GLOBAL INITIATIVE OF ACADEMIC NETWORKS

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Lecture 3: Temporal Logic and Model Checking

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Verification thus far

Given an **automaton** $\mathcal{A} = \langle X, \Theta, A, \mathcal{D} \rangle$ and a set of **unsafe states** $U \subseteq val(X)$ we can check whether $Reach_{\mathcal{A}}(\Theta) \cap U = \emptyset$?

Thus, far we looked at verification of invariant properties through reachability analysis

What about more general types of properties, e.g.,

- “Eventually the light turns red and prior to that the orange light blinks”

- “After failures, eventually there is just one token in the system”

How to express and verify such properties?

Plan

- CTL model checking
  - Setup
  - CTL syntax and semantics
  - Model checking algorithms
  - Example

Introduction to temporal logics

Temporal logics give a formal language for representing, and reasoning about, propositions qualified in terms of time, or their validity in a sequence

Amir Pnueli received the ACM Turing Award (1996) for seminal work introducing temporal logic into computer science and for outstanding contributions to program and systems verification.

Large follow-up literature, e.g., different temporal logics MTL, MITL, PCTL, ACTL, STL

Setup

We have a set of atomic propositions (AP)

These are the properties that hold in each state, e.g., “light is green”, “has 2 tokens”

We have a labeling function that assigns to each state, a set of propositions that hold at that state

$L: Q \rightarrow 2^{AP}$

Notations (this lecture)

$\mathcal{A} = \langle Q, Q_0, T, L \rangle$

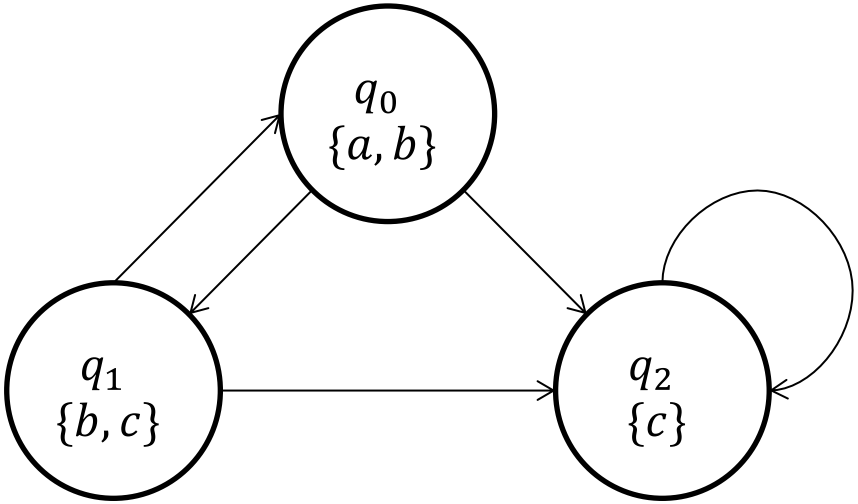
Executions $\alpha = q_0\ q_1 \ldots q_k = \alpha.lstate$

$\alpha[i] = q_i$

$Exec_{\mathcal{A}}$ set of all executions

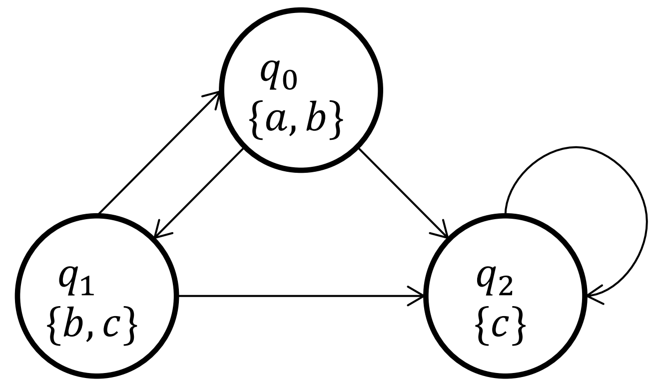
$AP = \{a, b, c\}$

$L(q_0) = \{a, b\}$

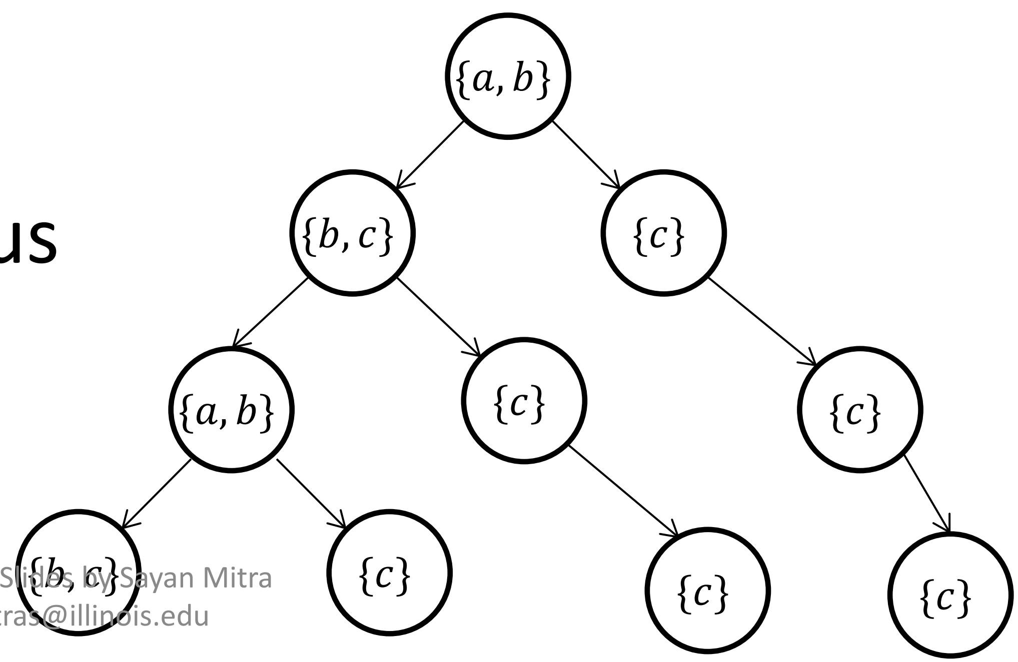


Computational tree logic (CTL)

Unfolding the automaton



We get a tree



A CTL formula allows us to specify subsets of paths in this tree

CTL quantifiers

Path quantifiers

- E: Exists some path
- A: All paths

Temporal operators

- X: Next state
- U: Until
- F: Eventually
- G: Globally (Always)

CTL syntax

CTL syntax

$State\ Formula\ (SF) ::= true \mid p \mid \neg f_1 \mid f_1 \wedge f_2 | E\ \phi \mid A\ \phi$
$Path\ Formula\ (PF) ::= X f_1 | \ f_1 U\ f_2\ | G f_1 |\ F\ f_1$
where $p \in AP, f_1, f_2 \in SF, \phi \in PF$

Depth of formula: number of production rules used

Examples (depth)

E$X\ a$; AXE$X\ a$; AXEXa U b; AG AF green; AF AG single token
Depth 3, 5, …

Non-examples

AX$X\ a$; path and state operators must alternate in CTL

CTL semantics

Given automaton $\mathcal{A} = \langle Q, \mathrm{Q}_0, T, L \rangle, q \in Q$ and a CTL formula $\phi, q \vDash \phi$ denotes that q satisfies ϕ; $\alpha \vDash \phi$ denotes that path (execution) α satisfies ϕ. The relation $\vDash$ is defined inductively as:

$$
\begin{aligned}
&\mathcal{A}, q \vDash p && \Leftrightarrow p \in L(q) \text{ for } p \in AP \\
&\mathcal{A}, q \vDash \neg f_1 && \Leftrightarrow \mathcal{A}, q \nvDash f_1 \\
&\mathcal{A}, q \vDash f_1 \wedge f_2 && \Leftrightarrow \mathcal{A}, q \vDash f_1 \wedge \mathcal{A}, q \vDash f_2 \\
&\mathcal{A}, q \vDash E\phi && \Leftrightarrow \exists\, \alpha, \alpha.fstate = q, \mathcal{A}, \alpha \vDash \phi \\
&\mathcal{A}, q \vDash A\phi && \Leftrightarrow \forall\, \alpha, \alpha.fstate = q, \mathcal{A}, \alpha \vDash \phi \\
&\mathcal{A}, q \vDash Xf && \Leftrightarrow \mathcal{A}, \alpha[1] \vDash f \\
&\mathcal{A}, \alpha \vDash f_1\, U\, f_2 && \Leftrightarrow \exists i \geq 0, \mathcal{A}, \alpha[i] \vDash f_2 \ and\ \forall j < i\ \alpha[j] \vDash f_1 \\
&\mathcal{A}, \alpha \vDash F\, f_1 && \Leftrightarrow \exists i \geq 0, \mathcal{A}, \alpha[i] \vDash f_1 \\
&\mathcal{A}, \alpha \vDash G\, f_1 && \Leftrightarrow \forall i \geq 0, \mathcal{A}, \alpha[i] \vDash f_1
\end{aligned}
$$

Automaton satisfies property: $\mathcal{A} \vDash f$ iff $\forall q \in Q_0, \mathcal{A}, q \vDash f$

Universal CTL oprators

X, U, G can be used to derive other operators

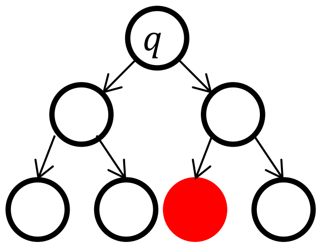
$true\ U\ f \equiv F\ f$

$Gf \equiv \neg F(\neg f)$

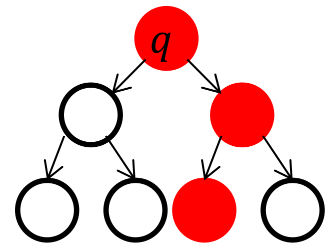
All ten combinations can be expressed using EX, EU, EG

AXf	AGf	AFf	AUf	ARf
$\neg EX(\neg f)$	$\neg EF(\neg f)$	$\neg EG(\neg f)$		
EX	EG	EF	EU	ER
EX	EG	$E(true\ U\ f)$	EU	

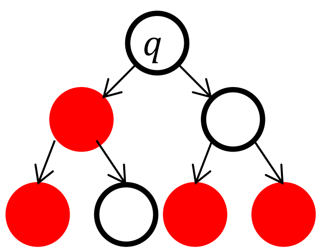
Visualizing semantics



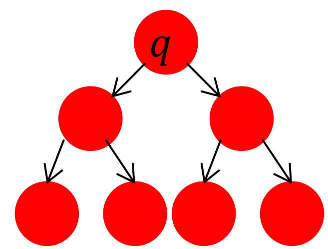
$q \vDash EF\ red$



$q \vDash EG\ red$



$q \vDash AF\ red$



$q \vDash AG\ red$

Algorithm for deciding $\mathcal{A} \vDash f$

Algorithm works by structural induction on the depth of the formula

Explicit state model checking

Compute the subset $Q' \subseteq Q$ such that $\forall q \in Q'$ *we have* $\mathcal{A}, \mathrm{q} \vDash f$

If $Q_0 \subseteq Q'$ $\cancel{Q' \subseteq Q_0}$ then we can conclude $\mathcal{A} \vDash f$

Induction on depth of formula

Algorithm computes a function $label: Q \to CTL(AP)$ that labels each state with a CTL formula

- Initially, $label(q) = L(q)$ for each $q \in$ Q
- At i^{th} iteration $label(q)$ contains all sub-formulas of f of depth $(i-1)$ that q satisfies

At termination $f \in label(q) \Leftrightarrow \mathcal{A}, q \vDash f$

Structural induction on formula

Six cases to consider based on structure of f

$f = p,$	for some $p \in AP, \forall q, label(q) \coloneqq label(q) \cup f$
$f = \neg f_1$	if $f_1 \notin label(q)$ then $label(q) \coloneqq label(q) \cup f$
$f = f_1 \wedge f_2$	if $f_1, f_2 \in label(q)$ then $label(q) \coloneqq label(q) \cup f$
$f = EXf_1$	if $\exists q' \in$ Q such that $(q, q') \in T$ and $f_1 \in label(q')$ then $label(q) \coloneqq label(q) \cup f$
$f = E[f_1 U f_2]$	$CheckEU(f_1, f_2, Q, T, L)$ [next slide]
$f = EGf_1$	$CheckEG(f_1, Q, T, L)$ [next slide]

$CheckEU(f_1, f_2, Q, T, L)$

Let $S = \{q \in Q \mid f_2 \in label(q)\}$
for each $q \in S$
  $label(q) := label(q) \cup \{E[f_1 U f_2]\}$
while $S \neq \emptyset$
  for each $q' \in S$
    $S := S \setminus \{q'\}$
    for each $q \in T^{-1}(q')$
      if $f_1 \in label(q)$ then
        $label(q) := label(q) \cup \{E[f_1 U f_2]\}$
          $S := S \cup \{q\}$

Proposition. For any state $label(q) \ni E[f_1 U f_2]$ iff $q \vDash E[f_1 U f_2]$.

Proposition. Finite Q therefore terminates and in $O(|Q| + |T|)$ steps.

$CheckEG(f_1, Q, T, L)$

From $\mathcal{A}$ we construct a new automaton $\mathcal{A}' = \langle Q', T', L' \rangle$ such that

$Q' = \{q \in Q \mid f_1 \in label(q)\}$

$T' = \{\langle q_1, q_2 \rangle \in T \mid q_1 \in Q'\} = T \mid Q'$

$L': Q' \to 2^{AP} \ \forall q' \in Q', L'(q') := L(q')$

Claim. $\mathcal{A}$, q $\vDash EG f_1$ iff

(1) $q \in Q'$

(2) $\exists \alpha \in Execs_{\mathcal{A}}$, with $\alpha.fstate = q$ and $\alpha.lstate$ is in a nontrivial SCC C of the graph $\langle Q', T' \rangle$

Claim. $\mathcal{A}$, q $\vDash EGf_1$ iff

(1) $q \in Q'$ and

(2) $\exists \alpha \in Execs_{\mathcal{A}}$, with $\alpha.fstate = q$ and $\alpha.lstate$ is in a nontrivial SCC C of the graph $\langle Q', T' \rangle$

Proof. Suppose $\mathcal{A}$, q $\vDash EGf_1$

Consider any execution α with $\alpha.fstate = q$. Obviously, $q \vDash f_1$ and so, $q \in Q'$.

Since Q is finite α can be written as $\alpha = \alpha_0\alpha_1$ where α_0 is finite and every state in α_1 repeats infinitely many times.

Let C be the states in α_1. $C \in Q'$.

Consider any two q_1 and q_2 states in C, we observe that $q_1 \rightleftarrows q_2$, and therefore C is a SCC.

Consider (1) and (2). We will construct a path $\alpha = \alpha_0\alpha_1$ such that $\alpha_0.fstate = q$ and $\alpha_0 \in Q'$and $\alpha_1$visits some states infinitely often.

$CheckEG(f_1, Q, T, L)$

Let $Q' = \{q \in Q \mid f_1 \in label(q)\}$
Let $\mathbb{C}$ be the set of nontrivial SCCs of $\langle Q', T' \rangle$
$\boldsymbol{T} = \cup_{C \in \mathbb{C}} \{q \mid q \in C\}$
for each $q \in \boldsymbol{T}$
  $label(q) \coloneqq label(q) \cup \{EGf_1\}$
while $\boldsymbol{T} \neq \emptyset$
  for each $q' \in \boldsymbol{T}$
    $\boldsymbol{T} \coloneqq \boldsymbol{T} \setminus \{q'\}$
    for each $q' \in Q'$ such that $(q', q) \in T'$
      if $\mathrm{E}Gf_1 \notin label(q')$ then
        $label(q') \coloneqq label(q') \cup \{EGf_1\}$
          $\boldsymbol{T} \coloneqq \boldsymbol{T} \cup \{q\}$

Proposition. For any state $label(q) \ni EGf_1$ iff $q \vDash EGf_1$.

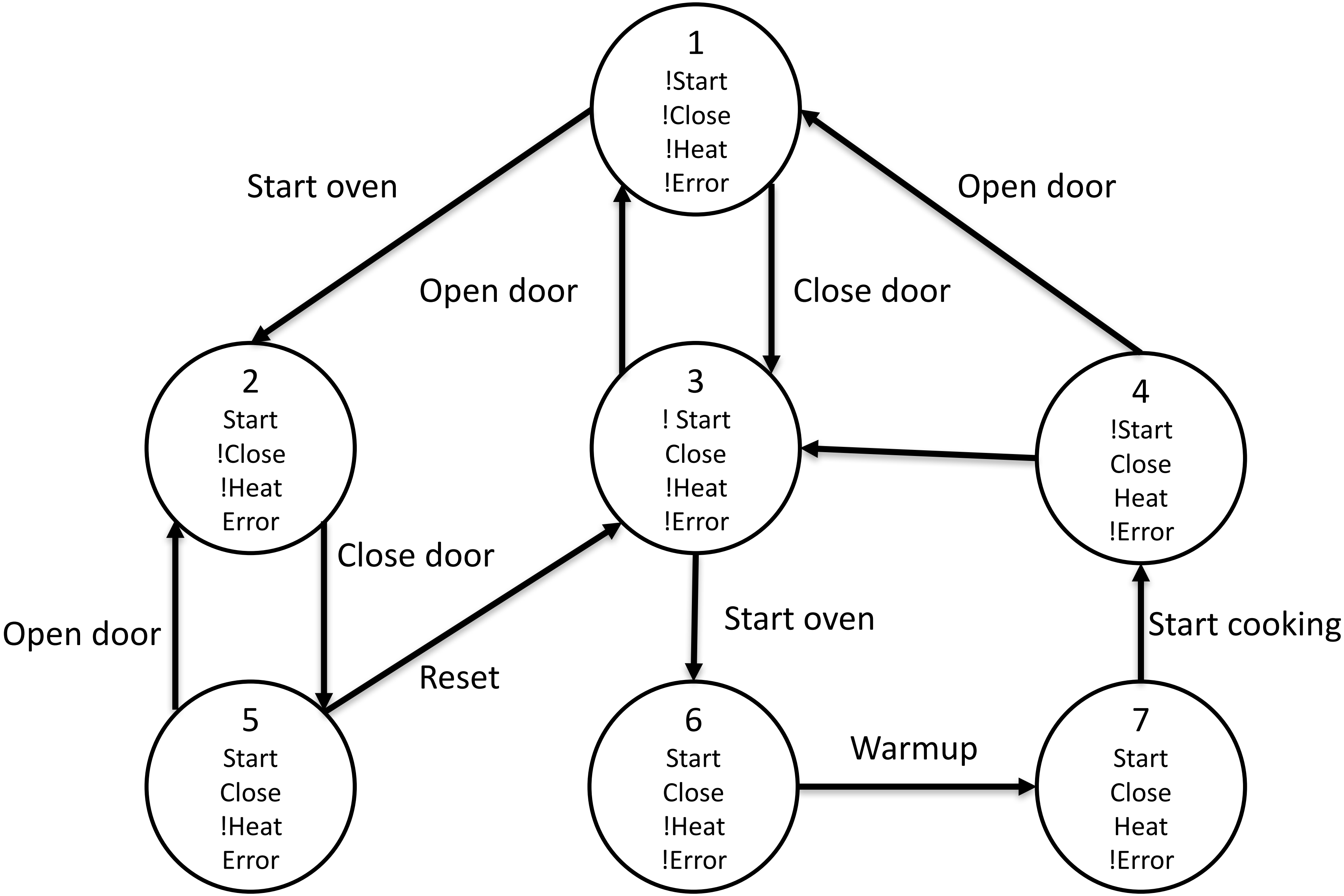
Proposition. Finite Q therefore terminates and in $O(|Q| + |T|)$ steps.

Summary

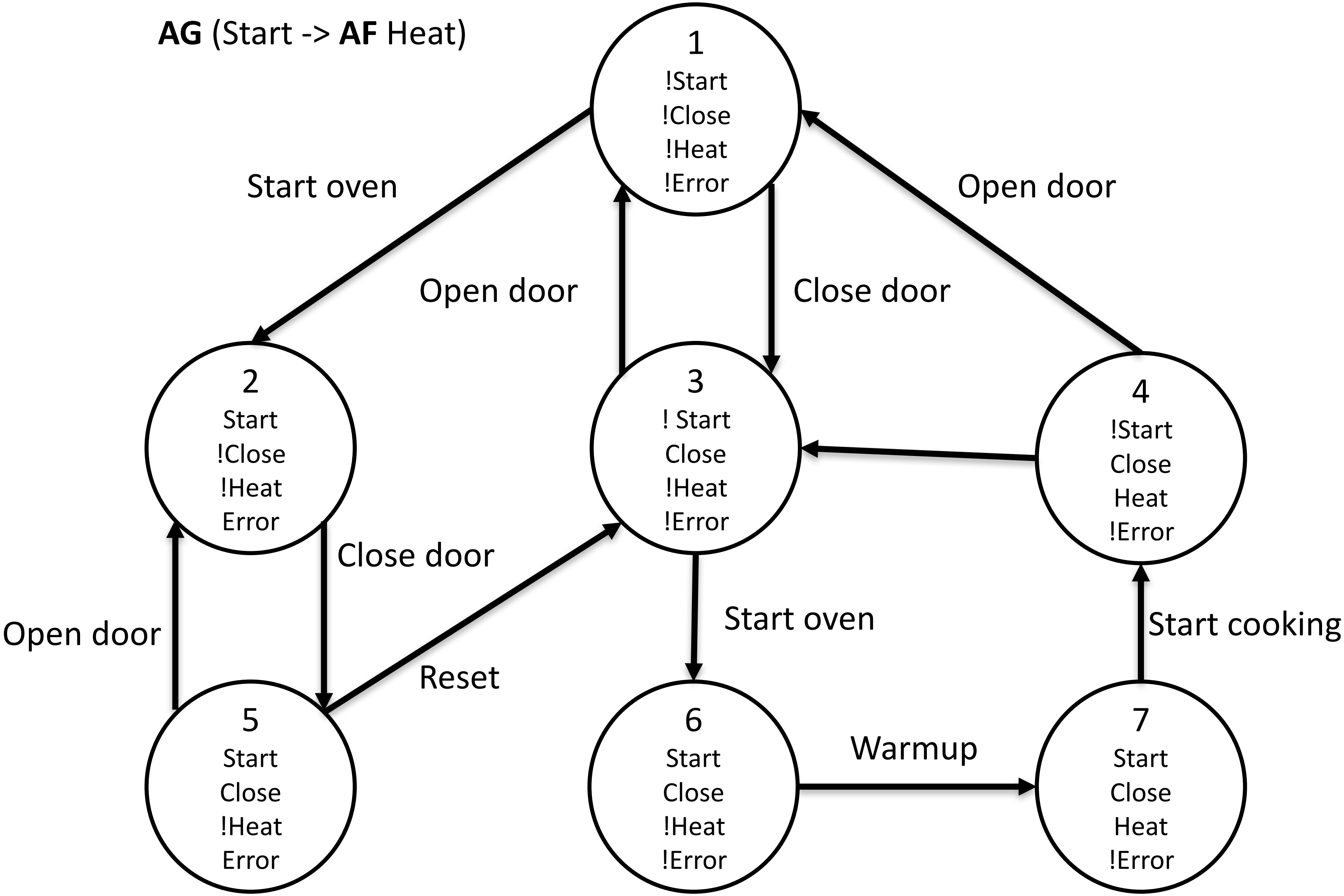
Explicit model checking algorithm input $\mathcal{A} \vDash f$?
Structural induction over CTL formula

$f = p,$	for some $p \in AP, \forall q, label(q) := label(q) \cup \{p\}$
$f = \neg f_1$	if $f_1 \notin label(q)$ then $label(q) := label(q) \cup f$
$f = f_1 \wedge f_2$	if $f_1, f_2 \in label(q)$ then $label(q) := label(q) \cup f$
$f = EXf_1$	if $\exists q' \in$ Q such that $(q, q') \in T$ and $f_1 \in label(q')$ then $label(q) := label(q) \cup f$
$f = E[f_1 U f_2]$	$CheckEU(f_1, f_2, Q, T, L)$
$f = EGf_1$	$CheckEG(f_1, Q, T, L)$

Proposition. Overall complexity of CTL model checkign $O(|f|(|Q| + |T|))$ steps.

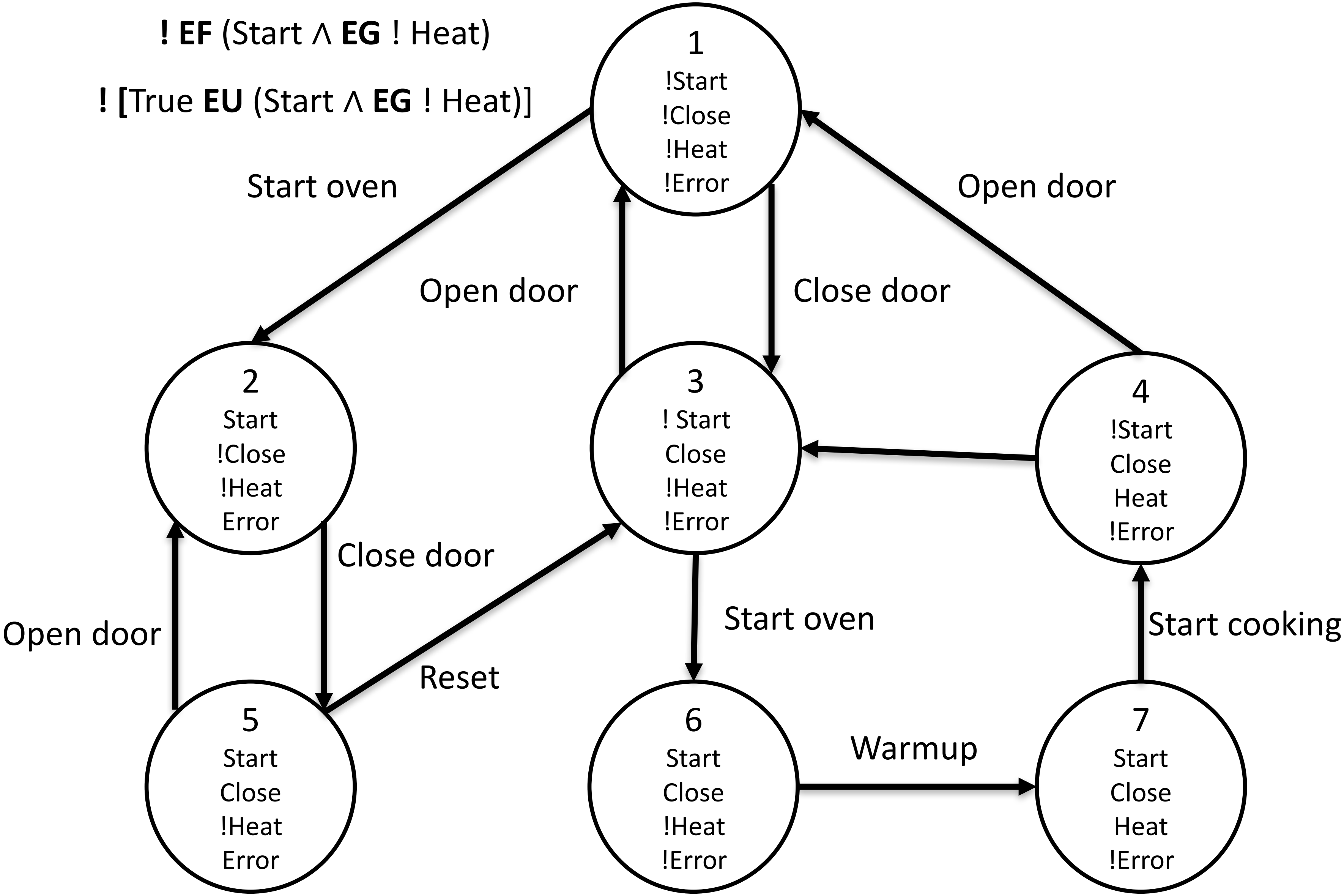

1
!Start
!Close
!Heat
!Error
Start oven
Open door
Close door
Open door
2
Start
!Close
!Heat
Error
3
! Start
Close
!Heat
!Error
4
!Start
Close
Heat
!Error
Close door
Open door
Reset
Start oven
Start cooking
5
Start
Close
!Heat
Error
6
Start
Close
!Heat
!Error
Warmup
7
Start
Close
Heat
!Error

AG (Start -> **AF** Heat)



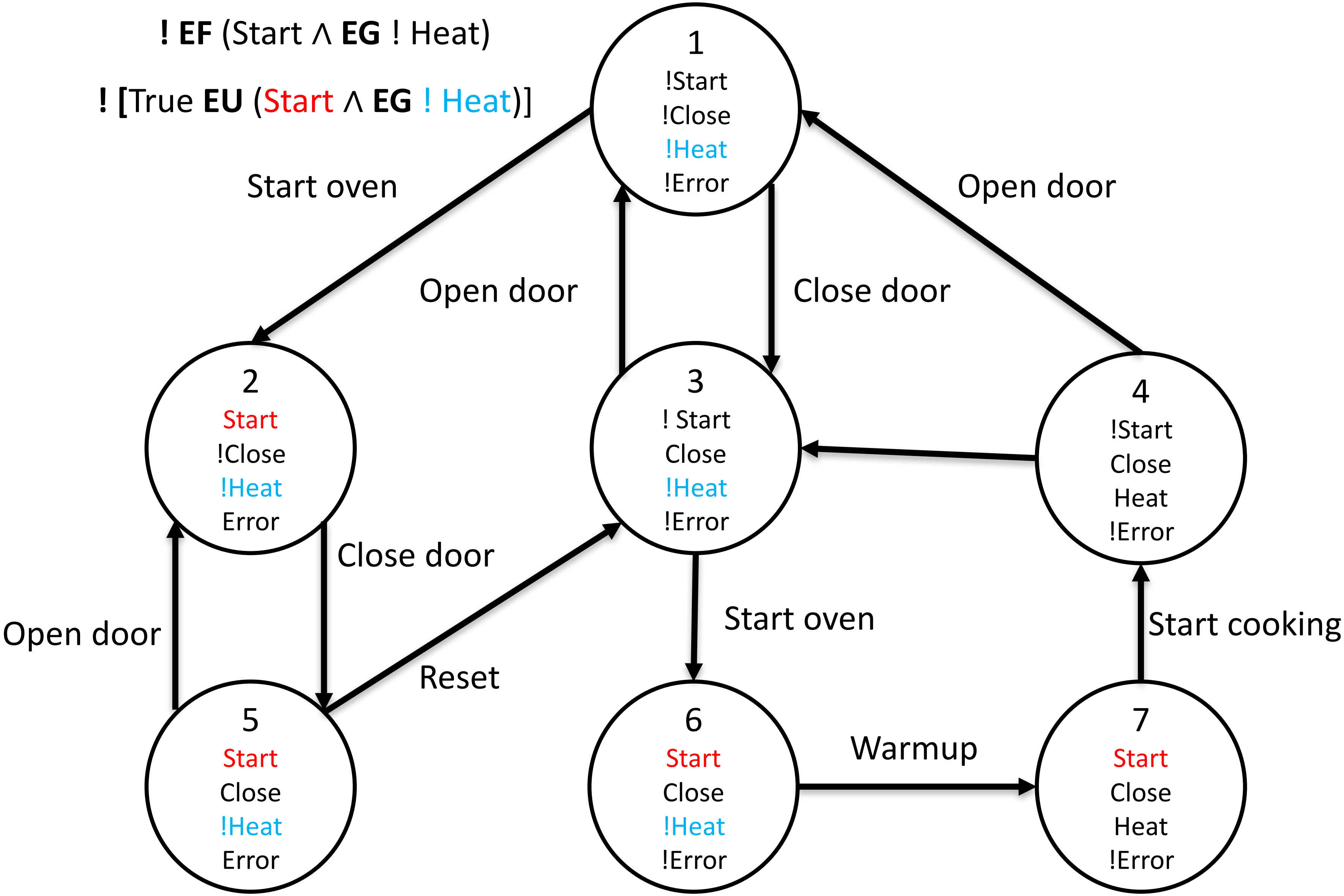
$$\mathbf{!\ EF}\ (\text{Start} \wedge \mathbf{EG}\ !\ \text{Heat})$$

$$\mathbf{!\ [}\text{True}\ \mathbf{EU}\ (\text{Start} \wedge \mathbf{EG}\ !\ \text{Heat})]$$



$! \, \mathbf{EF} \, (\text{Start} \wedge \mathbf{EG} \, ! \, \text{Heat})$

$! \, [\text{True} \, \mathbf{EU} \, (\text{Start} \wedge \mathbf{EG} \, ! \, \text{Heat})]$

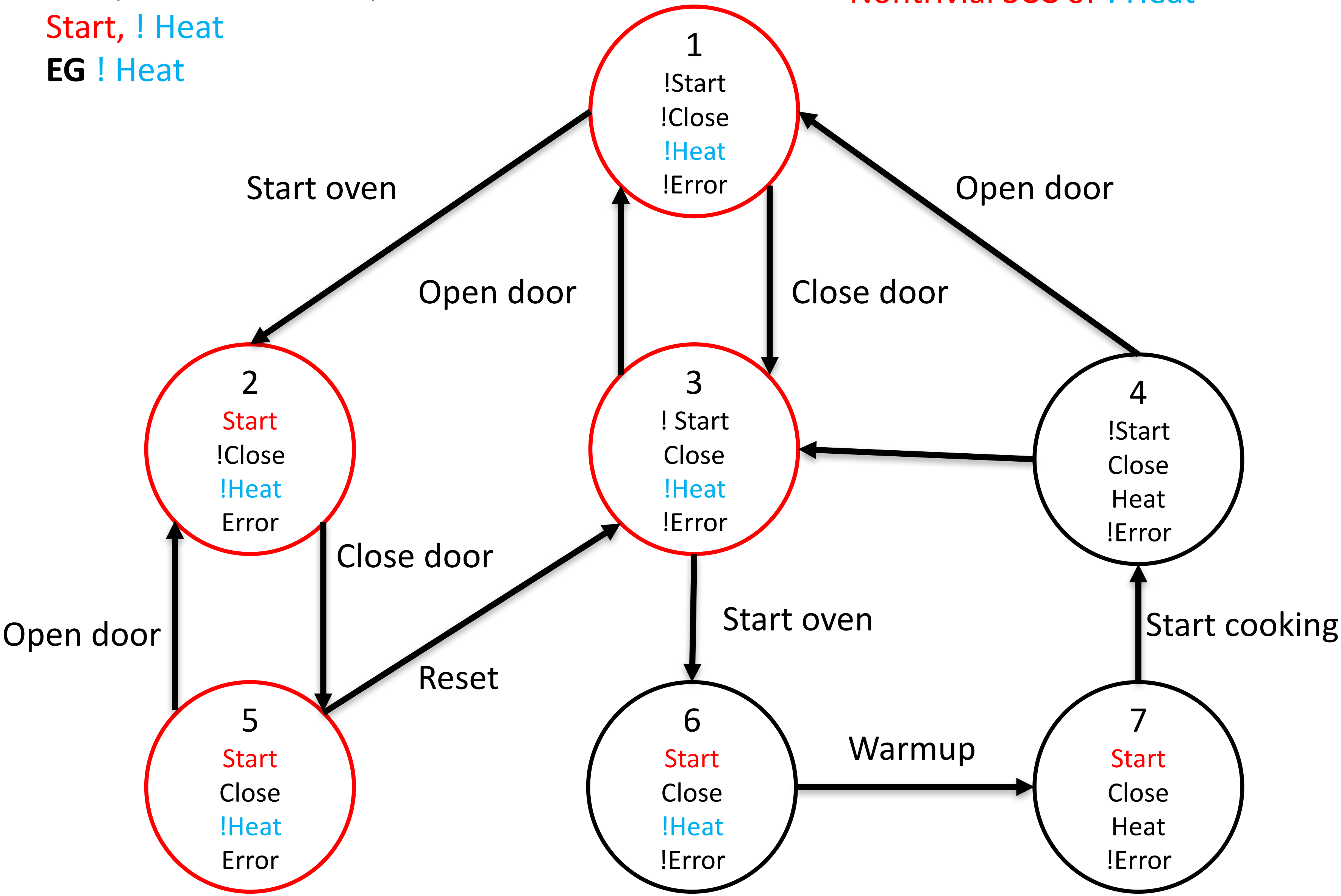


! EF (Start ∧ **EG** ! Heat)

Start, ! Heat

EG ! Heat

Nontrivial SCC of ! Heat



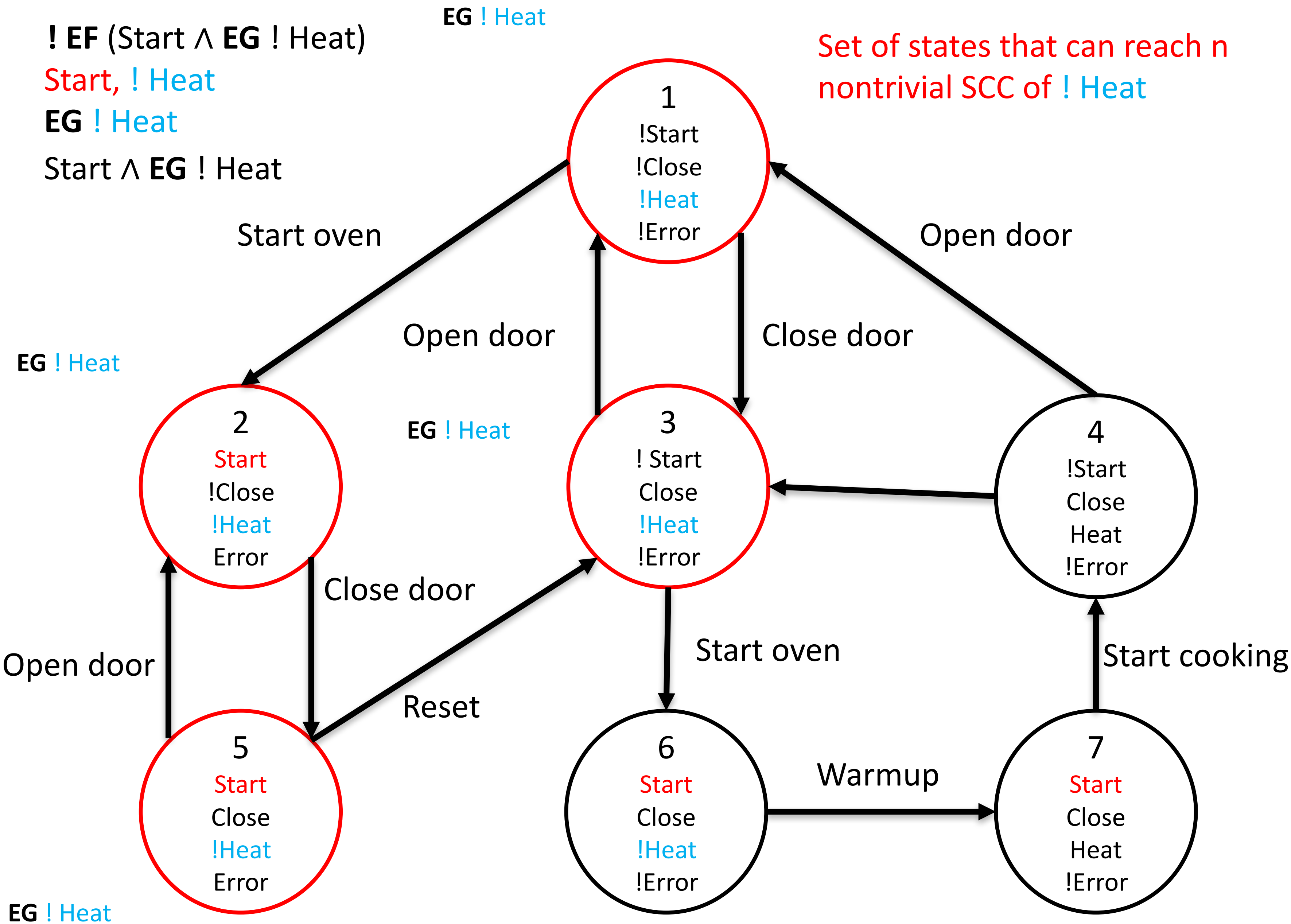
! EF (Start ∧ **EG** ! Heat)

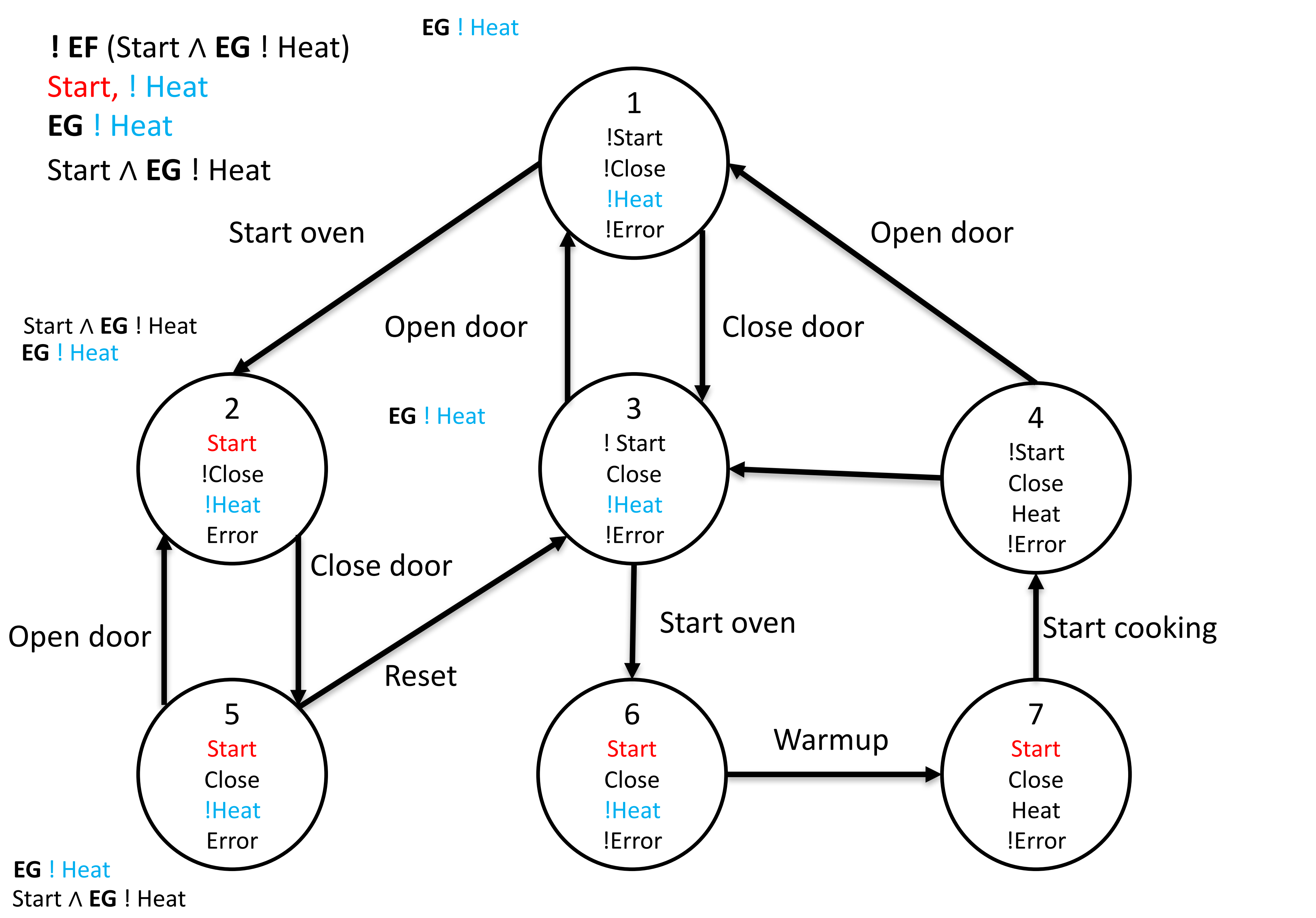
Start, ! Heat

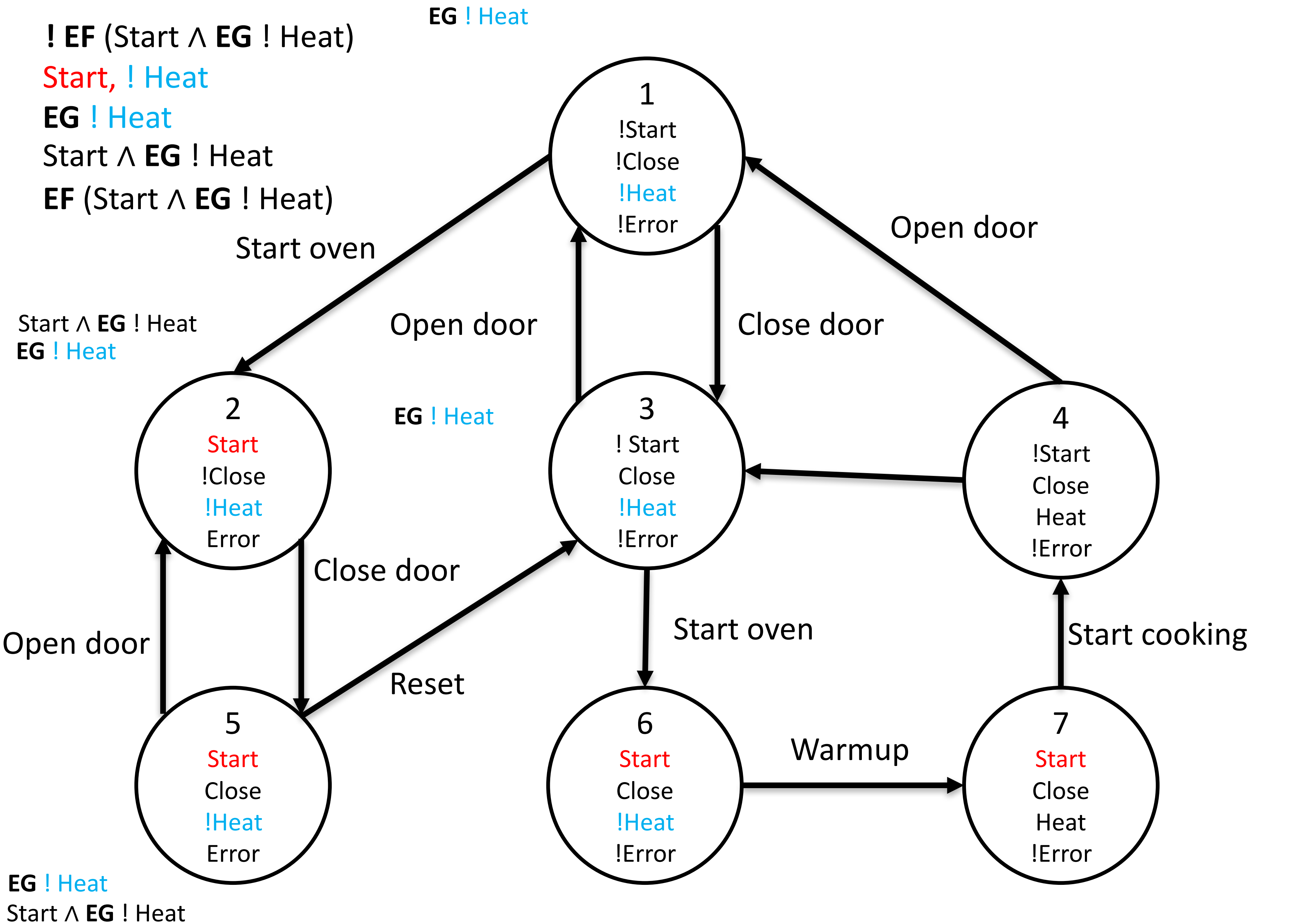
EG ! Heat

Start ∧ **EG** ! Heat

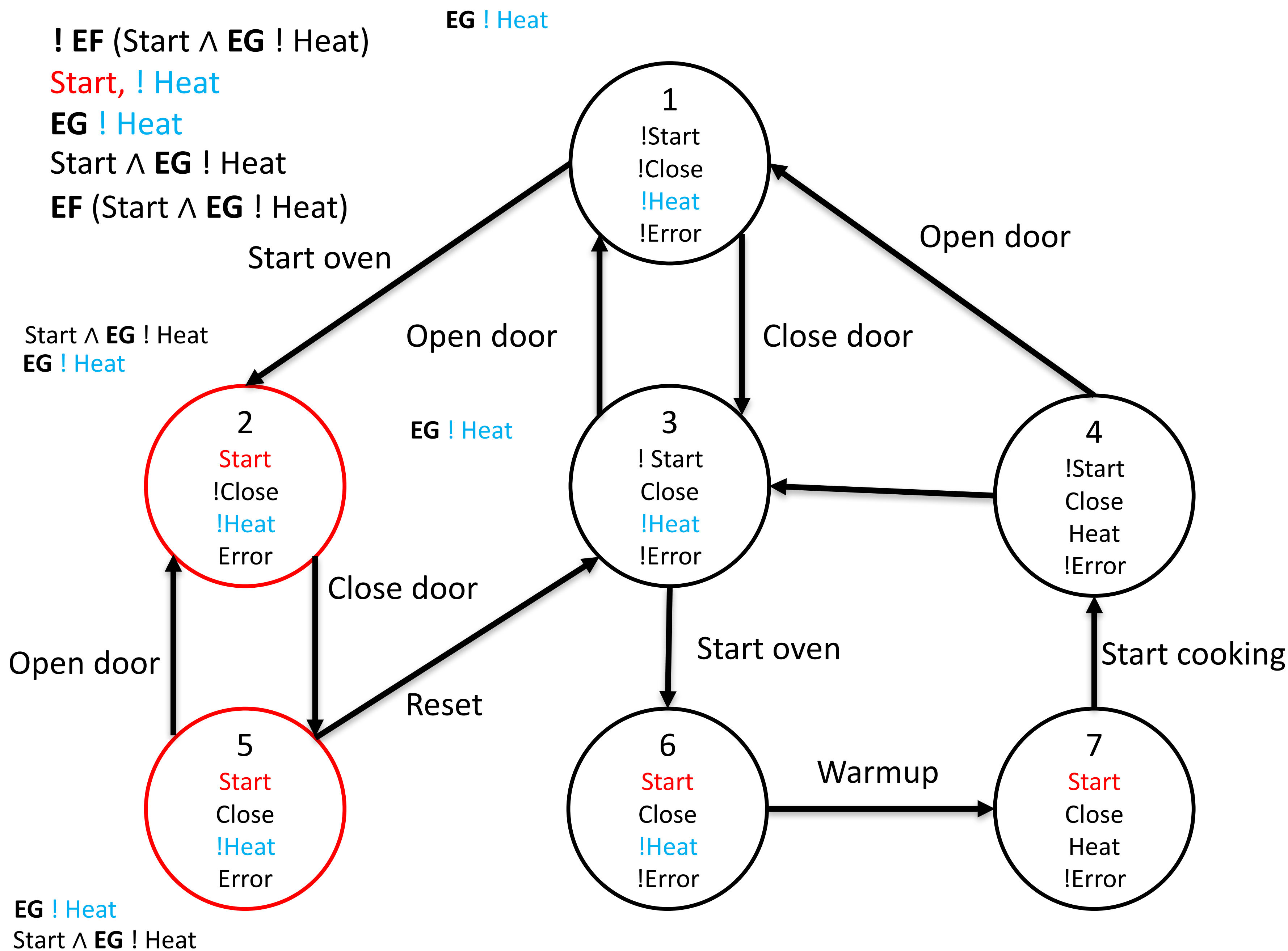
Set of states that can reach n nontrivial SCC of ! Heat




! EF (Start ∧ EG ! Heat)
Start, ! Heat
EG ! Heat
Start ∧ EG ! Heat
EG ! Heat
1
!Start
!Close
!Heat
!Error
Start oven
Open door
Close door
Open door
Start ∧ EG ! Heat
EG ! Heat
2
Start
!Close
!Heat
Error
EG ! Heat
3
! Start
Close
!Heat
!Error
4
!Start
Close
Heat
!Error
Close door
Open door
Start oven
Start cooking
Reset
5
Start
Close
!Heat
Error
6
Start
Close
!Heat
!Error
Warmup
7
Start
Close
Heat
!Error
EG ! Heat
Start ∧ EG ! Heat



! EF (Start ∧ EG ! Heat)
Start, ! Heat
EG ! Heat
Start ∧ EG ! Heat
EF (Start ∧ EG ! Heat)
EG ! Heat
1
!Start
!Close
!Heat
!Error
Start oven
Open door
Close door
Open door
Start ∧ EG ! Heat
EG ! Heat
2
Start
!Close
!Heat
Error
EG ! Heat
3
! Start
Close
!Heat
!Error
4
!Start
Close
Heat
!Error
Close door
Start oven
Start cooking
Open door
Reset
5
Start
Close
!Heat
Error
6
Start
Close
!Heat
!Error
Warmup
7
Start
Close
Heat
!Error
EG ! Heat
Start ∧ EG ! Heat



! EF (Start ∧ EG ! Heat)
Start, ! Heat
EG ! Heat
Start ∧ EG ! Heat
EF (Start ∧ EG ! Heat)
EG ! Heat
1
!Start
!Close
!Heat
!Error
Start oven
Open door
Close door
Open door
Start ∧ EG ! Heat
EG ! Heat
2
Start
!Close
!Heat
Error
EG ! Heat
3
! Start
Close
!Heat
!Error
4
!Start
Close
Heat
!Error
Close door
Start oven
Start cooking
Open door
Reset
Warmup
5
Start
Close
!Heat
Error
6
Start
Close
!Heat
!Error
7
Start
Close
Heat
!Error
EG ! Heat
Start ∧ EG ! Heat

! **EF** (Start ∧ **EG** ! Heat)

Start, ! Heat

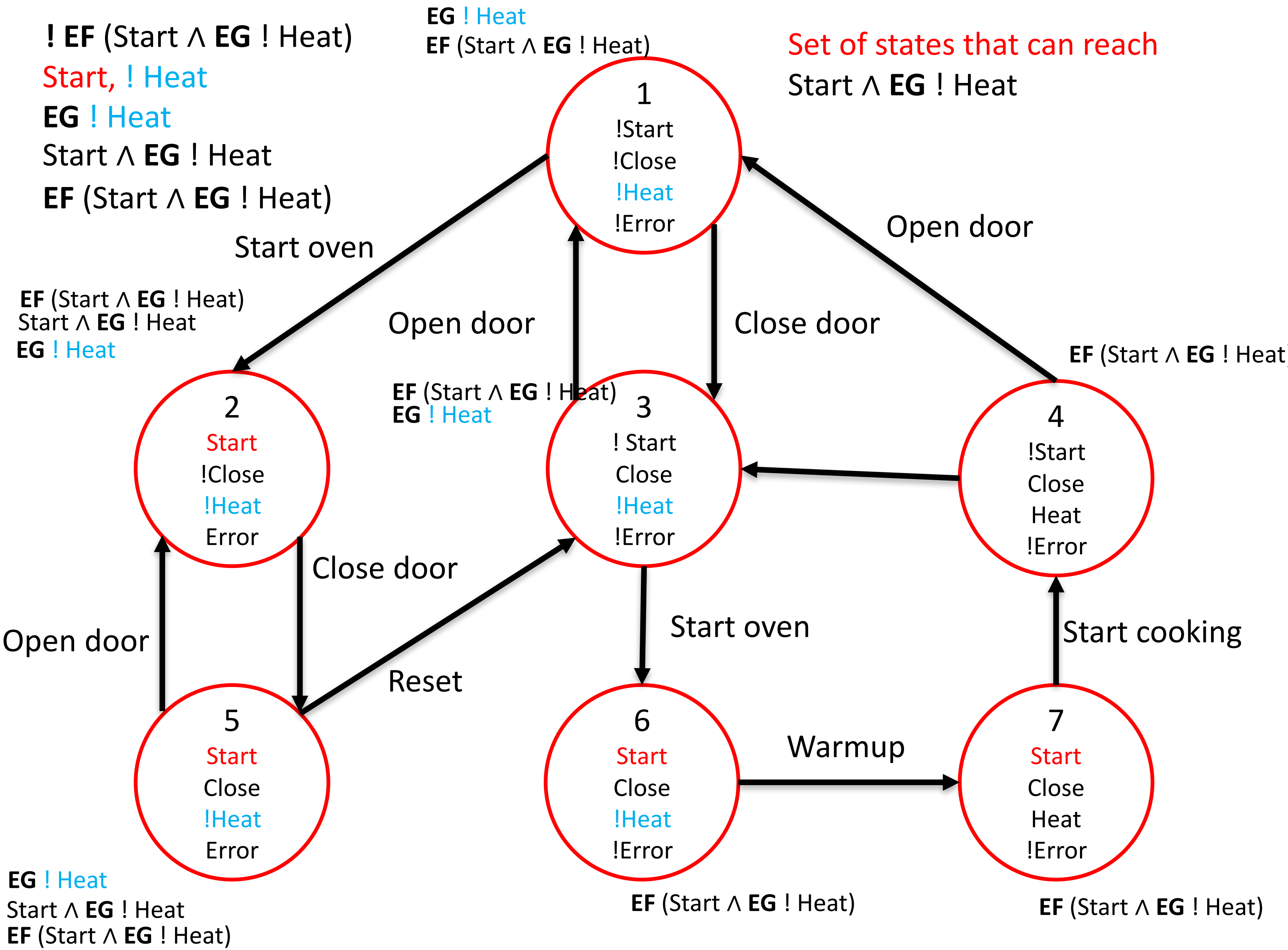
EG ! Heat

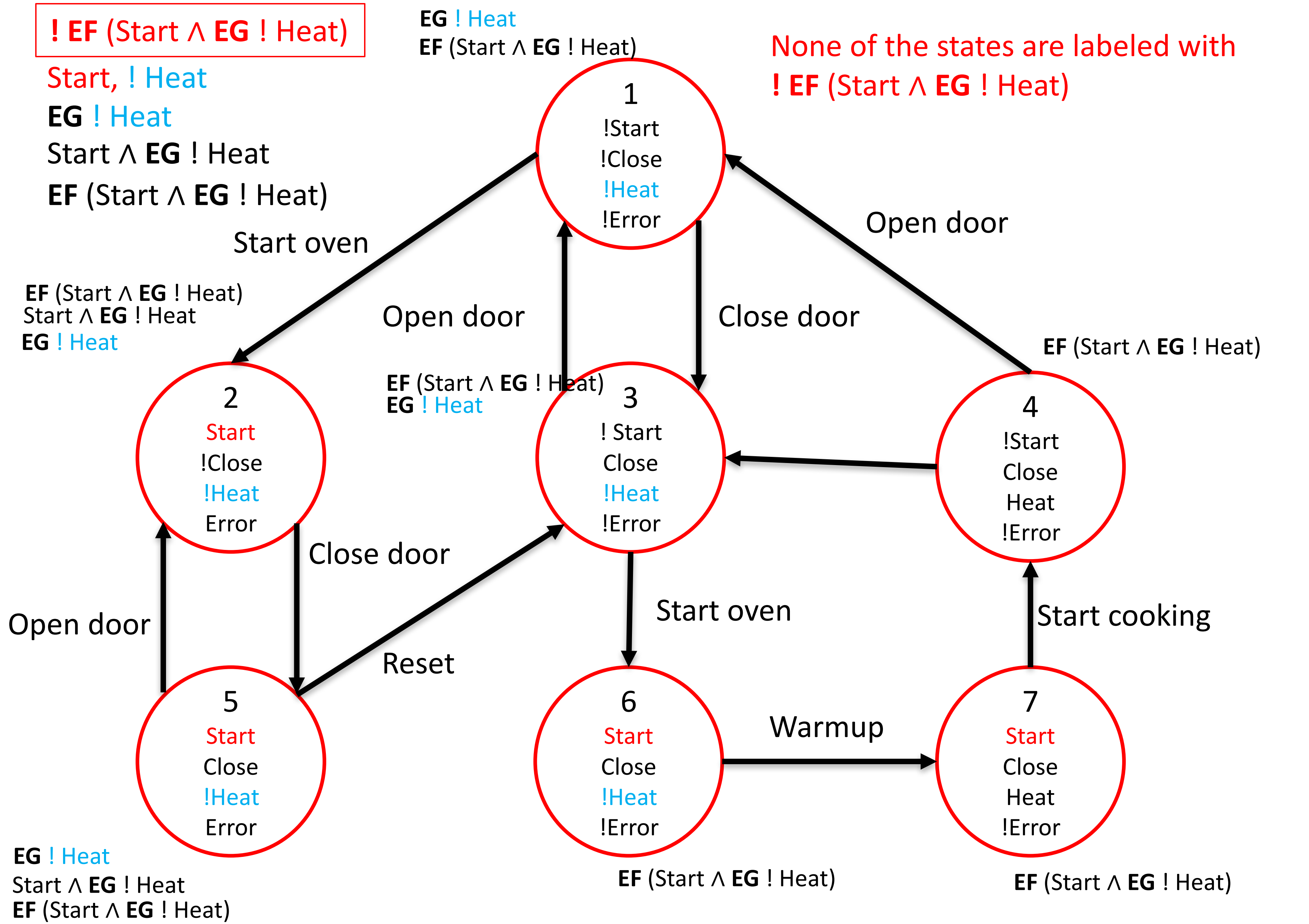
Start ∧ **EG** ! Heat

EF (Start ∧ **EG** ! Heat)

Set of states that can reach

Start ∧ **EG** ! Heat




! EF (Start ∧ EG ! Heat)
Start, ! Heat
EG ! Heat
Start ∧ EG ! Heat
EF (Start ∧ EG ! Heat)
EG ! Heat
EF (Start ∧ EG ! Heat)
None of the states are labeled with
! EF (Start ∧ EG ! Heat)
1
!Start
!Close
!Heat
!Error
Start oven
Open door
Close door
Open door
EF (Start ∧ EG ! Heat)
Start ∧ EG ! Heat
EG ! Heat
EF (Start ∧ EG ! Heat)
EG ! Heat
EF (Start ∧ EG ! Heat)
2
Start
!Close
!Heat
Error
3
! Start
Close
!Heat
!Error
4
!Start
Close
Heat
!Error
Close door
Start oven
Start cooking
Open door
Reset
5
Start
Close
!Heat
Error
6
Start
Close
!Heat
!Error
Warmup
7
Start
Close
Heat
!Error
EG ! Heat
Start ∧ EG ! Heat
EF (Start ∧ EG ! Heat)
EF (Start ∧ EG ! Heat)
EF (Start ∧ EG ! Heat)