# A review of a paper discussing an application of queuing theory.

Paper considered: "An application of queuing theory to the design of a message switching computer system", Jack Gostl and Irwin Greenberg, Communications of the ACM.

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## Introduction:

This paper aims to estimate the buffer sizes in a message switching computer system such that the probability of a system overload because of a shortage of buffers is ‘sufficiently’ small. A description of the message-switching system, including the read/write operations is given and then waiting times for the different message types are calculated using queuing probabilities and Tchebycheff’s inequality. Next, buffer requirements are established, confirming a mean time before failure due to buffer overload of about 14 years-a more than satisfactory resolution.

## Description of the message-switching system:

Messages are generated either by subscribers or by the computer for transmission to subscribers. The peak arrival rates for these 2 types of messages are 7.92 and 5.28 messages per second respectively. Every message generates two transactions: writing the message on disk when it is received and later reading the message for delivery to the final destination. Thus, in addition to the 7.92 + 5.28 = 13.20 messages written per second, an equal number of messages written earlier will be read each second, resulting in an overall transaction rate of 26.40 messages per second.

When the message switch detects a valid message start sequence from a subscriber station, it allocates a buffer from the message buffer pool and places the message characters in that buffer. Buffers are 128 characters in length, the first 8 characters of which are for buffer-pool management. In the first buffer, 20 characters are reserved for the queue entry: therefore, the first buffer will contain 100 message characters, and subsequent buffers 120 characters each. Examination of system traffic logs over a two-day period yielded the message distribution shown in the first five lines of Table I. Type A messages are from subscribers to the computer; type B, consisting of sub-classes Bl, B2, and B3, are from subscriber to subscriber; and type C are from computer to subscriber. The number of characters per message determines the number of disk accesses; every 580 characters requires one access.The various message characteristics are listed below in Table 1.

**TABLE I. Message Characteristics**

| Message type | Percent of total | Arrival rate per second | Length in characters | Disk accesses |
|---|---|---|---|---|
| A | 15.000 | 3.960 | <80 | 1 |
| B1 | 14.250 | 3.762 | 100–220 | 1 |
| B2 | 0.675 | 0.178 | 700–820 | 2 |
| B3 | 0.075 | 0.020 | 1540–1660 | 4 |
| C | 20.000 | 5.280 | 100–220 | 1 |
| a | 15.000 | 3.960 | 80 | 1 |
| b1 | 14.250 | 3.762 | 100–220 | 2 |
| b2 | 0.675 | 0.178 | 700–820 | 7 |
| b3 | 0.075 | 0.020 | 1540–1660 | 14 |
| c | 20.000 | 5.280 | 100–220 | 2 |
| | | 26.400 | | |

**Write Operation**: The write operation consists of various operations like set-up, latency and seek; and it takes 21.11 ms on average. This is a sum of the various times: set-up time of 2 ms, “Latency” of 11.11 ms and seek time of 8 ms. Each message type has its own characteristic pattern of buffer usage and service as shown in Figure 1: The horizontal lines represent buffers used, the number above the buffer represents the time (in seconds) the buffer is held, and W refers to the write operation. Type **A** messages each utilize a single buffer for 9.5 seconds, and then a write operation occurs. Type **B** messages use variable numbers of buffers. Type **C** messages require no buffering and are written immediately.

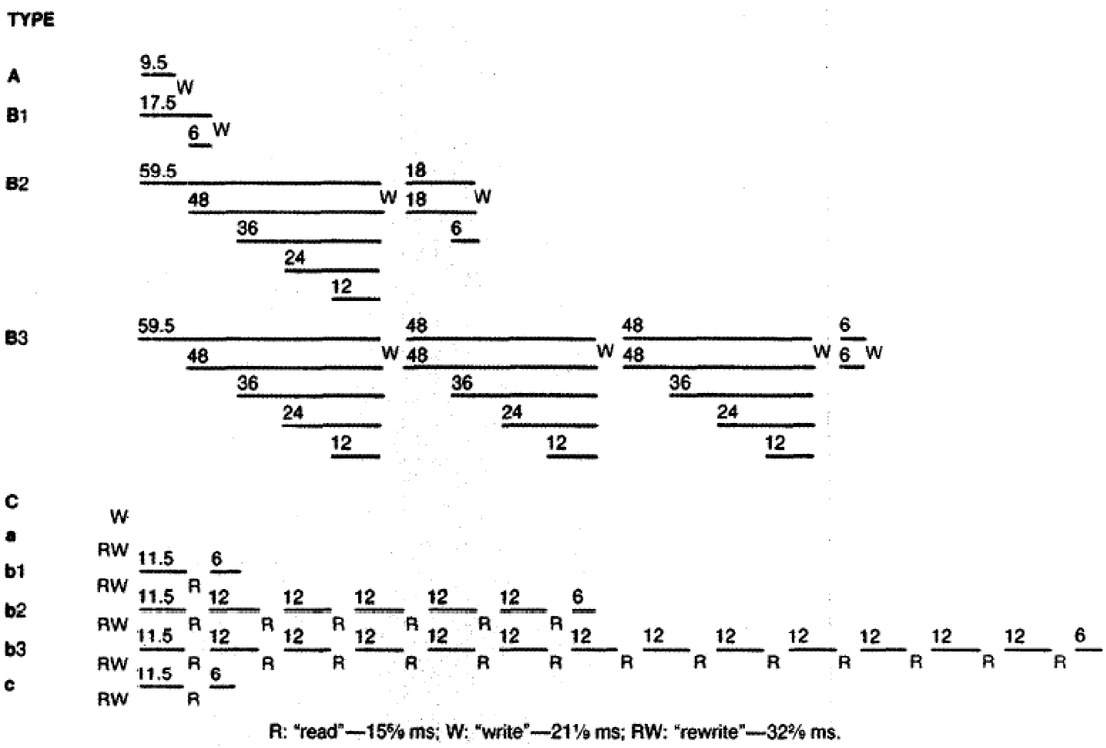


FIGURE 1. Service Characteristics

**Read Operation**: Messages for transmission utilize only a single buffer. A read operation (R) has the same set-up and seek times as a write, 2 ms and 8 ms, respectively. Latency time is 5.55 ms. Thus, the total transmission time is 15.55 ms. Each transmission begins with a rewrite operation (RW) consisting of read, modify, and write back taking 32.22 ms. The characteristics of transmitted messages are shown in the last five lines of Table I. The lowercase message types (e.g., **b1**) are the transmission counterparts of the uppercase message types received earlier, land therefore the percents, arrival rates, and lengths are the same for lowercase message types as their counterparts. The number of disk accesses is equal to the number of buffers required-one buffer for the first 100 characters plus one buffer for each additional 120 characters.These service characteristics are shown in the last five lines of Figure 1.

## Assumptions Made:

- The service times are constant and different for different type of service.
- The peak period arrival rates are used for calculation to get the worst case estimate.
- Arrivals are Poison in nature during peak periods.
- Successive demands for service by the same message are considered to be independent of one another.
- Some results like waiting time distribution etc, are used from single server queue model.

## Waiting Time Calculation

The first two columns of Table II summarize estimated total demands per second for each of the three types of service demands (A, B,and C). The actual service times for each demand are random variables with expectations given by the values in column one, but this analysis treats service times as the constants shown in column one. The probability that a service time is a given

one of the three values equals the demand rate of that value divided by the total demand rate, as given in column three. The probability that a demand for service will encounter some number of other demands ahead of it can be calculated by means of a technique devised by Greenberg[1] and modified by Gross and Harris[2] .
The product of the P and t columns in Table II is headed Pt, and its sum (23.52) is the mean service time.The product of P and the square oft is headed $Pt^2$, and its sum (599.9790) is the second moment of service-time distribution. The variance of the service times is var(service) = $(599.9790) - (23.52)^2 = 46.89$. Thus, in standard queueing notation,
$\lambda = 37.008$ per second = 0.037 per ms,
$l/\mu = 23.52$ ms,
$\sigma^2 = 46.89$ ms',
$\rho = \lambda (1/\mu) = 0.872$.
The probability that a demand for service will find some number n ahead of it (including the demand, if any, currently being serviced) can be calculated sequentially. Calling this probability $p_n$

$p_o = l - \rho$
$p_1 = (1 - \rho)(\lambda/c_0 - 1)$
$p_2 = (1 - \rho)\, \lambda/c_0\, ((\lambda - c_1)/c_0 - 1)$
$p_3 = (1 - \rho)\, \lambda/c_0\, (((\lambda - c_1)/c_0 ((\lambda - c_1)/c_0 - 1) - 1) - c_2/2!c_0)$

$$p_n = \frac{\lambda - c_1}{c_0} p_{n-1} - \frac{\lambda}{c_0} \frac{c_{n-1}}{c_0(n-1)!} (1 - \rho) - \sum_{j=2}^{n-2} \frac{c_j}{c_0 j!} p_{n-j}, \quad n = 4, 5, 6, \cdots$$

where

$$c_j = \sum_t (\lambda t)^j e^{-\lambda t} f(t)$$

t

and f(t) is the number of demands per ms (ie., column two of Table II divided by 1000). The summation is taken over the three t values in Table II. The first 15 probabilities are given in Table III.

**TABLE II. Service-Time Calculations**

| Service time (t) | Demands per second | Probability (P) | (Pt) | ($Pt^2$) | ($Pt^3$) |
|---|---|---|---|---|---|
| 15⅝ ms | 10.370 | 0.2802 | 4.3587 | 67.8015 | 1,054.6897 |
| 21⅛ ms | 13.438 | 0.3631 | 7.6654 | 161.8260 | 3,416.3277 |
| 32⅜ ms | 13.200 | 0.3567 | 11.4937 | 370.3515 | 11,933.5476 |
| | 37.008 | | 23.5178 | 599.9790 | 16,404.5650 |

**TABLE III. Steady-State Queueing Probabilities**

| | | |
|---|---|---|
| $p_0 = 0.130$ | $p_5 = 0.073$ | $p_{10} = 0.019$ |
| $p_1 = 0.170$ | $p_6 = 0.056$ | $p_{11} = 0.014$ |
| $p_2 = 0.152$ | $p_7 = 0.043$ | $p_{12} = 0.011$ |
| $p_3 = 0.121$ | $p_8 = 0.033$ | $p_{13} = 0.008$ |
| $p_4 = 0.094$ | $p_9 = 0.025$ | $p_{14} = 0.006$ |

The mean waiting time for any queue with Poisson arrivals is [3]:
$W = (\lambda^2 \sigma^2 + \rho^2)/2\lambda(1 - \rho)$
Using the service-time values calculated earlier W= 87 ms. Riordan[4] has shown that, for single server queues with Poisson arrivals, the variance of the waiting-time distribution is:
$Var(wait) = W^2 + \lambda v_3/3(1 - \rho)$

where $v_3$ is thta third moment of the service-time distribution, or the sum of the $Pt^3$ in Table II. Performing the calculations using Riordan's formula yields a standard deviation of 94.3 ms for waiting times. Tchebycheff's inequality is thus
Prob(87 - 94.3K < waiting time < 87 + 94.3K)
For K = √10, the 90 percent bound is obtained, and for K = √20, the 95 percent bound is obtained. Thus,
Prob(waiting time < 3841) > 0.90;
Prob(waiting time < 5071) > 0.95.

## Buffer Requirements:

Assuming that arrivals of each type of message are Poisson, the expected number of buffers in use at any instant is $m = \sum_{j=1}^{5} \sum_{k} j\lambda_k t_{jk}$ where $\lambda_k$ is the arrival rate for type k messages (column three of Table I). The variance is

$$\mathrm{Var(buffers)} = \sum_{j=1}^{5} \sum_{k} j^2 \lambda_k t_{jk}.$$

The $t_{jk}$ for the different types of messages are based on the service characteristic values given in Figure 1. The number B of buffers required such that there will be an overload only one out of every N messages on the average can be calculated from

$$\frac{B - m}{\sqrt{\mathrm{Var(buffers)}}} = \frac{B - 365}{\sqrt{543}} = z(1/N)$$

where z(I/N) is the standard normal variable such that I/N of the area lies to its right. Values of B for given values of N are given in the table below.

| N | B | N | B | N | B |
|---|---|---|---|---|---|
| $10^4$ | 452 | $10^6$ | 476 | $10^8$ | 496 |
| $10^5$ | 464 | $10^7$ | 486 | $10^9$ | 505 |

The specific design under consideration provides 512 buffers:
$z(1/512) = 6.6 \times 10^{-9}$
or one overload every $1.5 \times 10^8$ requests, During one second, there are 32.858 buffer requests. Thus, during the daily 15-minute peak period, there will be 32.858 X 60 X 15
= 29,572.2 buffer requests. The one failure per $1.5 \times 10^8$ requests is equivalent to approximately one failure in 5,000 15-minute peaks-a mean time between failure due to buffer overload of about 14 years.

## Conclusions:

Based on the assumptions listed above a system was designed with 512 buffers, and an estimate mean time between failure due to buffer overload of about 14 years.
This system was supposedly implemented and was seen to give satisfactory results.

## References:

1. Greenberg, I. Distribution-free analysis of M/G/l and G/M/l queues. Oper. Res. 21, 2 (Mar.-Apr. 1973), 629-635. A solution to queueing problems when the arrival-time or service-time distribution consists of individual mass points.
2. Gross, D.. and Harris, C. Fundnmentals of @wing Theory. John Wiley and Sons, New York, 1974. One of the best-known comprehensive texts on queueing.
3. Riordan. J. Stochastic Service Systems. John Wiley and Sons, New York, 1962. One of the early books on queueing oriented toward communication delays.