Lecture 4: Dynamical and Hybrid Systems

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Plan

- Dynamical system models
  - notions of solutions
  - Linear dynamical systems

Introduction to dynamical systems

Behaviors of physical processes are described in terms of instantaneous laws

Common notation: $\frac{dx(t)}{dt} = f(x(t), u(t), t)\ - (1)$, where time $t \in \mathbb{R}$; state $x(t) \in \mathbb{R}^n$; $input\ \ u(t) \in \mathbb{R}^m$; $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$

Initial value problem: Given system (1) and initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}$, and input u: $\mathbb{R} \rightarrow \mathbb{R}^m$, find a state trajectory or *solution* of (1).

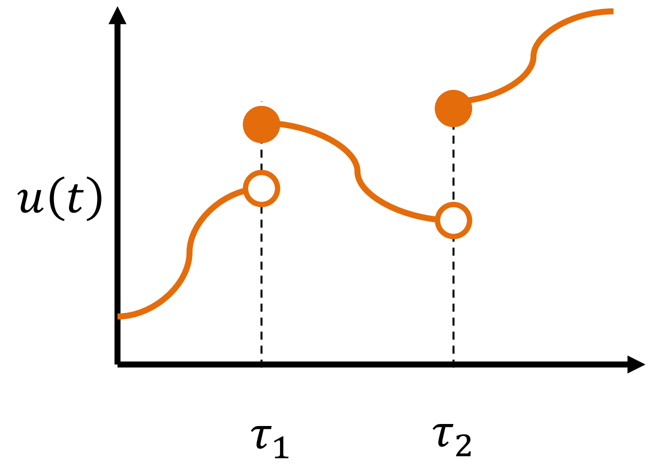
Notions of solution

What is a solution? Many different notions.

Definition 1. (First attempt) Given x_0 and u, ξ: $\mathbb{R} \rightarrow \mathbb{R}^n$ is a solution or trajectory iff (1) $\xi(t_0) = x_0$ and (2) $\frac{d}{dt}\xi(t) = f(\xi(t), u(t), t)), \forall t \in \mathbb{R}$.

Mathematically makes sense, but too restrictive. Assumes that ξ is not only continuous, but also differentiable. This disallows $\mathrm{u}(t)$ to be discontinuous, which is often required for optimal control.

Modified notion



Definition. $u(\cdot)$ is a piece-wise continuous with set of discontinuity points $D \subseteq \mathbb{R}^m$ if

(1) $\forall \tau \in D, \lim_{t \to \tau^+} u(t) < \infty; \lim_{t \to \tau^-} u(t) < \infty$

(2) Continuous from right $\lim_{t \to \tau^+} u(t) = u(t)$

(3) $\forall\ t_0 < t_1$, $[t_0, t_1] \cap D$ is finite

$PC([t_0, t_1], \mathbb{R}^m)$ is the set of all piece-wise continuous functions over the domain $[t_0, t_1]$

Define $p(\xi(t), t) = f(x(t), u(t), t)$, for a given $u(t)$. Since $u(t)$ is PC in t so is p in the second argument.

Definition 2. Given x_0 and u, ξ: $\mathbb{R} \to \mathbb{R}^n$ is a solution or trajectory iff (1) $\xi(t_0) = x_0$ and (2) $\frac{d}{dt}\xi(t) = p(\xi(t), t), \forall t \in \mathbb{R} \setminus \mathrm{D}$.

Is PC input $u(t)$ adequate for guaranteeing existence of solutions?

Example. $\dot{x}(t) = -sgn\big(x(t)\big); x_0 = c; t_0 = 0; c > 0$
Solution: $\xi(t) = c - t$ for $t \leq c$; check $\dot{\xi} = -1$
Problem: f discontinuous is x

Example. $\dot{x}(t) = x^2; x_0 = c; t_0 = 0; c > 0$
Solution: $\xi(t) = \frac{c}{1-tc}$ works for $t < 1/c$; check $\dot{\xi}$
Problem: As $t \to \frac{1}{c}$ then $x(t) \to \infty$; p grows too fast

Lipschitz continuity

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz continuous if there exist $L > 0$ such that for any pair $x, x' \in \mathbb{R}^n$, $\left|\left|f(x) - f(x')\right|\right| \leq L\left|\left|x - x'\right|\right|$

Examples: $6x + 4$; $|x|$; all differentiable functions with bounded derivatives

Non-examples: $\sqrt{x}$; x^2 (locally Lipschitz)

Existence and uniqueness of solutions

Theorem. If $p(x(t), x)$ is Lipschitz continuous in the first argument then (1) has unique solutions.

Transition system model

Linear time-varying systems

In general, for nonlinear dynamical systems we do not have closed form solutions for $\xi(t)$, but there are numerical solvers like CAPD, VNODE

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ --- (2)
$y(t) = C(t)x(t) + D(t)u(t)$

continuous everywhere except D_x

Theorem. Let $\xi(t, t_0, x_0, u)$ be the solution for (2) with points of discontinuity , D_x

1. $\forall t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(\cdot, t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable $\forall\ t \in \mathbb{R} \setminus D_x$
2. $\forall t, t_0 \in \mathbb{R}, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, t_0, \cdot, u): \mathbb{R}^n \to \mathbb{R}^n$ is continuous
3. $\forall t, t_0 \in \mathbb{R}, x_{01}, x_{02} \in \mathbb{R}^n, u_1, u_2 \in PC(\mathbb{R}, \mathbb{R}^m), a_1, a_2 \in \mathbb{R}, \xi(t, t_0, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 \xi(t, t_0, x_{01}, u_1) + a_2 \xi(t, t_0, x_{02}, u_2)$
4. $\forall t, t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, t_0, x_0, u) = \xi(t, t_0, x_0, \mathbf{0}) + \xi(t, t_0, 0, u)$

Linear system and solutions

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

For a given initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}\ \ \ and\ u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$ the *solution* is a function $\xi(., t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$

We studied several properties of ξ in the last lecture: continuity with respect to first and third argument, linearity, decomposition

Linear system and solutions

- Since $\xi(.\,, t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is a linear function of the initial state and input,
- $\xi(t, t_0, x_0, u) = \xi(t, t_0, 0, u) + \xi(.\,, t_0, x_0, 0)$
- Let us focus on the linear function $\xi(.\,, t_0, x_0, 0)$
- Define $\Phi(.\,, t_0)x_0 = \xi(.\,, t_0, x_0, u)$
- This $\Phi(.\,, t_0): \mathbb{R} \to \mathbb{R}^{n \times n}$ is called the <u>state transition matrix</u>

Properties of Φ

- $\Phi(.,t_0): \mathbb{R} \rightarrow \mathbb{R}^{n\times \mathrm{n}}$ is the unique solution of (2) and is defined by a (Peano-Baker) infinite sequence of integrals
- $\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0)$ with $\Phi(t,t) = I$
  - Continuous everywhere
  - Differentiable everywhere except D_x ($A(t)$ isn't)
- $\forall t_0, t_1, t\ \Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0)$
- $\Phi(t,t_0)$ is invertible $[\Phi(t,t_0)]^{-1} = \Phi(t_0,t)$

Solution of linear systems in Φ

Theorem.

$$\xi(t, t_0, x_0, u)$$
$$= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

Linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ \dots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$

Theorem. $\Phi(t, t_0) = e^{A(t-t_0)}$, that is

$$\xi(t, t_0, x_0, u) = x_0 \mathrm{e}^{A(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau) d\tau$$

Discrete time models / discrete transition systems

- $x(t+1) = f\big(x(t), u(t)\big)$
- $x(t+1) = f(x(t))$ autonomous
- Execution: $x_0, f(x_0), f^2(x_0), \ldots$
- $\boldsymbol{A} = \langle Q, Q_0, T \rangle$
  - $Q = \mathbb{R}^n, Q_0 = \{x_0\}$
  - $T: \mathbb{R}^n \to \mathbb{R}^n; \mathrm{T}(x) = f(x)$

Discretized or sampled-time model

- $\dot{x}(t) = f\big(x(t), u(t)\big)$
- Assume: $u \in PC(\mathbb{R}, U)$ $where\ U \subseteq \mathbb{R}^m$ is a finite set
- $\xi(t, t_0, x_0, u)$
- Fix a sampling period $\delta > 0$
- $\boldsymbol{A_\delta} = \langle Q, Q_0, U, T \rangle$
  - $Q = \mathbb{R}^n, Q_0 = \{x_0\}, Act = U,$
  - $T \subseteq \mathbb{R}^n \times U \times \mathbb{R}^n; (x, u, x') \in \mathrm{T}\ iff\ x' = \xi(\delta, 0, x, u)$

Properties for dynamical systems

What type of properties are we interested in?

- Invariance
- State remains bounded
- Converges to target
- Bounded input gives bounded output (BIBO)

Aleksandr M. Lyapunov

Aleksandr Mikhailovich Lyapunov (Russian: June 6 1857–November 3, 1918) was a Russian mathematician and physicist.

His methods, which he developed in 1899, make it possible to define the stability of sets of ordinary differential equations. He created the modern theory of the stability of a dynamic system. In the theory of probability, he generalized the works of Chebyshev and Markov, and proved the Central Limit Theorem under more general conditions than his predecessors.

Requirements: Stability

- We will focus on time invariant autonomous systems (closed systems, systems without inputs)
- $\dot{x}(t) = f\big(x(t)\big), x_0 \in \mathbb{R}^n, t_0 = 0$ –*(1)*
- $\xi(t)$ is the solution
- $|\xi(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume **0** to be an equilibrium point of (1) with out loss of generality

Example: Pendulum

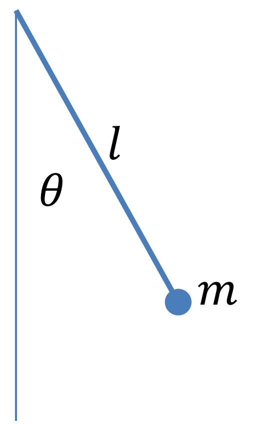
Pendulum equation

$x_1 = \theta \;\; x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$

$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$



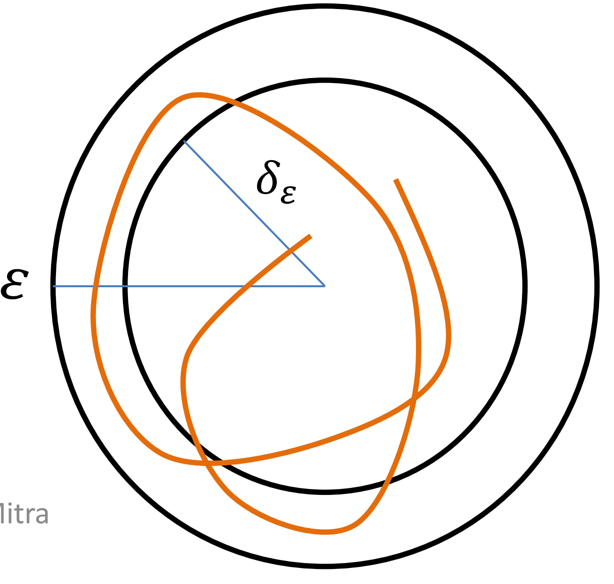
k: friction coefficient

Two equilibrium points: $(0,0), (\pi, 0)$

Lyapunov stability

Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every if $|\xi(0)| \leq \delta_\varepsilon$ then for all t $\geq 0, |\xi(t)| \leq \varepsilon$.

How is this related to invariants and reachable states ?



Asymptotically stability

The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and there exists $\delta_2 > 0$ such that for every if $|\xi(0)| \leq \delta_2$ then $t \rightarrow \infty, |\xi(t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**

Example: Pendulum

Pendulum equation

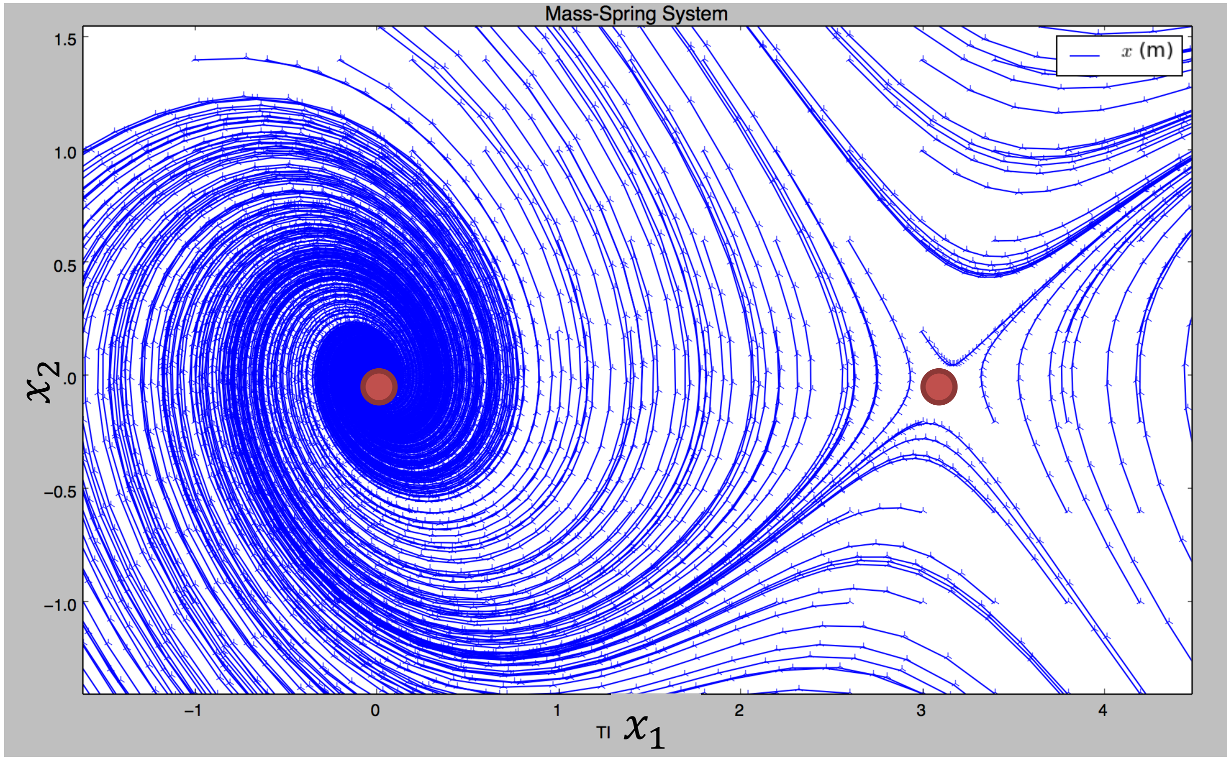
$x_1 = \theta \;\; x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

$$\begin{bmatrix}\dot{x_2}\\ \dot{x_1}\end{bmatrix} = \begin{bmatrix}-\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2\\ x_2\end{bmatrix}$$

Two equilibrium points: $(0,0), (\pi, 0)$



$x = (0,0)$
asymptotically stable

$x = (\pi, 0)$
unstable

Example: Pendulum

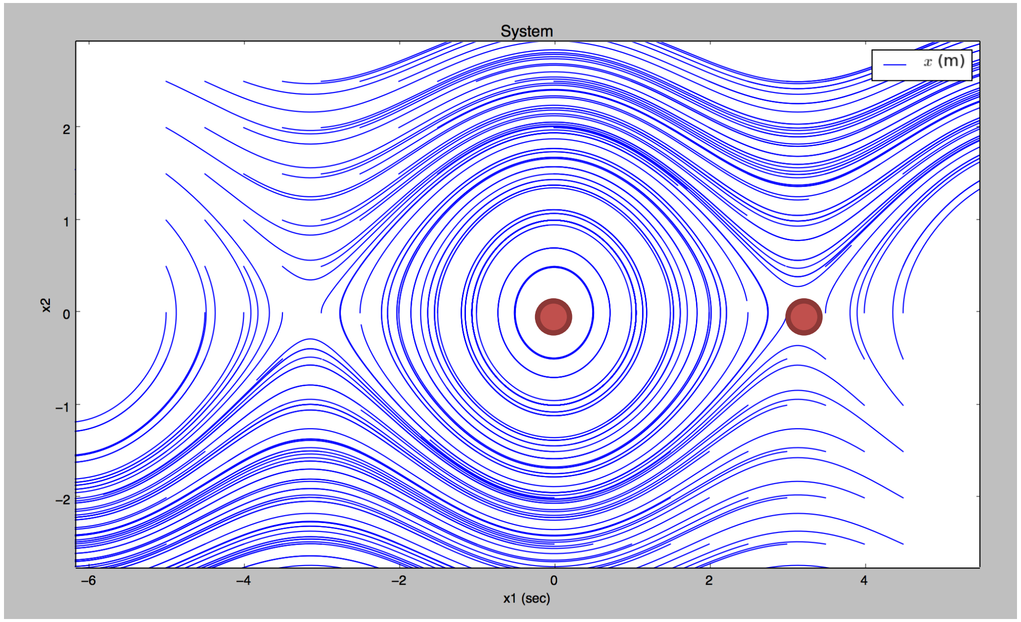
Pendulum equation

$x_1 = \theta \;\; x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$

$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$



$k = 0$ no friction

$x^* = (0, 0)$ stable but not asymptotically stable

$x^* = (\pi, 0)$ unstable

Van der pol oscillator

Van der pol oscillator

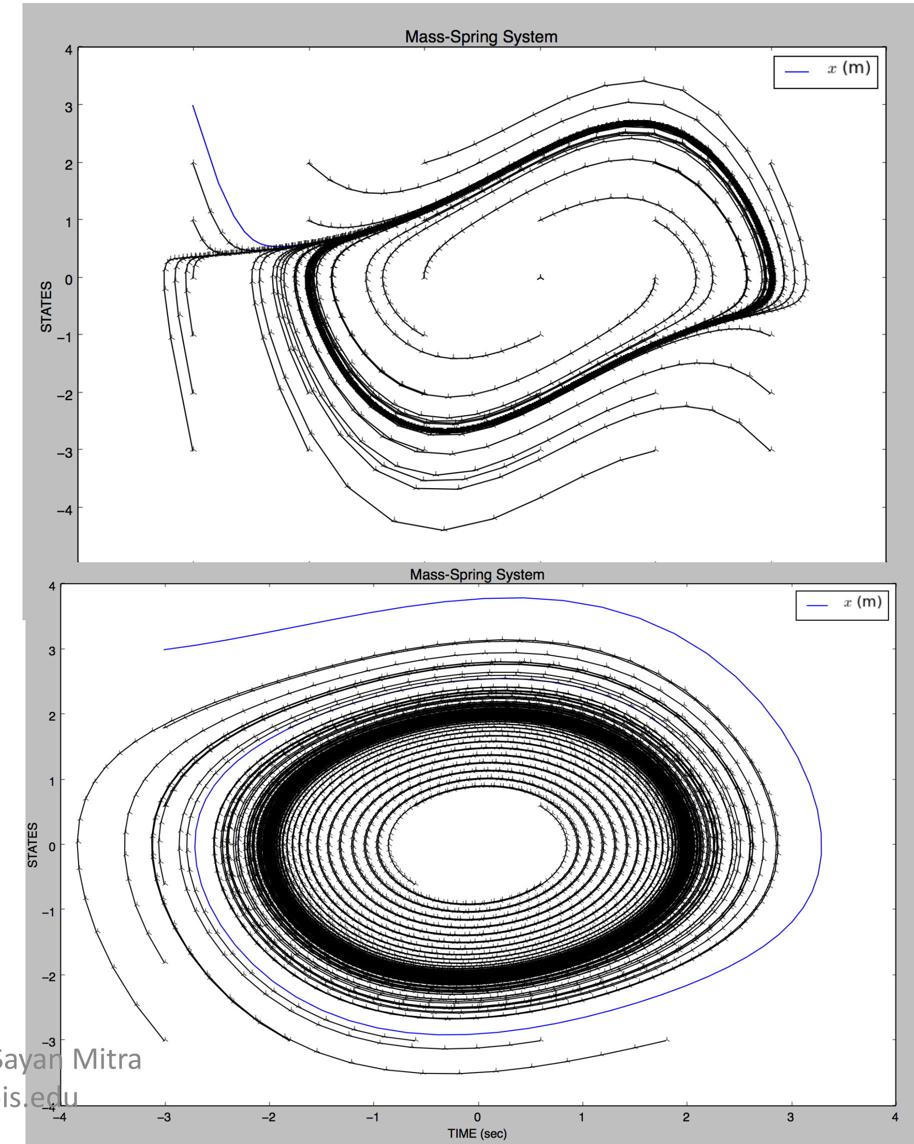
$$\frac{dx^2}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

$x_1 = x; x_2 = \dot{x}_1;$

coupling coefficient $\mu = 2 \;\; 0.1$

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} \mu(1 - x_1^2)x_2 - \;\; x_1 \\ x_2 \end{bmatrix}$$

stable ?



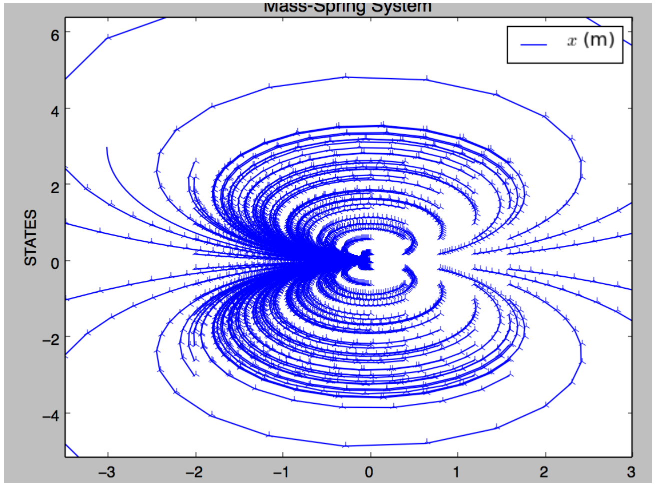
Stability of solutions* (instead of points)

- For any $\xi \in \mathrm{PC}(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$ define the s-norm $||\xi||_s = \sup_{t \in \mathbb{R}} ||\xi(t)||$
- A dynamical system can be seen as an operator that maps initial states to signals $T: \mathbb{R}^n \to PC(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$
- Lyapunov stability required that this operator is continuous
- The solution ξ^* is ***Lyapunov stable*** if T is continuous as $\xi^*(0)$. i.e., for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ if $|\xi^*(0) - x_0| \leq \delta_\varepsilon$ then $\left|\left|T\left(\xi^*(t)\right) - T(x_0)\right|\right|_s \leq \varepsilon$.

*Not discussed in class

Butterfly*

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$



All solutions converge to 0 but the equilibrium point (0,0) is not Lyapunov stable

*Not discussed in class

Verifying Stability for Linear Systems

Consider the linear system $\dot{x} = Ax$

Theorem.

1. It is asymptotically stable iff all the eigenvalues of *A* have **strictly** negative real parts (*Hurwitz*).

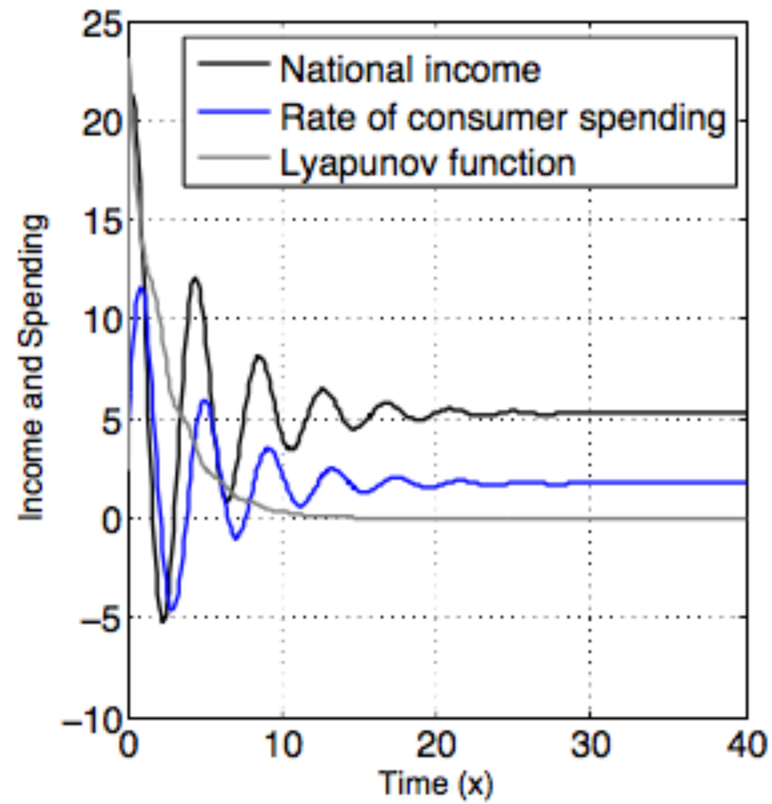
1. It is Lyapunov stable iff all the eigen values of *A* have real parts that are either zero or negative and the Jordan blocks corresponding to the eigenvalues with zero real parts are of size 1.

Example 1: Simple linear model of an economy

- x: national income y: rate of consumer spending; g: rate government expenditure
- $\dot{x} = x \ - \alpha y$
- $\dot{y} = \beta(x - y - g)$
- $g = g_0 + kx$ α, β, k are positive constants
- What is the equilibrium?
- $x^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha}\, y^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha}$
- Dynamics:
- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example: Simple linear model of an economy

- $\alpha = 3, \beta = 1, k = 0$
- $\lambda_1, \lambda_1^* = (-.25 \pm i\ 1.714)$
- Negative real parts, therefore, asymptotically stable and the national income and consumer spending rate converge to $x = 1.764\ y = 5.294$



Stability of nonlinear systems

- For any **positive definite** function of state $V: \mathbb{R}^n \to \mathbb{R}$
  - $V(x) \geq 0$ and $V(x) = 0$ iff $x = 0$
- Sub level sets of $L_p = \{x \in \mathbb{R}^n \mid V(x) \leq p\}$
- $V(\xi(t))$

V differentiable with continuous first derivative

- $\dot{V} = \ d\frac{V(\xi(t))}{dt} = ?$
- $\frac{\partial V}{\partial x}.\frac{d}{dt}(\xi(t)) = \frac{\partial V}{\partial x}.f(x)$ is also continuous
- V is radially unbounded if $||x|| \to \infty \Rightarrow V(x) \to \infty$

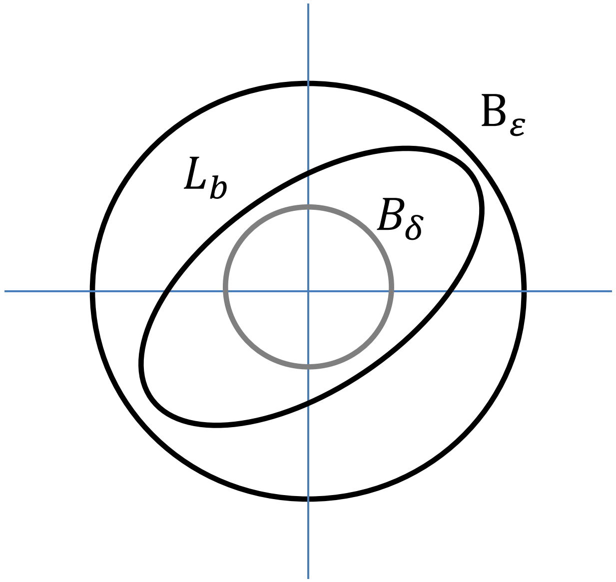
Verifying Stability

Theorem. (Lyapunov) Consider the system (1) with state space $\xi(t) \in \mathbb{R}^n$ and suppose there exists a positive definite, continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$. The system is:

1. Lyapunov stable if $\dot{V}\big(\xi(t)\big) = \frac{\partial V}{\partial x} f(x) \leq 0$
2. Asymptotically stable if $\dot{V}\big(\xi(t)\big) < 0$
3. It is globally AS if *V* is also radially unbounded.

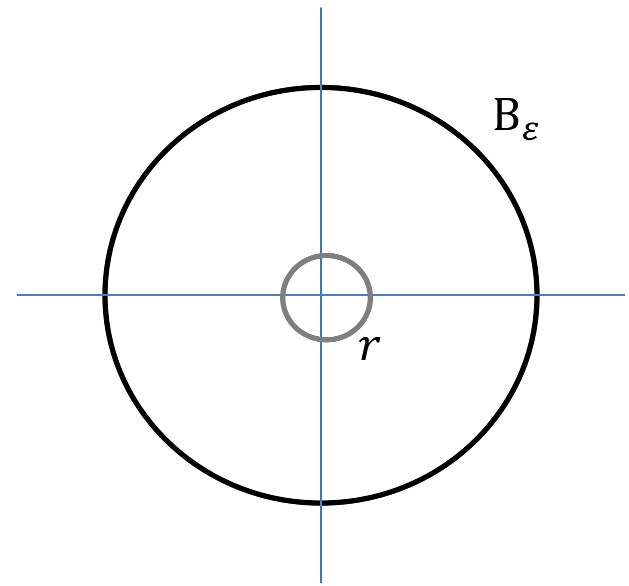
Proof sketch: Lyapunov stable if $\dot{V} \leq 0$

- Assume $\dot{V} \leq 0$
- Consider a ball B_ε around the origin of radius $\varepsilon > 0$.
- Pick a positive number $b < \min_{|x|=\varepsilon} V(x)$.
- Let δ be a radius of ball around origin which is inside $\mathrm{B}_\delta = \{x \mid V(x) \leq b\}$
- Since along all trajectories V is non-increasing, starting from B_δ each solution satisfies $V\big(\xi(t)\big) \leq b$ and therefore remains in B_ε



Proof sketch: Asymptotically stable if $\dot{V}(\xi(t)) < 0$

- Assume $\dot{V} < 0$
- Take arbitrary $|\xi(0)| \leq \delta$, where this δ comes from some ε for Lyapunov stability
- Since $V(\xi(.)) > 0$ and decreasing along ξ it has a limit $c \geq 0$ at $t \to \infty$
- It suffices to show that this limit is actually 0
- Suppose not, c > 0 then the solution evolves in the compact set $S = \{x \mid r \leq |x| \leq \varepsilon\}$ for some sufficiently small r
- Let $d = \max_{x \in S} \dot{V}(x)$ [slowest rate]
- This number is well-defined and negative
- $\dot{V}(\xi(t)) \leq d$ for all t
- $V(t) \leq V(0) + dt$
- But then eventually $V(t) < c$



Example 2

- $\dot{x}_1 = -x_1 + g(x_2); \dot{x}_2 = -x_2 + h(x_1)$
- $|g(u)| \leq \frac{|u|}{2}, |h(u)| \leq \frac{|u|}{2}$
- Use $V = \frac{1}{2}(x_1^2 + x_2^2) \geq 0$
- $\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$

$$= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1)$$

$$\leq -x_1^2 - x_2^2 + \frac{1}{2}(|x_1 x_2| + |x_2 x_1|)$$

$$\leq -\frac{1}{2}(x_1^2 + x_2^2) = -V$$

$$(|x_1| - |x_2|)^2 \geq 0$$

$$x_1^2 + x_2^2 \geq 2|x_1 x_2|$$

$$|x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$$

We conclude global asymptotic stability (in fact global exponential stability) without knowing solutions

Proposition. Every sublevel set of V is an invariant

Proof. $V\big(\xi(t)\big) =$

$$= V\big(\xi(0)\big) + \int_0^t \dot{V}\big(\xi(\tau)\big)d\tau$$

$$\leq V(\xi(0))$$

An aside: Checking inductive invariants

- $\boldsymbol{A} = \langle X, Q_0, T \rangle$
  - X: set of variables
  - $Q_0 \subseteq val(X)$
  - $T \subseteq val(X) \times val(X)$ written as a program $x' \subseteq T(x)$
- How do we check that $I \subseteq val(X)$ is an inductive invariant?
  - $Q_0 \Rightarrow I(X)$
  - $I(X) \Rightarrow I(T(X))$
- Implies that $Reach_{\boldsymbol{A}}(Q_0) \subseteq I$ without computing the executions or reachable states of ***A***
- The key is to find such I

Finding Lyapunov Functions

- The key to using Lyapunov theory is to *find* a Lyapunov function and verify that it has the properties
- In general for nonlinear systems this is hard
- There are several approaches
  - Linear quadratic Lyapunov functions for linear systems
  - Decide the form/template of the function (e.g., quadratic), parameterized by some parameters
  - Try to find values of the parameters so that the conditions hold

Linear autonomous systems

- $\dot{x} = Ax, A \in \mathbb{R}^{n \times n}$
- The Lyapunov equation: $A^T P + PA + Q = 0$

  where $P, Q \in \mathbb{R}^{n \times n}$ are symmetric

- Interpretation: $V(x) = x^T P x$ then

$$\dot{V}(x) = (Ax)^T P x + x^T P (Ax)$$

[using $\frac{\partial u^T P v}{\partial t} = \frac{\partial u}{\partial t} P v + \frac{\partial v}{\partial t} P^T u$]

= $x^T (A^T P + PA) x = -x^T Q x$

- If $x^T P x$ is the generalized energy then $-x^T Q x$ is the associated dissipation

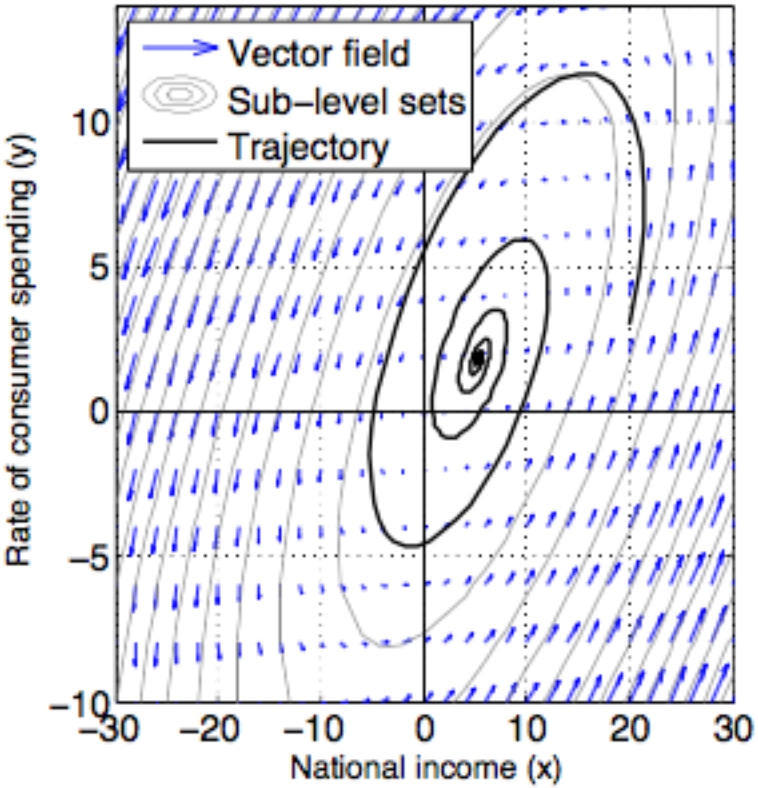
Quadratic Lyapunov Functions

- If $P > 0$ (positive definite)
- $V(x) = x^T P x = 0 \Leftrightarrow x = 0$
- The sub-level sets are ellipsoids
- If $Q > 0$ then the system is globally asymptotically stable

Same example

Lyapunov equations are solved as a set of $\frac{n(n+1)}{2}$ equations in $n(n+1)/2$ variables. Cost O(n^6)

Choose $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ solving Lyapunov equations we get $P = \begin{bmatrix} 2.59 & -2.29 \\ -2.29 & 4.92 \end{bmatrix}$ and we get the quadratic Lyapunov function $(x - x^*)P(x - x^*)^T$ an a sequence of invariants



Converse Lyapunov

Converse Lyapunov theorems show that conditions of the previous theorem are also necessary. For example, if the system is asymptotically stable then there exists a positive definite, continuously differentiable function V, that satisfies the inequalities.

For example if the LTI system $\dot{x} = Ax$ is globally asymptotically stable then there is a quadratic Lyapunov function that proves it.