Bibliographic Remarks

For a complete and concise presentation of propositional and first-order logic, see Smullyan's text *First-Order Logic* [87]. The semantic argument method is similar to Smullyan's tableau method. Also, the proofs of completeness of the semantic argument method, the Compactness Theorem, and the Craig Interpolation Lemma are inspired by Smullyan's presentation.

The history of the development of mathematical logic is rich. For an overview, see [98] and related articles in *The Stanford Encyclopedia of Philosophy*. We mention in particular *Hilbert's program* of the 1920s — see, for example, [38] — to find a consistent and complete axiomatization of arithmetic. Gödels two *incompleteness theorems* proved that such a goal is impossible. The first incompleteness theorem, which Gödel presented in a lecture in September, 1930, and then in [36], states that any axiomatization of arithmetic contains theorems that are not provable within the theory. The second, which Gödel had proved by October, 1930, states that a theory such as Peano arithmetic cannot prove its own consistency unless it is itself inconsistent. Earlier, Gödel proved that first-order logic is complete [35]: every theorem has a proof. However, Church — and, independently, Turing — proved that satisfiability in first-order logic is undecidable [13]. Thus, while every theorem of first-order logic has a finite proof, invalid formulae need not have a finite proof of their invalidity.

For an introduction to formal languages, decidability, and complexity theory, see [85, 72, 41].

Exercises

2.1 (English and FOL). Encode the following English sentences into FOL.

(a) Some days are longer than others.
(b) In all the world, there is but one place that I call home.
(c) My mother's mother is my grandmother.
(d) The intersection of two convex sets is convex.

2.2 (FOL validity & satisfiability). For each of the following FOL formulae, identify whether it is valid or not. If it is valid, prove it with a semantic argument; otherwise, identify a falsifying interpretation.

(a) $(\forall x, y.\ p(x,y) \ \rightarrow\ p(y,x)) \ \rightarrow\ \forall z.\ p(z,z)$
(b) $\forall x, y.\ p(x,y) \ \rightarrow\ p(y,x) \ \rightarrow\ \forall z.\ p(z,z)$
(c) $(\exists x.\ p(x)) \ \rightarrow\ \forall y.\ p(y)$
(d) $(\forall x.\ p(x)) \ \rightarrow\ \exists y.\ p(y)$
(e) $\exists x, y.\ (p(x,y) \ \rightarrow\ (p(y,x) \ \rightarrow\ \forall z.\ p(z,z)))$

2.3 (Semantic argument). Use the semantic argument method to prove the following formula schemata.

(a) $\neg(\forall x.\ F) \ \Leftrightarrow\ \exists x.\ \neg F$
(b) $\neg(\exists x.\ F) \ \Leftrightarrow\ \forall x.\ \neg F$
(c) $\forall x, y.\ F \ \Leftrightarrow\ \forall y, x.\ F$
(d) $\exists y.\ \forall x.\ F \ \Rightarrow\ \forall x.\ \exists y.\ F$
(e) $\exists x.\ F \lor G \ \Leftrightarrow\ (\exists x.\ F)\ \lor\ (\exists y.\ G)$
(f) $\exists x.\ F \rightarrow G \ \Leftrightarrow\ (\forall x.\ F)\ \rightarrow\ (\exists x.\ G)$
(g) $\exists x.\ F \lor G \ \Leftrightarrow\ (\exists x.\ F)\ \lor\ G$, provided $x \notin \mathsf{free}(G)$
(h) $\forall x.\ F \lor G \ \Leftrightarrow\ (\forall x.\ F)\ \lor\ G$, provided $x \notin \mathsf{free}(G)$
(i) $\exists x.\ F \land G \ \Leftrightarrow\ (\exists x.\ F)\ \land\ G$, provided $x \notin \mathsf{free}(G)$
(j) $\forall x.\ F \rightarrow G \ \Leftrightarrow\ (\exists x.\ F)\ \rightarrow\ G$, provided $x \notin \mathsf{free}(G)$

2.4 (Normal forms). Put the following formulae into prenex normal form.

(a) $(\forall x.\ \exists y.\ p(x, y))\ \rightarrow\ \forall x.\ p(x, x)$
(b) $\exists z.\ (\forall x.\ \exists y.\ p(x, y))\ \rightarrow\ \forall x.\ p(x, z)$
(c) $\forall w.\ \neg(\exists x, y.\ \forall z.\ p(x, z)\ \rightarrow\ q(y, z))\ \land\ \exists z.\ p(w, z)$

2.5 (*Characteristic formula). Why is the characteristic formula of a line on a closed branch of a semantic argument valid?