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# Representation of Near-Semirings and Approximation of Their Categories

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**Abstract.** This work observes that $S$-semigroups are essentially the representations of near-semirings to proceed to establish categorical representation of near-semirings. Further, this work addresses some approximations to find a suitable category in which a given near-semiring is primitive.

**Keywords:** Near-semiring; $S$-Semigroup; Representation; Category.

## 1. Introduction

An algebraic structure $(S, +, \cdot)$ is said to be a *near-semiring* if

1. $(S, +)$ is semigroup with identity 0,
2. $(S, \cdot)$ is semigroup,
3. $(x + y)z = xz + yz$ for all $x, y, z \in S$, and
4. $0x = 0$ for all $x \in S$.

The standard examples of near-semirings are typically of the form $\mathfrak{M}(\Gamma)$, the set of all mappings on a semigroup $(\Gamma, +)$ with identity zero, with respect to pointwise addition and composition of mappings, and certain subsets of this set.

Two important subsets of $\mathfrak{M}(\Gamma)$ are the set of constant mappings, and the set of mappings which fix zero. In fact, these two sets are subnear-semirings of $\mathfrak{M}(\Gamma)$ in the usual sense. In an arbitrary near-semiring $S$, these substructures can be defined as *constant part* $S_c = \{s \in S \mid s0 = s\}$ and *zero-symmetric part* $S_0 = \{s \in S \mid s0 = 0\}$. A near-semiring $S$ is said to be a *zero-symmetric near-semiring*[*] (*constant near-semiring*) if $S = S_0$ ($S = S_c$, respectively). Another example of near-semiring that generalizes $\mathfrak{M}(\Gamma)$ is: let $\Sigma \subseteq End(\Gamma)$, the set of endomorphisms on $\Gamma$, and define

$$\mathfrak{M}_\Sigma(\Gamma) = \{f : \Gamma \to \Gamma \mid f\alpha = \alpha f, \forall \alpha \in \Sigma\}.$$

Then $\mathfrak{M}_\Sigma(\Gamma)$ is a near-semiring. Indeed, $\mathfrak{M}(\Gamma) = \mathfrak{M}_{\{id_\Gamma\}}(\Gamma)$.

A semigroup $(\Gamma, +)$ with zero $o$ is said to be an *$S$-semigroup* if there exists a composition $(x, \gamma) \mapsto x\gamma$ of $S \times \Gamma \longrightarrow \Gamma$ such that

1. $(x + y)\gamma = x\gamma + y\gamma$,
2. $(xy)\gamma = x(y\gamma)$, and
3. $0\gamma = o$, for all $x, y \in S, \gamma \in \Gamma$.

It is clear that $\Gamma$ is an $S$-semigroup with $S = \mathfrak{M}(\Gamma)$. Also, the semigroup $(S, +)$ of a near-semiring $(S, +, \cdot)$ is an $S$-semigroup.

For further details on near-semirings or $S$-semigroups one may refer [6, 8, 10, 11]. In what follows $S$ always denotes a near-semiring, and an additive semigroup with zero is simply referred as semigroup.

In this work we first observe that the notion of $S$-semigroup gives an algebraic representation of near-semirings which further helps us to establish a categorical representation. This enables one to make use of the special properties of the near-semirings, and provides a practical approach to the problem of classifying certain classes of near-semirings. We also made an attempt to approximate categories in which a given arbitrary near-semiring is primitive, as an extension of the work of Holcombe [3] and that of Clay [2] for near-rings.

## 2. Representations

Let $\Gamma, \Gamma'$ be two $S$-semigroups. A mapping $f : \Gamma \longrightarrow \Gamma'$ is said to be an *$S$-homomorphism* if

$$f(x + y) = f(x) + f(y);\ \ f(ax) = af(x)$$

for all $a \in S$ and all $x, y \in \Gamma$. Near-semiring homomorphism can be defined in usual way.

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[*]In the literature, zero-symmetric near-semirings are often referred as seminearrings [4, 6, 7, 10].

Following Jacobson [5], we define a *representation* of a near-semiring $S$ as a homomorphism of $S$ into the near-semiring of mappings of some semigroup with zero.

Let us recall the following embedding theorem from [4] before going to observe that $S$-semigroups are precisely the representations of near-semirings.

**Embedding Theorem.** *For every near-semiring $S$ there exists a semigroup $\Gamma$, such that $S$ can be embedded in $\mathfrak{M}(\Gamma)$.*

From this theorem one can ascertain that every near-semiring can be embedded into a near-semiring with unity.

Let $a \mapsto \bar{a}$ be a representation of $S$ that acts on a semigroup $\Gamma$. Define a composition from $S \times \Gamma$ to $\Gamma$ by $ax = \bar{a}(x)$, for $x \in \Gamma$ and $a \in S$, so that $\Gamma$ is an $S$-semigroup. Hence, every representation of a near-semiring $S$ determines an $S$-semigroup.

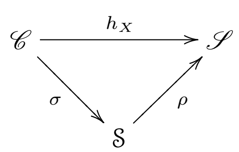
On the other hand, every $S$-semigroup $\Gamma$ determines a representation of the near-semiring $S$. Indeed, for $a \in S$, define a mapping $a_S$ on $\Gamma$ by $a_S(x) = ax$ for all $x \in \Gamma$. Then $\tau : S \longrightarrow \mathfrak{M}(\Gamma)$ given by $\tau(a) = a_S$ is a near-semiring homomorphism. Hence $\tau$ is a representation of $S$.

This discussion can be summarized as follows.

**Theorem 2.1.** *The concepts of $S$-semigroup and representation of a near-semiring $S$ are equivalent.*

In the following we obtain a representation of near-semirings in a more general way using the theory of categories. Let $\mathscr{C}$ be a category; write $X \in \mathscr{C}$ to indicate that $X$ is an object of $\mathscr{C}$. For any $X, Y \in \mathscr{C}$, the set of morphisms in $\mathscr{C}$ from $X$ to $Y$ is written as $[X, Y]_{\mathscr{C}}$. The category of sets and mappings is denoted by $\mathscr{S}$; $\mathcal{S}$ denotes the category of semigroups and their homomorphisms. The category of $S$-semigroups and $S$-homomorphisms for a fixed near-semiring $S$ will be denoted by $\mathcal{S}_S$. The contravariant representable functor $h_X : \mathscr{C} \longrightarrow \mathscr{S}$ is given by $h_X(Y) = [Y, X]_{\mathscr{C}}$, $h_X(u) = vu$ for any $v : Z \longrightarrow X$, where $u : Y \longrightarrow Z$. The forgetful functor from $\mathcal{S}$ to $\mathscr{S}$ will be denoted by $\rho : \mathcal{S} \longrightarrow \mathscr{S}$, and it is $\bar{\rho} : \mathcal{S}_S \longrightarrow \mathscr{S}$. For other terminology and fundamental concepts of category theory that are used in the rest of the paper, one may refer [1, 9].

An object $X \in \mathscr{C}$ is said to be a *semigroup object* in $\mathscr{C}$ if and only if there exists a functor $\sigma : \mathscr{C} \longrightarrow \mathcal{S}$ such that the following functor diagram commutes.




It is more practical to deal with morphisms rather than functors in some circumstances. For that purpose, Lemma 2.2 is formulated in the similar lines of a theorem for group objects (cf. Theorem 4.1 of [1]).

**Lemma 2.2.** *Let $\mathscr{C}$ be a category with finite products and a final object $e$, and let $X \in \mathscr{C}$. Let $\eta$ be the unique element of $[X, e]_{\mathscr{C}}$. $X$ is a semigroup object in $\mathscr{C}$ if and only if there exist morphisms $m \in [X \times X, X]_{\mathscr{C}}$, and $\varepsilon \in [e, X]_{\mathscr{C}}$ such that the diagrams*

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times 1_X} & X \times X \\ {\scriptstyle 1_X \times m}\downarrow & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow[m]{} & X \end{array} \qquad\qquad \begin{array}{ccc} X \times X & \xrightarrow{(\varepsilon\eta) \times 1_X} & X \times X \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{1_X} & X \end{array}$$

*are commutative, where $\Delta$ is the 'diagonal' morphism.*

**Theorem 2.3.** *Let $(X, \sigma)$ be a semigroup object in $\mathscr{C}$, a category with finite products and a final object. Then $[X, X]_{\mathscr{C}}$ is a near-semiring, say $S$, and for any $Y \in \mathscr{C}$, $[Y, X]_{\mathscr{C}}$ is an $S$-semigroup.*

*Proof.* Let $S = [X, X]_{\mathscr{C}}$ and $a, b \in S$. By Lemma 2.2, there is a semigroup structure $(S, +)$ defined by $a + b = m\{a, b\}$, where $\{a, b\}$ is the unique morphism making the following diagram (1) commutative with $p_1, p_2 : X \times X \longrightarrow X$ canonical projections, and $m$ is obtained as in Lemma 2.2.

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times X & \xrightarrow{p_2} & X \\ & {\scriptstyle a}\nwarrow & \uparrow{\scriptstyle \{a,b\}} & \nearrow{\scriptstyle b} & \\ & & X & & \end{array} \qquad -(1)$$

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times X & \xrightarrow{p_2} & X \\ & {\scriptstyle a}\nwarrow & \uparrow{\scriptstyle \{a,b\}} & \nearrow{\scriptstyle b} & \\ {\scriptstyle ac}\nwarrow & & X & & \nearrow{\scriptstyle bc} \\ & & \uparrow{\scriptstyle c} & & \\ & & X & & \end{array} \qquad -(2)$$

Clearly, $S$ is a semigroup under the composition of morphisms in $\mathscr{C}$. Right distributivity follows from the commutative diagram (2), so that $S$ is a near-semiring.

Again, since $(X, \sigma)$ is a semigroup object in $\mathscr{C}$, for any $Y \in \mathscr{C}$, $\Gamma = [Y, X]_{\mathscr{C}}$ is a semigroup, where addition $+$ on $\Gamma$ is given by $\alpha + \beta = m\{\alpha, \beta\}$ for $\alpha, \beta \in \Gamma$ and $\{\alpha, \beta\}$ is the unique morphism, such that the following diagram commutes.

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times X & \xrightarrow{p_2} & X \\ & {\scriptstyle \alpha}\nwarrow & \uparrow{\scriptstyle \{\alpha,\beta\}} & \nearrow{\scriptstyle \beta} & \\ & & Y & & \end{array}$$

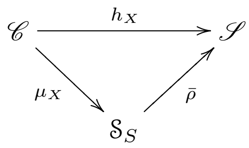
Define an action from $S \times \Gamma$ to $\Gamma$ by $(a, \alpha) \mapsto a\alpha$, for $a \in S$ and $\alpha \in \Gamma$. By a similar argument to the first part of this proof we see that

$$(a+b)\alpha = a\alpha + b\alpha, \quad \text{and} \quad (ab)\alpha = a(b\alpha)$$

for all $a, b \in S$ and $\alpha \in \Gamma$, so that $\Gamma$ is an $S$-semigroup. ∎

Given the situation of Theorem 2.3, we call $[X, X]_{\mathscr{C}}$ the *endomorphism near-semiring* of $X$ in $\mathscr{C}$.

*Remark 2.4.* Let $(X, \sigma)$ be a semigroup object in a category $\mathscr{C}$ with finite products and final object. If $S = [X, X]_{\mathscr{C}}$ then there is a contravariant functor $\mu_X : \mathscr{C} \longrightarrow \mathcal{S}_S$, such that the following diagram commutes,

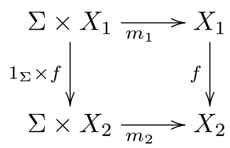



where $\bar{\rho}$ is the forgetful functor from $\mathcal{S}_S$ to $\mathscr{S}$.

*Example 2.5.* In the category of sets and mappings $\mathscr{S}$, semigroup objects are just semigroups. Then the endomorphism near-semiring of a semigroup $\Gamma$ in $\mathscr{C}$ is the set of all mappings of $\Gamma$ into itself.

*Example 2.6.* In the category $\mathscr{S}^*$ of pointed sets, let us consider the zero of semigroup objects $\Gamma^*$ as distinguished element. The endomorphism near-semiring of $\Gamma^*$ is the set of zero-preserving maps of $\Gamma^*$ into itself. This near-semiring is zero-symmetric.

*Example 2.7.* Let $\Sigma$ be a semigroup. The category $\mathscr{S}_\Sigma$, of $\Sigma$-sets, has objects as pairs $(X, m)$, where $X$ is a set and $m : \Sigma \times X \longrightarrow X$ is a mapping with the property that $m(\alpha\beta, x) = m(\alpha, m(\beta, x))$ for all $x \in X$ and $\alpha, \beta \in \Sigma$. A morphism $f : (X_1, m_1) \longrightarrow (X_2, m_2)$ is a mapping $f : X_1 \longrightarrow X_2$, such that the following diagram commutes.




The endomorphism near-semiring of $X$, a semigroup object in $\mathscr{S}_\Sigma$, is the set of mappings $f$ of $X$ into itself, such that $f(m(\alpha, x)) = m(\alpha, f(x))$ for all $x \in X$ and $\alpha \in \Sigma$. This example generalizes near-semirings of the form $\mathfrak{M}_\Sigma(X)$ and $S$-semigroups.

*Example 2.8.* Let $\mathscr{C}_1$, $\mathscr{C}_2$ be two categories with $\mathscr{C}_2$ a subcategory of $\mathscr{C}_1$. The category $\langle \mathscr{C}_1, \mathscr{C}_2 \rangle$ is defined to have objects $f : B \longrightarrow A$, where $B \in \mathscr{C}_2, A \in \mathscr{C}_1$ and $f \in [B, A]_{\mathscr{C}_1}$. A morphism from $f : B \longrightarrow A$ to $g : C \longrightarrow D$ is a pair $(a, b)$ such that $bf = ga$, where $a \in [B, C]_{\mathscr{C}_2}$, $b \in [A, D]_{\mathscr{C}_1}$, i.e. a morphism from $f$ to $g$ can be given by a commutative diagram as below.

$$\begin{array}{ccc} B & \xrightarrow{a} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{b} & D \end{array}$$

Further, let $\mathscr{C}_3$ be a subcategory of $\mathscr{C}_1$ and define $\langle \mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3 \rangle$ to have objects $(f, f')$, where $f : B \longrightarrow A$, $f' : C \longrightarrow A$ and $A \in \mathscr{C}_1$, $B \in \mathscr{C}_2$, $C \in \mathscr{C}_3$, $f \in [B, A]_{\mathscr{C}_1}$, $f' \in [C, A]_{\mathscr{C}_1}$; and morphisms from $(f, f')$ to $(g, g')$ are the commutative diagrams,

$$\begin{array}{ccccc} B & \xrightarrow{f} & A & \xleftarrow{f'} & C \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \xrightarrow{g} & A_1 & \xleftarrow{g'} & C_1 \end{array}$$

where $A_1 \in \mathscr{C}_1$, $B_1 \in \mathscr{C}_2$, $C_1 \in \mathscr{C}_3$, $g \in [B_1, A_1]_{\mathscr{C}_1}$, $g' \in [C_1, A_1]_{\mathscr{C}_1}$. A natural extension of these ideas give a category $\langle \mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k \rangle$, where $\mathscr{C}_j$ is a subcategory of $\mathscr{C}_1$ for $j = 2, 3, \ldots, k$. In a more general case, suppose $F : \mathscr{C} \longrightarrow \mathscr{C}'$ is an embedding functor, we can construct the category $\langle \mathscr{C}', F(\mathscr{C}) \rangle$.

As an example of the construction, let $\Sigma$ be a semigroup and $\Sigma'$ be a subsemigroup of $\Sigma$. There exists an embedding $F : \mathscr{S}^*_\Sigma \longrightarrow \mathscr{S}^*_{\Sigma'}$, where $\mathscr{S}^*_\Sigma$ (and $\mathscr{S}^*_{\Sigma'}$) is pointed $\Sigma$-sets ($\Sigma'$-sets respectively). Let $X$ be a semigroup object of $\mathscr{S}^*_{\Sigma'}$ and $Y$ a subsemigroup of $X$ which is also a semigroup object of $\mathscr{S}^*_\Sigma$. Then the endomorphism near-semiring of $(F(Y) \subseteq X)$ is the set of all mappings $f : X \longrightarrow X$, such that $f(Y) \subseteq Y$, $f(m(\alpha', x)) = m(\alpha', f(x))$, $f(m(\alpha, y)) = m(\alpha, f(y))$ for all $x \in X, y \in Y, \alpha' \in \Sigma'$, and $\alpha \in \Sigma$. These near-semirings are examples of an important class of near-semirings which deserves study in its own right.

Several other examples come in the same line. So far it is observed how the near-semirings arise in essentially the same way as endomorphism sets of semigroup objects in particular categories.

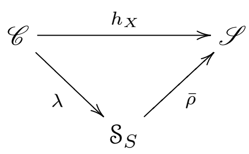
Let $S$ be a near-semiring and $\mathscr{C}$ be a category with finite products. An object $X$ is said to be an *$S$-semigroup object* in $\mathscr{C}$ if and only if there exist

1. a functor $\sigma$ such that $(X, \sigma)$ is a semigroup object in $\mathscr{C}$, and
2. a near-semiring homomorphism $\tau : S \longrightarrow [X, X]_{\mathscr{C}}$.

In this case, $(X, \sigma, \tau)$ denotes an $S$-semigroup object in $\mathscr{C}$. An $S$-semigroup object $(X, \sigma, \tau)$ is said to be *faithful* in $\mathscr{C}$ if and only if $\tau$ is one-one.

If $\mathscr{C}$ is the category of sets and mappings then a semigroup object in $\mathscr{C}$ is simply a semigroup and the concept of an $S$-semigroup in $\mathscr{C}$ coincides with the natural definition of an $S$-semigroup. Therefore, $S$-semigroups are special cases of the concept of $S$-semigroups in a category $\mathscr{C}$.

**Theorem 2.9.** *Let $S$ be a near-semiring and let $\mathscr{C}$ be a category with finite products and final object. Then for $X \in \mathscr{C}$, $X$ is an $S$-semigroup object in $\mathscr{C}$ if and only if there exists a contravariant functor $\lambda : \mathscr{C} \longrightarrow \mathbb{S}_S$ such that the following diagram*




*is commutative, where $\bar{\rho} : \mathbb{S}_S \longrightarrow \mathscr{S}$ is the forgetful functor.*

*Proof.* Suppose $(X, \sigma, \tau)$ is an $S$-semigroup object in $\mathscr{C}$. Let $Y \in \mathscr{C}$, and write $\Gamma = [Y, X]_{\mathscr{C}}$. Consider the structure of an $S$-semigroup to $\Gamma$ as follows: define $s \cdot \gamma = \tau(s)\gamma$ for $\gamma \in \Gamma, s \in S$. Thus there exists a functor $\lambda : \mathscr{C} \longrightarrow \mathbb{S}_S$, such that $\lambda(Y)$ is the $S$-semigroup $\Gamma = [Y, X]_{\mathscr{C}}$.

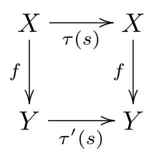
Conversely, suppose $\lambda$ exists, and that $\rho^* : \mathbb{S}_S \longrightarrow \mathbb{S}$ and $\rho : \mathbb{S} \longrightarrow \mathscr{S}$ are forgetful functors. Then $(X, \rho^* \circ \lambda)$ is a semigroup object in $\mathscr{C}$. A near-semiring homomorphism $\tau : S \longrightarrow [X, X]_{\mathscr{C}}$ is defined as follows. Since $\lambda(Y)$ is an $S$-semigroup, one may construct a near-semiring homomorphism

$$\bar{\tau} : S \longrightarrow [h_X(Y), h_X(Y)]_{\mathscr{S}}$$

for any $Y \in \mathscr{C}$. For each $s \in S$, the homomorphism $\bar{\tau}(s)$ induces a natural transformation $T_s : h_X \longrightarrow h_X$. As a consequence of Yoneda lemma, one may find a unique morphism $g_s \in [X, X]_{\mathscr{C}}$ in natural correspondence with $T_s$. Now define $\tau : S \longrightarrow [X, X]_{\mathscr{C}}$ by $\tau(s) = g_s$ for all $s \in S$. This gives the required near-semiring homomorphism. ∎

*Remark 2.10.* It is possible to define an $S$-homomorphism between $S$-semigroups in the same category $\mathscr{C}$. For instance, given a near-semiring $S$ and a category $\mathscr{C}$ with finite products and final object, let $(X, \sigma, \tau), (Y, \sigma', \tau')$ be $S$-semigroups in $\mathscr{C}$. A morphism $f : X \longrightarrow Y$ in $\mathscr{C}$ is an $S$-homomorphism in $\mathscr{C}$ if and only if

– for all $s \in S$ the following diagram commutes




– there exists a natural transformation $\xi : \sigma \longrightarrow \sigma'$ such that the induced natural transformation $T_\xi : h_X \longrightarrow h_Y$ corresponds via the Yoneda lemma to the morphism $f : X \longrightarrow Y$ in $\mathscr{C}$.

Thus, one can define a category of $S$-semigroups and $S$-homomorphisms in a category with finite products.

## 3. Approximation Theorems

First we formulate the notions: transparent $S$-subsemigroups, minimality and primitivity in categories for near-semirings as an extension of those parallel notions for near-rings given by Holcombe [3]. Then we proceed to approximate categories in which the given near-semiring is primitive. Unless otherwise stated, in the following $\mathscr{C}$ is a category with finite products and a final object, also there exists a forgetful functor $\mathscr{U} : \mathscr{C} \longrightarrow \mathscr{S}$.

Suppose $(X, \sigma, \tau)$ and $(Y, \sigma', \tau')$ are $S$-semigroups in $\mathscr{C}$ and $u : Y \longrightarrow X$ is an $S$-homomorphism in $\mathscr{C}$. We call $(Y, u)$ an *$S$-subsemigroup* of $X$ in $\mathscr{C}$ if and only if $u$ is a monomorphism in $\mathscr{C}$, and $\mathscr{U}(u)$ is an inclusion in $\mathscr{S}$. An $S$-subsemigroup $(Y, u)$ of $X$ is called *transparent* if and only if

$$[Y, X]_{\mathscr{C}} = \{uf \mid f \in [Y, Y]_{\mathscr{C}}\},$$

i.e. any morphism in $[Y, X]_{\mathscr{C}}$ can be decomposed into the composition of a morphism in the near-semiring $[Y, Y]_{\mathscr{C}}$ with $u$. Let $X$ be an $S$-semigroup in $\mathscr{C}$ and $f \in [K, X]_{\mathscr{C}}$ a monomorphism, for $K \in \mathscr{C}$. We call $(K, f)$ is a *generator of* $X$ if and only if $\mathscr{U}(f)$ is a set inclusion, and for every $a \in [K, X]_{\mathscr{C}}$, there exists $s_a \in S$ such that $\tau(s_a)f = a$. An $S$-semigroup $X$ in $\mathscr{C}$ is called *$\mathscr{C}$-minimal* if and only if given a nontrivial monomorphism $f \in [K, X]_{\mathscr{C}}$ with $\mathscr{U}(f)$ a set inclusion, either $(K, f)$ is a generator of $X$, or there exists a transparent $S$-subsemigroup $(Y, u)$ of $X$ such that $f$ factors through $u$ in the following way: there exists $f' \in [K, Y]_{\mathscr{C}}$ such that $\mathscr{U}(f')$ is a set inclusion and $f = uf'$. Further, a near-semiring $S$ is said to be *$\mathscr{C}$-primitive* for some $\mathscr{C}$ if there exists a $\mathscr{C}$-minimal $S$-semigroup $X$ in $\mathscr{C}$ which is faithful.

Naturally there may exist near-semirings which are not $\mathscr{C}$-primitive for any $\mathscr{C}$. Though finding a suitable category $\mathscr{C}$ such that given a near-semiring $S$ is $\mathscr{C}$-primitive is difficult, it is often possible to find a category $\mathscr{C}$ over which $S$ can be represented in a useful way. For example there may be representations of $X$ over $\mathscr{C}$ such that $\tau : S \longrightarrow [X, X]_{\mathscr{C}}$ is one-one. Now replace $\mathscr{C}$ by other categories so that the representations are preserved, and at the same time to make the homomorphism $\tau$ nearer to being an isomorphism, which is clearly a desirable objective.

Let $(X, \tau)$ be an $S$-semigroup in $\mathscr{C}$, where $\tau : S \longrightarrow [X, X]_{\mathscr{C}}$ the near-semiring homomorphism. Suppose $G$ is $Aut_S(X)$, the group of all invertible $S$-homomorphisms in $\mathscr{C}$. Construct a category $\mathscr{C}_G$ in which objects are the pairs $(A, \alpha)$, where $A \in \mathscr{C}$ and $\alpha : G \longrightarrow [A, A]_{\mathscr{C}}$ is a semigroup homomorphism. A morphism $\xi$ of $\mathscr{C}_G$, say $(A, \alpha) \xrightarrow{\xi} (B, \beta)$, is a morphism $\xi \in [A, B]_{\mathscr{C}}$ such that $\beta(g)\xi = \xi\alpha(g)$ for all $g \in G$.

*Remark 3.1.* $\mathscr{C}_G$ is a category with finite products and final object. Moreover, there exists a forgetful functor $\mathscr{U}_G : \mathscr{C}_G \longrightarrow \mathscr{S}$.

The objects $X \in \mathscr{C}$ may be equipped with the structure of an object $X_G \in \mathscr{C}_G$ by defining $X_G = (X, id_X)$.

*Remark 3.2.* $(X_G, \tau_G)$ is an $S$-semigroup in $\mathscr{C}_G$, where $\tau_G : S \longrightarrow [X_G, X_G]_{\mathscr{C}_G}$ is defined by $\tau_G(s) = \tau(s)\ \forall s \in S$. Moreover, if $\tau$ is one-one then $\tau_G$ is one-one.

**Theorem 3.3.** *If $X$ is $\mathscr{C}$-minimal then $X_G$ is $\mathscr{C}_G$-minimal.*

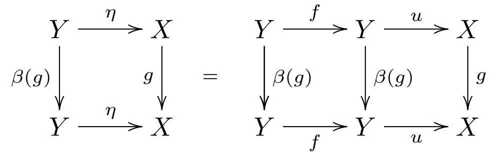
*Proof.* Let $f_G \in [K_G, X_G]_{\mathscr{C}_G}$ be a monomorphism and $\mathscr{U}_G(f_G)$ be a set inclusion. On forgetting the $G$-structure, we obtain a monomorphism $f \in [K, X]_{\mathscr{C}}$. Since $X$ is $\mathscr{C}$-minimal, there are two cases.

Consider the case when $(K, f)$ is generator of $X$ in $\mathscr{C}$. Suppose $a_G \in [K_G, X_G]_{\mathscr{C}_G}$, and consider the corresponding morphism $a \in [K, X]_{\mathscr{C}}$. There exists $s_a \in S$ such that $\tau(s_a)f = a$. Since $\tau_G(s_a) = \tau(s_a)$ we have $\tau_G(s_a)f = a$ and so $\tau_G(s_a)f_G = a_G$. Hence $(K_G, f_G)$ is a generator of $X_G$ in $\mathscr{C}_G$.

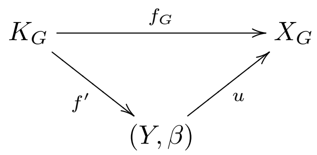
On the other hand, suppose $(Y, u)$ is a transparent $S$-subsemigroup of $X$ in $\mathscr{C}$ and $f' \in [K, Y]_{\mathscr{C}}$ is such that $\mathscr{U}(f')$ is a set inclusion and $f = uf'$. We turn $Y$ into an object of $\mathscr{C}_G$ as follows. Let $g \in G$, then $gu \in [Y, X]_{\mathscr{C}}$ and hence, by transparency of $Y$, $gu = uf_g$ for some unique $f_g \in [Y, Y]_{\mathscr{C}}$. Define a mapping

$$\beta : G \longrightarrow [Y, Y]_{\mathscr{C}}$$

by $\beta(g) = f_g$ for all $g \in G$. For $g, g' \in G$ we have $\beta(gg') = f_{gg'}$ and $uf_{gg'} = gg'u = guf_{g'} = uf_g f_{g'}$, so that $\beta$ is a semigroup homomorphism and hence $(Y, \beta)$ is an object of $\mathscr{C}_G$. Now we shall prove that $(Y, \beta)$ is transparent $S$-subsemigroup of $X_G$ in $\mathscr{C}_G$. Let $(Y, \beta) \stackrel{\eta}{\longrightarrow} X_G$ be any morphism in $\mathscr{C}_G$. Then for each $g \in G$, the following left side diagram is commutative. Since $Y$ is transparent $S$-subsemigroup of $X$ in $\mathscr{C}$ we have $\eta = uf$, for some $f \in [Y, Y]_{\mathscr{C}}$, so that the outer square of the following right side diagram is equals to left side diagram and hence commutes.




Note that the right hand square of the right side diagram also commutes and hence, because $u$ is a monomorphism, the left hand square of right diagram commutes, i.e. $\beta(g)f = f\beta(g)$ for all $g \in G$, so that $(Y, \beta) \stackrel{f}{\longrightarrow} (Y, \beta)$ is a morphism of $\mathscr{C}_G$. Thus $(Y, \beta)$ is transparent in $X_G$ in the category $\mathscr{C}_G$. Finally, the diagram




commutes in $\mathscr{C}_G$ from similar considerations. Hence $X_G$ is $\mathscr{C}_G$-minimal. ∎

Though the results are valid with the semigroup structure of $[X, X]_{\mathscr{C}}$ in place of the group $G$, by choosing the group $G$ we could further narrow down the category to $\mathscr{C}_G$. As we can embed $[X_G, X_G]_{\mathscr{C}_G}$ in $[X, X]_{\mathscr{C}}$, Theorem 3.3 gives us an approximation theorem without disturbing the special nature of the representation $X$ of $S$.

If $X$ has any $G$-closed $S$-subsemigroup we can produce a better approximation to $S$. Here, an $S$-subsemigroup $(Y, u)$ of $X$ is referred as *$G$-closed* if and only if given $g \in G$ there exists a unique $f \in [Y, Y]_{\mathscr{C}}$ such that $gu = uf$.

*Remark 3.4.* Since $u$ is monomorphism, if $(Y, u)$ is transparent in $\mathscr{C}$ then $(Y, u)$ is $G$-closed.

**Lemma 3.5.** *Let $(Y, u)$ be a $G$-closed $S$-subsemigroup of $X$ in $\mathscr{C}$. Define $G' = Aut_{S/_{\ker \tau'}}(Y)$, where $\tau' : S \longrightarrow [Y, Y]_{\mathscr{C}}$ is the $S$-semigroup structure near-semiring homomorphism. There is an embedding functor $F : \mathscr{C}_{G'} \longrightarrow \mathscr{C}_G$.*

*Proof.* $F$ can be obtained by defining a semigroup monomorphism $\theta : G \longrightarrow G'$. For that, let $g \in G$; then $g \in [X, X]_{\mathscr{C}}$, $g$ is invertible and $\tau(s)g = g\tau(s)$ for any $s \in S$. Also, since $(Y, u)$ is $G$-closed there is unique $f \in [Y, Y]_{\mathscr{C}}$ such that $gu = uf$. Define $\theta : G \longrightarrow G'$ by setting $\theta(g) = f$, for $g \in G$. We shall ascertain that $f \in G'$. Since $\theta(1)$ is the identity morphism on $Y$, it follows that $\theta(g)$ is invertible. To show that $f\bar{\tau}'(\bar{s}) = \bar{\tau}'(\bar{s})f$ for all $\bar{s} \in S/_{\ker \tau'}$, we have to prove that $f\tau'(s) = \tau'(s)f$ $\forall s \in S$. Since $u$ is $S$-homomorphism in $\mathscr{C}$, we have:

$$\begin{aligned} uf\tau'(s) &= gu\tau'(s) = g\tau(s)u \\ &= \tau(s)gu = \tau(s)uf = u\tau'(s)f \end{aligned}$$

and thus $f\tau'(s) = \tau'(s)f$ $\forall s \in S$. It is easy to see that $\theta$ is a semigroup monomorphism, as desired.

Let $(A, \alpha) \in \mathscr{C}_{G'}$, so that $\alpha : G' \longrightarrow [A, A]_{\mathscr{C}}$ is a semigroup homomorphism. Set $F((A, \alpha)) = (A, \alpha\theta)$ so that $F((A, \alpha)) \in \mathscr{C}_G$, and $F$ is an embedding functor. ∎

Consider the category $\mathscr{D} = \langle \mathscr{C}_G, F(\mathscr{C}_{G'}) \rangle$ (cf. Example 2.8 for notation). Note that this is a category with finite products, final object, and there exists a forgetful functor. The object $X_G \in \mathscr{C}_G$ can naturally be equipped with the structure, $X_*$, defined to be $(F(Y) \subseteq X_G)$, and the near-semiring homomorphism $\tau_* : S \longrightarrow [X_*, X_*]_{\mathscr{D}}$ is defined by $\tau_*(s) = \tau(s)$ for all $s \in S$. Thus $X_*$ is an $S$-semigroup object of $\mathscr{D}$. If $X$ is faithful in $\mathscr{C}$ then $X_*$ is faithful in $\mathscr{D}$. Further, if $X_G$ is $\mathscr{C}_G$-minimal, then in a similar way to that of Theorem 3.3, one can finalize that $X_*$ is $\mathscr{D}$-minimal.

This can be summarized as the second approximation theorem as follows:

**Theorem 3.6.** *The object $X_*$ of $\mathscr{D}$ is an $S$-semigroup object and if $X$ is faithful then $X_*$ is faithful. Moreover, if $X$ is $\mathscr{C}$-minimal then $X_*$ is $\mathscr{D}$-minimal.*

Further, if $X$ has $G$-closed $S$-subsemigroups in $\mathscr{C}$ then each of which gives an approximation theorem in the following way.

**Theorem 3.7.** *Let $(Y_i, u_i)$ be $G$-closed $S$-subsemigroups of $X$ for $i = 1, 2, \ldots, k$ and $G_i = Aut_{S/_{\ker \tau_i}}(Y_i)$ for each $i = 1, 2, \ldots, k$. Let $F_i : \mathscr{C}_{G_i} \longrightarrow \mathscr{C}_G$ be an appropriate embedding functor for each $i = 1, 2, \ldots, k$. Consider the category*

$$\mathscr{D}_k = \langle \mathscr{C}_G, F_1(\mathscr{C}_{G_1}), F_2(\mathscr{C}_{G_2}), \ldots, F_k(\mathscr{C}_{G_k}) \rangle.$$

*If $X$ is a $\mathscr{C}$-minimal, faithful $S$-semigroup in $\mathscr{C}$, then $X$ can be given the structure of a $\mathscr{D}_k$-minimal faithful $S$-semigroup in $\mathscr{D}_k$.*

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