- *Complete induction* is presented in the context of integer arithmetic. The induction principle relies on the well-foundedness of the $<$ predicate. Rather than assuming that the desired property holds for one element n and proving the property for the case $n+1$ as in stepwise reduction, one assumes that the property holds for all elements $n' < n$ and proves that it holds for n. This stronger assumption sometime yields easier or more concise proofs.
- *Well-founded induction* generalizes complete induction to other theories; it is presented in the context of lists and lexicographic tuples. The induction principle requires a well-founded relation over the domain.
- *Structural induction* is an instance of well-founded induction in which the domain is formulae and the well-founded relation is the strict subformula relation.

Besides being an important tool for proving first-order validities, induction is the basis for both verification methodologies studied in Chapter 5. Structural induction also serves as the basis for the quantifier elimination procedures studied in Chapter 7.

Bibliographic Remarks

The induction proofs in Examples 4.1, 4.3, and 4.9 are taken from the text of Manna and Waldinger [55].

Blaise Pascal (1623–1662) and Jacob Bernoulli (1654–1705) are recognized as having formalized stepwise and complete induction, respectively. Less formal versions of induction appear in texts by Francesco Maurolico (1494–1575); Rabbi Levi Ben Gershon (1288–1344), who recognized induction as a distinct form of mathematical proof; Abu Bekr ibn Muhammad ibn al-Husayn Al-Karaji (953–1029); and Abu Kamil Shuja Ibn Aslam Ibn Mohammad Ibn Shaji (850–930) [97]. Some historians claim that Euclid may have applied induction informally.

Exercises

4.1 (T^{+}_{cons}). Prove the following in T^{+}_{cons}:

(a) $\forall u, v.\ \mathit{flat}(u)\ \wedge\ \mathit{flat}(v)\ \rightarrow\ \mathit{flat}(\mathit{concat}(u, v))$
(b) $\forall u.\ \mathit{flat}(u)\ \rightarrow\ \mathit{flat}(\mathit{rvs}(u))$

4.2 ($T^{\mathsf{PA}}_{\mathsf{cons}}$). Prove or disprove the following in $T^{\mathsf{PA}}_{\mathsf{cons}}$:

(a) $\forall u.\ u \preceq_{\mathsf{c}} u$
(b) $\forall u, v, w.\ \mathsf{cons}(u, v) \preceq_{\mathsf{c}} w\ \rightarrow\ v \preceq_{\mathsf{c}} w$
(c) $\forall u, v.\ v \prec_{\mathsf{c}} \mathsf{cons}(u, v)$