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Holonomy Decomposition of Seminearrings

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Abstract. This work extends the Holcombe's holonomy decomposition of near-rings to seminearrings employing the techniques of Eilenberg for studying the structure of transformation semigroups. This work investigates structural properties of certain types of seminearrings.

Keywords: Seminearring; Transformation semigroup; Holonomy decomposition.

1. Introduction and Preliminaries

Eilenberg's holonomy decomposition theorem for transformation semigroups is a sophisticated version of Krohn-Rhodes decomposition theorem of automata [1]. The work of [5], which investigates the role of a broader class of seminearrings in certain types of automata, motivated us to think about the holonomy decomposition of seminearrings. In [2] Holcombe studies the structure of a class of 2-primitive near-rings using Eilenberg's techniques for decomposition of transformation semigroups. The current work extends the results of Holcombe to seminearrings. The later part of this section recalls some fundamental concepts of transformation semigroups from Eilenberg's work. For further details of transformation semigroups one may refer [1]. In Section 2, we introduce the seminearring $\mathfrak{M}_{G^0}(\Gamma)$ and study its skeleton. Section 3 deals with the main result of the work, holonomy decomposition of $\mathfrak{M}_{G^0}(\Gamma)$.

A *seminearring* is an algebraic structure with two binary operations $(S, +, \cdot)$

such that

1. $(S, +)$ is a semigroup with identity 0,
2. $(S, \cdot)$ is a semigroup,
3. $(x + y)z = xz + yz \ \forall x, y, z \in S$, and
4. $0s = s0 = 0 \ \forall s \in S$.

A semigroup $(\Gamma, +)$ with identity 0_Γ is said to be an *S-semigroup* if there exists a composition $(x, \gamma) \mapsto x\gamma$ of $S \times \Gamma \longrightarrow \Gamma$ such that

1. $(x + y)\gamma = x\gamma + y\gamma$,
2. $(xy)\gamma = x(y\gamma)$, and
3. $0\gamma = 0_\Gamma \quad$ for all $x, y \in S, \gamma \in \Gamma$.

Note that the semigroup $(S, +)$ of a seminearring $(S, +, \cdot)$ is an S-semigroup. One may refer [3, 4, 6, 7] for fundamentals and details on seminearrings and S-semigroups.

A pair (P, S) with a nonempty set P and a semigroup with identity S is called a *transformation semigroup* if there is an embedding $\phi : S \hookrightarrow \mathfrak{M}(P)$, where $\mathfrak{M}(P)$ is the semigroup of all mappings on P with respect to composition. Let us denote the action of $s \in S$ on $p \in P$ as ps, rather than $p\phi(s)$. For $p \in P$, let $\bar{p}$ be the constant function on P which takes the value p, i.e. $\bar{p}(q) = p, \forall q \in P$. The *closure* of a transformation semigroup (P, S) is defined as $\overline{(P, S)} = (P, \bar{S})$, where $\bar{S}$ is the semigroup generated by $S \cup \bigcup_{p \in P} \{\bar{p}\}$. Associated with (P, S), $\mathscr{I}^0$ is the space of all s-images of P, where s-image of P, denoted by Ps, is $\{ps \mid p \in P\}$. The *skeleton space* $\mathscr{I}$ of (P, S) is

$$\mathscr{I}^0 \cup \{P\} \cup \bigcup_{p \in P} \{\{p\}\}$$

with the preorder $\leq$ defined by: for $A, B \in \mathscr{I}$, $A \leq B$ if and only if $A \subseteq Bs$ for some $s \in S$. Define an equivalence relation $\sim$ on $\mathscr{I}$ by putting $A \sim B$ if and only if $A \leq B$ and $B \leq A$. For $A \in \mathscr{I}^0$ write $K(A)$ to denote the set of elements of S that behave as units on A, i.e.

$$K(A) = \{f \in S \mid \exists g \in S \text{ with } f(A) = A \text{ and } fg(a) = gf(a) = a, \ \forall a \in A\}.$$

Define *paving of* A, denoted by $B(A)$, to be the set of maximal elements (with respect to set inclusion) of $\mathscr{I}$ that are contained in A, i.e.

$$\{B \in \mathscr{I} \mid B \subseteq A \text{ and if } C \in \mathscr{I} \text{ with } B \subseteq C \subseteq A \text{ then } C = B \text{ or } C = A\}.$$

Each $s \in K(A)$ acts as a permutation on $B(A)$ and the set $\mathscr{G}(A)$ of the distinct permutations of $B(A)$ induced by the elements of $K(A)$ is called the *holonomy* *group* of A. The group $\mathscr{G}(A)$ acts as a transformation group on the paving of A, $B(A)$.

Due to Eilenberg, any transformation semigroup of finite height can be covered by a wreath product of holonomy transformation groups [1]. More precisely,

Theorem (Holonomy decomposition of transformation semigroups) *If* (P, S) *is a transformation semigroup of finite height and* $h : \mathscr{J} \longrightarrow \mathbb{Z}$ *is a height function then*

$$(P, S) \prec \bar{\mathscr{H}_n} \circ \bar{\mathscr{H}_{n-1}} \circ \ldots \circ \bar{\mathscr{H}_1}$$

where $n = h(P)$ *and*

$$\mathscr{H}_i = \left(\prod_{j \in J} B(A_{ij}), \prod_{j \in J} \mathscr{G}(A_{ij}) \right)$$

in which $\{A_{ij} \mid j \in J\}$ *is the set representatives of equivalence classes (with respect to* $\sim$*) in* $\mathscr{J}(i)$*, the set of elements of* $\mathscr{J}$ *of height* i*.*

2. $\mathfrak{M}_{G^0}(\Gamma)$ and Its Skeleton

Let S be a seminearring. Let $\{\Gamma_i\}_{i \in \Delta}$ be a nonempty family of S- subsemigroups of an S-semigroup Γ such that $\gamma_i + \gamma_j = \gamma_j + \gamma_i$, where $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ for all $i, j \in \Delta$ with $i \neq j$. Then the set

$$\left\{ \sum_{i \in \Delta} \gamma_i \mid \gamma_i \in \Gamma_i \ \forall i \in \Delta \text{ and only finitely many of the } \gamma_i\text{'s are nonzero} \right\}$$

is said to be *direct sum* of $\{\Gamma_i\}_{i \in \Delta}$, denoted by $\bigoplus_{i \in \Delta} \Gamma_i$, if every element has unique representation.

Remark 2.1. For any family of S-subsemigroups $\{\Gamma_i\}_{i \in \Delta}$ of an S-semigroup Γ, $\bigoplus_{i \in \Delta} \Gamma_i$ is an S-semigroup.

Let $(\Gamma, +)$ be a semigroup with zero 0_Γ and G a group of automorphisms on Γ. Then

$$\mathfrak{M}_{G^0}(\Gamma) = \{f \in \mathfrak{M}_0(\Gamma) \mid fg = gf \ \forall g \in G\},$$

the set of mappings on Γ each of which fixes zero and commutes with every element of G is a seminearring with unity with respect to pointwise addition and composition of mappings. In what follows S denotes the seminearring $\mathfrak{M}_{G^0}(\Gamma)$.

Observe that (S, S) is a transformation semigroup with the action of right multiplication of S. Here S can be embedded in $\mathfrak{M}(S)$ by the assignment $f \mapsto F_f$, where $F_f : S \longrightarrow S$ is given by $F_f(h) = fh$ for all $h \in S$. The skeleton space, $(\mathscr{J}, \leq)$, of this transformation semigroup consists of all the principal right S-semigroups of S of the form Sa, for $a \in S$.

Eilenberg ascertained holonomy decomposition for transformation semigroups of finite height. Accordingly, to find such a decomposition for the seminearring S, we assume the number of orbits of Γ is finite when G acts on Γ. Let n be the number of nonzero orbits of Γ. Let us use $\mathbf{k}$ to denote the set of first k natural numbers $\{1, 2, \ldots, k\}$, whereas $\{0, 1, 2, \ldots, k\}$ will be denoted by $\mathbf{k}^0$.

Theorem 2.2. *Let $\{\gamma_i\}_{i\in\mathbf{n}}$ be a set of representatives of all the distinct nonzero orbits of Γ. Then there exist idempotents $e_i \in S$, for all $i \in \mathbf{n}$, such that*

$$S = \bigoplus_{i\in\mathbf{n}} Se_i.$$

Proof. Given $\Gamma = (0_\Gamma) \cup \left(\bigcup_{i\in\mathbf{n}} G\gamma_i \right)$. For $i \in \mathbf{n}$, define $e_i : \Gamma \longrightarrow \Gamma$ by

$$e_i(\gamma) = \begin{cases} \gamma, & \text{if } \gamma \in G\gamma_i; \\ 0_\Gamma, & \text{otherwise.} \end{cases}$$

Then e_i fixes zero and commutes with every element of G so that $e_i \in S$ for all $i \in \mathbf{n}$. Moreover e_i is an idempotent. It is clear that the elements of Se_i and Se_j commutes with each other for $i \neq j$. Now for every $a \in S$, $a = \sum_{i\in\mathbf{n}} ae_i$, so that $a \in \bigoplus_{i\in\mathbf{n}} Se_i$. Suppose there is another representation, say $\sum_{i\in\mathbf{n}} a'e_i$, for a. We claim that $ae_i = a'e_i$ for all i. For every $\gamma \in \Gamma$, if $\gamma \in G\gamma_k$ then

$$\left(\sum_{i\in\mathbf{n}} ae_i \right)(\gamma) = \left(\sum_{i\in\mathbf{n}} a'e_i \right)(\gamma) \Longrightarrow ae_k(\gamma) = a'e_k(\gamma);$$

otherwise, i.e. $\gamma \notin G\gamma_k$, $ae_k(\gamma) = 0_\Gamma = a'e_k(\gamma)$ so that $ae_k = a'e_k$. Hence $S = \bigoplus_{i\in\mathbf{n}} Se_i$. ∎

Remark 2.3. Since $e_ie_j = 0$ for all $i, j \in \mathbf{n}$ with $i \neq j$, $\{e_i\}_{i\in\mathbf{n}}$ forms a set of orthogonal idempotents.

Theorem 2.4. *For $m \in \mathbf{n}$, there exists $a \in S$ such that $\bigoplus_{i\in\mathbf{m}} (Se_{j_i}) = Sa$.*

Proof. We prove that $\sum_{i\in\mathbf{m}} e_{j_i} \in S$ serves our purpose of a. Since $\bigoplus_{i\in\mathbf{m}} (Se_{j_i})$ is an S-semigroup and $\sum_{i\in\mathbf{m}} e_{j_i} \in \bigoplus_{i\in\mathbf{m}} (Se_{j_i})$ we have $S \sum_{i\in\mathbf{m}} e_{j_i} \subseteq \bigoplus_{i\in\mathbf{m}} (Se_{j_i})$. Conversely, suppose $\sum_{i\in\mathbf{m}} a_{j_i}e_{j_i} \in \bigoplus_{i\in\mathbf{m}}(Se_{j_i})$, where $a_{j_i} \in S$. For each $i \in \mathbf{m}$, define elements $b_{j_i} \in S$ by

$$b_{j_i}(\gamma) = \begin{cases} a_{j_i}(\gamma), & \text{if } \gamma \in G\gamma_{j_i}; \\ 0_\Gamma, & \text{otherwise.} \end{cases}$$

For $\gamma \in \Gamma$, if $\gamma \in G\gamma_{j_k}$ for some $k \in \mathbf{m}$ then

$$\left(\sum_{i\in\mathbf{m}} b_{j_i} \sum_{i\in\mathbf{m}} e_{j_i}\right)(\gamma) = a_{j_k}(\gamma) = \left(\sum_{i\in\mathbf{m}} a_{j_i}e_{j_i}\right)(\gamma).$$

Otherwise

$$\left(\sum_{i\in\mathbf{m}} b_{j_i} \sum_{i\in\mathbf{m}} e_{j_i}\right)(\gamma) = 0_\Gamma = \left(\sum_{i\in\mathbf{m}} a_{j_i}e_{j_i}\right)(\gamma).$$

Therefore $\sum_{i\in\mathbf{m}} a_{j_i}e_{j_i} = \sum_{i\in\mathbf{m}} b_{j_i} \sum_{i\in\mathbf{m}} e_{j_i} \in S\sum_{i\in\mathbf{m}} e_{j_i}$. Hence $\bigoplus_{i\in\mathbf{m}}(Se_{j_i}) = S\sum_{i\in\mathbf{m}} e_{j_i}$, as desired. ■

Corollary 2.5. *For $m \in \mathbf{n}$, $\bigoplus_{i\in\mathbf{m}}(Se_{j_i}) \in \mathscr{J}$.*

Theorem 2.6. *If $m \in \mathbf{n}$, then*

$$S\sum_{i\in\mathbf{m}} e_{j_i} \sim S\sum_{i\in\mathbf{m}} e_{k_i}.$$

Proof. For $p, q \in \mathbf{n}$, define $f_{pq} : \Gamma \longrightarrow \Gamma$ by

$$f_{pq}(\gamma) = \begin{cases} g(\gamma_p), & \text{if } \gamma = g(\gamma_q) \text{ for some } g \in G; \\ 0_\Gamma, & \text{otherwise.} \end{cases}$$

Then f_{pq} fixes 0_Γ and commutes with every element of G, so that $f_{pq} \in S$. Set

$$y = (\sum_{i\in\mathbf{m}} f_{j_ik_i})(\sum_{i\in\mathbf{m}} e_{k_i})(\sum_{i\in\mathbf{m}} f_{k_ij_i}).$$

Then

$$y(\gamma) = \begin{cases} \gamma, & \text{if } \gamma = g(\gamma_{j_i}) \text{ for some } i \in \mathbf{m} \text{ and } g \in G; \\ 0_\Gamma, & \text{otherwise.} \end{cases}$$

Thus $y = \sum_{i\in\mathbf{m}} e_{j_i}$ and hence

$$S(\sum_{i\in\mathbf{m}} e_{j_i}) \subseteq S(\sum_{i\in\mathbf{m}} e_{k_i})(\sum_{i\in\mathbf{m}} f_{k_ij_i}).$$

So $S(\sum_{i\in\mathbf{m}} e_{j_i}) \leq S(\sum_{i\in\mathbf{m}} e_{k_i})$. By symmetry we can conclude that

$$S(\sum_{i\in\mathbf{m}} e_{j_i}) \sim S(\sum_{i\in\mathbf{m}} e_{k_i}).$$

■

Theorem 2.7. *For $a(\neq 0) \in S$, let $r \in \mathbf{n}$ be the number of distinct nonzero orbits of $a\Gamma$ and $\{\gamma_{l_i}\}_{i\in\mathbf{r}}$ the set of their representatives. Then*

$$Sa \sim S\bar{e}_r$$

where $\bar{e}_r = \sum_{i\in\mathbf{r}} e_{l_i}$.

Proof. Let $a(\neq 0) \in S$. Note that $\bar{e}_r a = a$. Indeed, if $\gamma \in \Gamma$ then $a(\gamma) = g(\gamma_{l_i})$ for some $i \in \mathbf{r}$ and $g \in G$. Then

$$\bar{e}_r a(\gamma) = \bar{e}_r g(\gamma_{l_i}) = g\bar{e}_r(\gamma_{l_i}) = g(\gamma_{l_i}) = a(\gamma).$$

Hence $Sa \subseteq S\bar{e}_r a$ and so $Sa \leq S\bar{e}_r$. For each $i \in \mathbf{r}$, let $a(\gamma_{l_i}) = g_i(\gamma_{l_{j_i}})$, where $g_i \in G$. Define a function $a' : \Gamma \longrightarrow \Gamma$ by

$$a'(\gamma) = \begin{cases} g(\gamma_{l_i}), & \text{if } \gamma = g(\gamma_{l_{j_i}}) \text{ for some } g \in G \text{ and } i \in \mathbf{r}; \\ 0_\Gamma, & \text{otherwise.} \end{cases}$$

Then a' fixes 0_Γ and commutes with every element of G so that $a' \in S$. Also for each $i \in \mathbf{r}$ define a function $s_i : \Gamma \longrightarrow \Gamma$ by

$$s_i(\gamma) = \begin{cases} gg_i^{-1}(\gamma_{l_{j_i}}), & \text{if } \gamma = g\gamma_{l_{j_i}} \text{ for some } g \in G; \\ 0_\Gamma, & \text{otherwise.} \end{cases}$$

Let $g \in G$ be arbitrary. For $\gamma \in \Gamma$, if $\gamma = g'(\gamma_{l_{j_i}})$ for some $g' \in G$, then

$$s_i g(\gamma) = s_i gg'(\gamma_{l_{j_i}}) = gg'g_i^{-1}(\gamma_{l_{j_i}}) = gs_i(g'(\gamma_{l_{j_i}})) = gs_i(\gamma),$$

otherwise, i.e. $\gamma \notin G\gamma_{l_{j_i}}$,

$$s_i g(\gamma) = 0_\Gamma = gs_i(\gamma).$$

Thus for each $i \in \mathbf{r}$, $s_i \in S$, as s_i fixes 0_Γ. We show that $(\sum_{i\in\mathbf{r}} s_i)aa' = \bar{e}_r$ so that $S\bar{e}_r \subseteq Saa'$, which in turn gives the reverse inequality $S\bar{e}_r \leq Sa$. For $\gamma \in \Gamma$, if $\gamma = g(\gamma_{l_{j_i}})$ for some $g \in G$ and $i \in \mathbf{r}$, then

$$\begin{aligned}(\sum_{i\in\mathbf{r}} s_i)aa'(\gamma) = (\sum_{i\in\mathbf{r}} s_i)ag(\gamma_{l_i}) &= (\sum_{i\in\mathbf{r}} s_i)ga(\gamma_{l_i}) \\ &= (\sum_{i\in\mathbf{r}} s_i)gg_i(\gamma_{l_{j_i}}) = gg_ig_i^{-1}(\gamma_{l_{j_i}}) = \gamma.\end{aligned}$$

Otherwise, for $\gamma \in \Gamma \setminus \bigcup_{i\in\mathbf{r}} G\gamma_{l_{j_i}}$, $(\sum_{i\in\mathbf{r}} s_i)aa'(\gamma) = 0_\Gamma$. Thus $(\sum_{i\in\mathbf{r}} s_i)aa' = \bar{e}_r$. Hence $S\bar{e}_r \sim Sa$. ∎

Corollary 2.8. $Sa \sim S\sum_{i\in\mathbf{r}} e_i$.

Corollary 2.9. *There exists a height function $h : \mathscr{J} \longrightarrow \mathbb{Z}$ such that the height of S is equals to the number of nonzero orbits of Γ under the action of G. Moreover, $h(Sa)$ is the number of nonzero orbits of $a\Gamma$.*

In the sequel, $\bar{e}_r$ always denotes the sum $\sum_{i\in\mathbf{r}} e_i$.

Corollary 2.10. *For $r \in \mathbf{n}$, there is only one equivalence class of height r, which is precisely the equivalence class containing $S\bar{e}_r$.*

3. Main Result

In order to describe the structure of $\mathfrak{M}_{G^0}(\Gamma)$ through holonomy decomposition, we need to study the properties of paving $B(S\bar{e}_r)$ and holonomy group $\mathscr{G}(S\bar{e}_r)$ of $S\bar{e}_r$. The following lemmas describe these properties to consolidate the structure of $\mathfrak{M}_{G^0}(\Gamma)$ in the main result, Theorem 3.5.

Lemma 3.1. *$Sa \in B(S\bar{e}_r)$ if and only if a can be represented by a pair (f, h) with $f : \mathbf{r}^0 \longrightarrow \mathbf{r}^0$, $h : \mathbf{r} \longrightarrow G$ such that $a(\gamma_i) = h(i)(\gamma_{f(i)})$ and $|f(\mathbf{r}^0)| = r$, where $f(0) = 0$ and $\gamma_0 = 0_\Gamma$.*

Proof. Suppose $Sa \in B(S\bar{e}_r)$. Then since $Sa \subsetneq S\bar{e}_r$, we have $a \in S\bar{e}_r$ and consequently $a(\gamma_i) = 0$ for all $i \in \mathbf{n} \setminus \mathbf{r}$. Also since Sa is maximal in $S\bar{e}_r$, either $a(\gamma_i)$ is nonzero for all $i \in \mathbf{r}$ or $a(\gamma_i) = 0$ for only one $i \in \mathbf{r}$. Accordingly the following cases arise:

1. There exist $i, j \in \mathbf{r}$ such that $a(\gamma_i)$ and $a(\gamma_j)$ are in the same orbit and the remaining $a(\gamma_k)$ are in distinct orbits.

2. $a(\gamma_i) = 0$ for one $i \in \mathbf{r}$ and the remaining $a(\gamma_k)$ are in distinct orbits.

Thus a can be represented by a pair (f, h) where $f : \mathbf{r}^0 \longrightarrow \mathbf{r}^0$, $h : \mathbf{r} \longrightarrow G$ so that $a(\gamma_i) = h(i)(\gamma_{f(i)})$, where $f(0) = 0$ and γ_0 is interpreted as 0_Γ. Then in either case $|f(\mathbf{r}^0)| = r$. Converse is straightforward, as any such pair will define an element a such that Sa is maximal in $S\bar{e}_r$. ∎

To find the distinct elements $B(S\bar{e}_r)$, let us define the relation $a \approx b$ if and only if $Sa = Sb$. Then we have the following.

Lemma 3.2. *Let $a = (f, h), b = (f', h')$ as in Lemma 3.1. Then $a \approx b$ if and only if one of the following holds:*

1. *$f(i) = f'(i) = 0$ for some $i \in \mathbf{r}$*
2. *$f(i) = f(j)(\neq 0) \Longleftrightarrow f'(i) = f'(j)(\neq 0)$ and $(h(i))^{-1}h'(i) = (h(j))^{-1}h'(j)$ with value $s(\gamma_{g(i)})$ at $\gamma_{\bar{g}(i)}$ for some $s \in S$, where if $g = f$ then $\bar{g} = f'$ and vice versa.*

Proof. If $a \approx b$ then $Sa = Sb$. Thus $a = sb, b = s'a$ for some $s, s' \in S$ so that

$$a(\gamma_i) = sb(\gamma_i), \quad \text{i.e.} \quad h(i)(\gamma_{f(i)}) = sh'(i)(\gamma_{f'(i)}) = h'(i)s(\gamma_{f'(i)})$$

and

$$b(\gamma_i) = s'a(\gamma_i), \quad \text{i.e.} \quad h'(i)(\gamma_{f'(i)}) = s'h(i)(\gamma_{f(i)}) = h(i)s'(\gamma_{f(i)})$$

for each $i \in \mathbf{r}$. In case $f(i) = 0$ for some $i \in \mathbf{r}$, then

$$\gamma_{f'(i)} = (h'(i))^{-1}h(i)s'(0_\Gamma) = 0_\Gamma,$$

so that $f'(i) = 0$. Otherwise, for $i \neq j$ if $f(i) = f(j)(\neq 0)$ then $s'(\gamma_{f(i)}) = (h(i))^{-1}h'(i)(\gamma_{f'(i)})$ and also $s'(\gamma_{f(i)}) = (h(j))^{-1}h'(j)(\gamma_{f'(j)})$. Thus $f'(i) = f'(j)$ and $(h(i))^{-1}h'(i) = (h(j))^{-1}h'(j)$. Consequently the second choice holds.

Converse is clear from the above calculation for the second choice. In case of first choice, if $f(i) = 0$ for some $i \in \mathbf{r}$ but $f(j) \neq 0$ for $j \in \mathbf{r} \setminus \{i\}$ then $a(\sum_{\substack{j\in\mathbf{r}\\ j\neq i}} e_j) = a$ and so $a \in S\sum_{\substack{j\in\mathbf{r}\\ j\neq i}} e_j$. Thus $Sa \subseteq S\sum_{\substack{j\in\mathbf{r}\\ j\neq i}} e_j$. But since Sa is maximal in $S\bar{e}_r$ we have $Sa = S\sum_{\substack{j\in\mathbf{r}\\ j\neq i}} e_j$. Since $f'(i) = 0$, in a similar way one can obtain $Sb = S\sum_{\substack{j\in\mathbf{r}\\ j\neq i}} e_j$. Hence $a \approx b$. ∎

Lemma 3.3. *$s \in K(S\bar{e}_r)$ if and only if $s(\bar{e}_r\Gamma) = \bar{e}_r\Gamma$ and s is a bijection on $\bar{e}_r\Gamma$.*

Proof. If $s \in K(S\bar{e}_r)$ then there exists $s' \in S$ such that $tss' = ts's = t$ for all $t \in S\bar{e}_r$. In particular, if $t = \bar{e}_r$ then $ss'(\gamma) = s's(\gamma) = \gamma$ for all $\gamma \in \bar{e}_r\Gamma$, so that

$s(\bar{e}_r\Gamma) = \bar{e}_r\Gamma$ and s is a bijection on $\bar{e}_r\Gamma$. For converse, let $s(\gamma_i) = g_i(\gamma_{j_i})$, $i \in \mathbf{r}$. Define $\bar{s} : \Gamma \longrightarrow \Gamma$ by

$$\bar{s}(\gamma) = \begin{cases} gg_i^{-1}(\gamma_i), & \text{if } \gamma = g(\gamma_{j_i}) \text{ for some } g \in G \text{ and } i \in \mathbf{r}; \\ 0_\Gamma, & \text{otherwise.} \end{cases}$$

Then $\bar{s} \in S$ and $\bar{s}s(\gamma) = s\bar{s}(\gamma) = \gamma$ for all $\gamma \in \bar{e}_r\Gamma$. Thus $a\bar{e}_r\bar{s}s(\gamma) = a\bar{e}_r s\bar{s}(\gamma) = a\bar{e}_r(\gamma)$ for all $\gamma \in \Gamma$, so that $s \in K(S\bar{e}_r)$. ∎

Lemma 3.4. *For $r \in \mathbf{n}$, the holonomy group of $S\bar{e}_r$ is*

$$\mathscr{G}(S\bar{e}_r) = \frac{S_r \circ G}{Z(G)}$$

where S_r is the symmetric group of degree r and $Z(G)$ is the center of G.

Proof. For each $r \in \mathbf{n}$, the holonomy group $\mathscr{G}(S\bar{e}_r)$ is the group induced by the elements of $K(S\bar{e}_r)$. Since each $s \in K(S\bar{e}_r)$ is a bijection on $\bar{e}_r\Gamma$ (cf. Lemma 3.3), s can be considered as a pair (α, β), with $\alpha : \mathbf{r} \longrightarrow \mathbf{r}$ and $\beta : \mathbf{r} \longrightarrow G$ defined by $\alpha(i) = j_i$ and $\beta(i) = g_i$, where

$$s(\gamma_i) = g_i(\gamma_{j_i}) \ \ \forall i \in \mathbf{r}.$$

Thus for $\mathscr{G}(S\bar{e}_r)$, we observe the group $S_r \circ G$, wreath product of S_r, the symmetric group of degree r, and G. This is because, in wreath product the action of (α, β) on $\mathbf{r} \times \Gamma$ is given by $(\alpha, \beta)(i, \gamma) = (\alpha(i), \beta(i)(\gamma))$, which can be identified as $\beta(i)(\gamma_{\alpha(i)})$. We define the relation $\equiv$ on $S_r \circ G$ by

$$(\alpha, \beta) \equiv (\alpha', \beta') \ \text{ if and only if } \ Sa(\alpha, \beta) = Sa(\alpha', \beta')$$

for all $Sa \in B(S\bar{e}_r)$. Now let $a = (f, h)$ where $f : \mathbf{r}^0 \longrightarrow \mathbf{r}^0$ with $|f(\mathbf{r}^0)| = r$ and $h : \mathbf{r} \longrightarrow G$. Then it is enough to observe $(f, h)(\alpha, \beta) \approx (f, h)(\alpha', \beta')$ for all (f, h), i.e.

$$(f(\alpha), h(\alpha) \cdot \beta) \approx (f(\alpha'), h(\alpha') \cdot \beta')$$

where $f(\alpha)(i) = f(\alpha(i)), (h(\alpha) \cdot \beta)(i) = h(\alpha(i))\beta(i)$ for each $i \in \mathbf{r}$. We claim that $\alpha = \alpha'$. On the contrary, assume $\alpha(i) \neq \alpha'(i)$ for some $i \in \mathbf{r}$. Then we can find $f : \mathbf{r}^0 \longrightarrow \mathbf{r}^0$ such that $|f(\mathbf{r}^0)| = r$, $f(\alpha(i)) = 0 \neq f(\alpha'(i))$ but $f(\alpha'(j)) = 0$ for some $j \in \mathbf{r}$ with $j \neq i$. Then by Lemma 3.2, $(f(\alpha), h(\alpha) \cdot \beta) \not\approx (f(\alpha'), h(\alpha') \cdot \beta)$, a contradiction. Thus $\alpha = \alpha'$. Now choose any $i, j \in \mathbf{r}$ with $i \neq j$. There exists $f : \mathbf{r}^0 \longrightarrow \mathbf{r}^0$ with $|f(\mathbf{r}^0)| = r$ and $f(i) = f(j) \neq 0$. So again by Lemma 3.2,

$$\begin{aligned} &(h(\alpha(i))\beta(i))^{-1}h(\alpha'(i))\beta'(i) = (h(\alpha(j))\beta(j))^{-1}h(\alpha'(j))\beta'(j) \\ \textit{i.e. } \ &\beta(i)^{-1}h(\alpha(i))^{-1}h(\alpha(i))\beta'(i) = \beta(j)^{-1}h(\alpha(j))^{-1}h(\alpha(j))\beta'(j) \end{aligned}$$

and thus

$$\beta(i)^{-1}\beta'(i) = \beta(j)^{-1}\beta'(j).$$

Since $\beta(j)^{-1}\beta'(j)$ takes the values of S (cf. Lemma 3.2), it commutes with every element of G so that $\beta(j)^{-1}\beta'(j)$ lies in $Z(G)$, the center of G. Consequently, $\beta'(i) = \beta(i)z$ for some $z \in Z(G)$. Hence by regarding $Z(G)$ as a subgroup of $S_r \circ G$ we see that

$$\mathscr{G}(S\bar{e}_r) = \frac{S_r \circ G}{Z(G)}.$$

■

Using above lemmas in holonomy decomposition theorem for transformation semigroups we get:

Theorem 3.5. *If $S = \mathfrak{M}_{G^0}(\Gamma)$ and the number of orbits of Γ under the action of G is finite, say n, then*

$$(S, S) \prec \bar{\mathscr{H}}_n \circ \bar{\mathscr{H}}_{n-1} \circ \cdots \circ \bar{\mathscr{H}}_1$$

where, for $r \in \mathbf{n}$,

$$\mathscr{H}_r = \left(B(S\bar{e}_r), \frac{S_r \circ G}{Z(G)} \right)$$

in which S_r is the symmetric group of degree r and $Z(G)$ is the center of G.

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