# MA 402 : Queueing Models for Performance Analysis

(Assignment)

Paper considered: "Approximate solution for multi-server queueing systems with Erlangian service times" Marcos Escobara, Amedeo R. Odonib and Emily Roth

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## Introduction

Multi-server queueing systems with Poisson arrivals and Erlangian service times ($M/E_k/n$ and variations) are among the most applicable of what are considered "easy" systems in queueing theory. The Erlangian family, $E_k$, of distributions encompasses service times ranging from the negative exponential ($k$=1) to constant (for infinite $k$), while the opportunity to set the number of parallel and independent servers, $n$, to an appropriate integer value adds flexibility to the model. Moreover, by selecting the proper value of $k$, Erlangian service times can be used to approximate reasonably well many general types of service times which have a unimodal distribution and a coefficient of variation less than or equal to 1.

In view of their practical importance, it may be surprising that the existing literature on $M/E_k/n$ systems is quite sparse. The probable reason is that, while it is indeed possible to represent these systems through a Markov process (hence the characterization as "easy"), serious difficulties arise because of

(1) the very large number of system states that may be present with increasing Erlang order, $k$, and increasing number of servers, $n$, and

(2) the complex state transition probabilities that one has to consider .

This makes it difficult to compute numerical solutions to systems with even modest values of $k$ and $n$—while obtaining general closed-form expressions seems intractable. The practitioner is thus left with little to go on.

This paper discuss the exact approach to the solution of $M/E_k/n/n{+}q$ systems. This is necessary in order, first, to explain the complications mentioned above and, second, to introduce state descriptions of these systems. the "traditional" description is more natural And also heuristic approach

## The $M(t)/E_k(t)/n$ and $M(t)/E_k(t)/n/n{+}q$ queueing systems

It has beendescribed  in detail how to obtain exact solutions for the $M(t)/E_k(t)/n$ and $M(t)/E_k(t)/n/n{+}q$ queueing systems .they started by reviewing the method of stages used to represent the Erlang distribution in the $M(t)/E_k(t)/1$ system to enable solution of such systems. Then, we extend this approach to the case of multiple servers with limited or unlimited queue size. Note that all the results in this section apply under both stationary and non-stationary conditions, unless otherwise specified.

## State description for single-server systems

Each customer that enters the system can be considered as a *package* of $k$ consecutive, exponentially distributed tasks to be performed by the service facility. The service rate for each stage is $k\mu(t)$, with a corresponding expected time of $1/k\mu(t)$ per stage. The service rate for completing all stages is $\mu(t)$ with a corresponding expected service time of $1/\mu(t)$. The usefulness of this approach is that we can derive a state transition diagram with independent, exponentially distributed transitions that completely describes the queue. Due to the memoryless property of Poisson processes, in any time increment $\delta t$, the state can change only as indicated in the diagram. In the $M(t)/E_k(t)/1$ system, the states are defined fully by the total number of stages (or tasks) remaining in the system to be completed for all customers

## State description for multi-server systems

f the system has multiple servers, the total number of stages is not sufficient to define the state of the system because, for a particular number of stages, the distribution of such stages among the servers may not be unique. Instead, the state must identify: the number of uncompleted stages in the system; the number of customers in service and in the queue; and the distribution of uncompleted stages among the busy servers.

We use two different, but equivalent, state representations in this work. The first, proposed recently is a $(k+1)$-tuple state description of the form $(a_k, a_{k-1}, \ldots, a_1, a_q)$ where $a_i$ indicates the number of servers with $i$ stages remaining to be completed, for $1 \leqslant i \leqslant k$, and $a_q$ indicates the number of customers in the queue, waiting for service. We shall refer to this as Description 1.

Using Description 1, the number $m$ of customers in the system is given by

$$m = \begin{cases} \sum_{i=1}^{k} a_i, & a_q = 0 \\ n + a_q, & a_q > 0 \end{cases}$$

and the total number of stages left in the system, $l$, is

$$l = \sum_{i=1}^{k} a_i i + a_q k.$$

## Exact solution technique

In this section, using Description 1 , we write the equations needed to obtain the state probabilities of the M(t)/Ek(t)/n and M(t)/Ek(t)/n/n+q systems. Let S0 be the array containing the state probabilities when m<n, and let S0(ak…,a1) be the probability of state (ak,…,a1,0) for which ?i=1kai<n. Let Q0 be the state probability array of states (ak,…,a1,0) when ?i=1kai=n, and m=n, with the state probabilities specified by Q0(ak,…,a1).

Similarly, let Qs be the state probability arrays for the case in which aq>0, and let Qs(ak,…,a1) be the state probabilities of state (ak,…,a1,s) when there are s customers waiting for service in the queue. The total number of customers in the system in this case is n+s. If the system has infinite queue size, then 1less-than-or-equals, slants<?. If the queue size is limited, 1less-than-or-equals, slantsless-than-or-equals, slantq, where q is the maximum number of customers that can wait for service and the array Qq has the state probabilities Qq(ak,…,q1) when there are n+q customers in the system.

### *Total number of states*

The total number of states in the system that need to be considered is given by

$$T_S = \binom{n+k}{n} + q\binom{n+k-1}{n}$$

Note that the first term indicates the number of states when the queue is empty, and the second term indicates the number of states for the customers waiting in the queue

## State-to-state transitions

A transition between states occurs due to a stage completion (rate $k\mu(t)$) or to an arrival of a new customer to the system (rate $\lambda(t)$). Table 2 shows the transitions for each type of state and their corresponding state-to-state transition rates. These state transitions lead directly to the Chapman–Kolmogorov equations for the $M(t)/E_k(t)/n$ and $M(t)/E_k(t)/n/n+q$ systems.

Table 2. State-to-state transitions for the exact solution technique

| From state | In | To state | In | With rate |
|---|---|---|---|---|
| $(a_k,\ldots,a_1,0)$ | $S_0$ | $(a_k+1,a_{k-1},\ldots,a_1,0)$ | $S_0$ | $\lambda(t)$ if $m < n-1$ |
| | | $(a_k+1,a_{k-1},\ldots,a_1,0)$ | $Q_0$ | $\lambda(t)$ if $m = n-1$ |
| | | $(a_k-1,a_{k-1}+1,a_{k-2},\ldots,a_1,0)$ | $S_0$ | $a_k k\mu(t)$ |
| | | $(a_k,a_{k-1}-1,a_{k-2}+1,\ldots,a_1,0)$ | $S_0$ | $a_{k-1}k\mu(t)$ |
| | | $(a_k,\ldots,a_i-1,a_{i-1}+1,\ldots,a_1,0)$ | $S_0$ | $a_i k\mu(t)$ for $i=k\ldots2$ |
| | | $(a_k,a_{k-1},\ldots,a_1-1,0)$ | $S_0$ | $a_1 k\mu(t)$ |
| $(a_k,\ldots,a_1,0)$ | $Q_0$ | $(a_k,\ldots,a_1,1)$ | $Q_1$ | $\lambda(t)$ |
| | | $(a_k,\ldots,a_i-1,a_{i-1}+1,\ldots,a_1,0)$ | $Q_0$ | $a_i k\mu(t)$ for $i=k\ldots2$ |
| | | $(a_k,a_{k-1},\ldots,a_1-1,0)$ | $S_0$ | $a_1 k\mu(t)$ |
| $(a_k,\ldots,a_1,s)$ $(1 \leqslant s < q)$ | $Q_s$ | $(a_k,\ldots,a_1,s+1)$ | $Q_{s+1}$ | $\lambda(t)$ |
| | | $(a_k,\ldots,a_i-1,a_{i-1}+1,\ldots,a_1,s)$ | $Q_s$ | $a_i k\mu(t)$ for $i=k\ldots2$ |
| | | $(a_k+1,a_{k-1},\ldots,a_1-1,s-1)$ | $Q_{s-1}$ | $a_1 k\mu(t)$ |
| $(a_k,\ldots,a_1,q)$ $(q < \infty)$ | $Q_q$ | $(a_k,\ldots,a_i-1,a_{i-1}+1,\ldots,a_1,q)$ | $Q_q$ | $a_i k\mu(t)$ for $i=k\ldots2$ |
| | | $(a_k+1,a_{k-1},\ldots,a_1-1,q-1)$ | $Q_{q-1}$ | $a_1 k\mu(t)$ |

## Conclusion

In this paper they also proposed heuristic to reduce the number of equations to be solved by combining each collection of states $(l,m,r)$ into a single state $(l,m)$ but I have studied only exact solution technique. They have shown that heuristic approximation scheme provides excellent, for most practical purposes, approximations to the typical quantities of interest; increases considerably the size of systems that can be solved with reasonable computational effort, as a result of the greatly reduced number of states required by the approximate model; and speeds up the solution of some large systems by a factor of 1000 or more.

Link to the paper http://dspace.mit.edu/handle/1721.1/9631