m}{n} \tag{2.3}$$

for positive integers m, n with no common prime factor. Now we consider two cases:

Case 1: [n is odd]. Then $2m$ and n also have no common prime factor, and therefore

$$q_0 = 2 \cdot \left(\frac{m}{n}\right) = \frac{2m}{n}$$

expresses q_0 in lowest terms, a contradiction.

Case 2: [n is even]. Any common prime factor of m and $n/2$ would also be a common prime factor of m and n. Therefore m and $n/2$ have no common prime factor, and so

$$q_0 = \frac{m}{n/2}$$

expresses q_0 in lowest terms, a contradiction.

Since the assumption that C is nonempty leads to a contradiction, it follows that C is empty—that is, there are no counterexamples. ■

Class Problems

Problem 2.4.

Use the *Well Ordering Principle* [2] to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2.4}$$

for all nonnegative integers n.

Problem 2.5.

Use the Well Ordering Principle to prove that there is no solution over the positive integers to the equation:

$$4a^3 + 2b^3 = c^3.$$

[2] Proofs by other methods such as induction or by appeal to known formulas for similar sums will not receive full credit.

Problem 2.6.
You are given a series of envelopes, respectively containing $1, 2, 4, \ldots, 2^m$ dollars. Define

> **Property** m: For any nonnegative integer less than 2^{m+1}, there is a selection of envelopes whose contents add up to *exactly* that number of dollars.

Use the Well Ordering Principle (WOP) to prove that Property m holds for all nonnegative integers m.

Hint: Consider two cases: first, when the target number of dollars is less than 2^m and second, when the target is at least 2^m.

Homework Problems

Problem 2.7.
Use the Well Ordering Principle to prove that any integer greater than or equal to 8 can be represented as the sum of nonnegative integer multiples of 3 and 5.

Problem 2.8.
Use the Well Ordering Principle to prove that any integer greater than or equal to 50 can be represented as the sum of nonnegative integer multiples of 7, 11, and 13.

Problem 2.9.
Euler's Conjecture in 1769 was that there are no positive integer solutions to the equation

$$a^4 + b^4 + c^4 = d^4.$$

Integer values for a, b, c, d that do satisfy this equation were first discovered in 1986. So Euler guessed wrong, but it took more than two centuries to demonstrate his mistake.

Now let's consider Lehman's equation, similar to Euler's but with some coefficients:

$$8a^4 + 4b^4 + 2c^4 = d^4 \tag{2.5}$$

Prove that Lehman's equation (2.5) really does not have any positive integer solutions.

Hint: Consider the minimum value of a among all possible solutions to (2.5).