Section 6.4 Formal Axiom Systems

By a formal axiom system we mean a specific set of axioms (a fixed set of premises) and proof rules. The aims of a formal axiom system are *soundness* and *completeness*::

Soundness: All proofs yield theorems that are tautologies.
Completeness: All tautologies are provable as theorems.

Frege-Lukasiewicz (F-L) Axiom System

Axiom 1: $A \to (B \to A)$.
Axiom 2: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.
Axiom 3: $(\neg A \to \neg B) \to (B \to A)$.

Proof Rule: MP.

Since the axioms of F-L are tautologies and MP maps tautologies to a tautology, the F-L system is sound. The F-L system is also complete, but that takes a bit of proof (see the text).

Example (*Lemma)*. Use the F-L system to prove $A \to A$.

Proof:
1. $A \to ((A \to A) \to A)$ — Axiom 1
2. $(A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$ — Axiom 2
3. $(A \to (A \to A)) \to (A \to A)$ — 1, 2, MP
4. $A \to (A \to A)$ — Axiom 1
5. $A \to A$ — 3, 4, MP

QED.

Deduction Theorem (The CP Rule)
If A is a premise in a proof of B, then there is a proof of $A \to B$ that does not use A as a premise.

Proof Idea: Assume the proof has the form

$$A = B_0, \ldots, B_n = B.$$

If $n = 0$, then $A = B$. So we must find a proof of $A \to B = A \to A$ that does not use A as a premise. A proof was given in the previous example (lemma). Let $n > 0$ and assume that for each k in the range $0 \le k < n$ there is a proof of $A \to B_k$ that does not use A as a premise. Show that there is a proof of $A \to B_n$ that does not use A as a premise. If B_n is a premise or an axiom, then we have the following proof that does not use A as a premise:

1. B_n	Premise or Axiom	
2. $B_n \to (A \to B_n)$	Axiom 1	
3. $A \to B_n$	1, 2, MP	QED.

If B_n is neither a premise nor an axiom, then it is inferred by MP from B_i and $B_j = B_i \to B_n$, where $i < n$ and $j < n$. So we obtain the following proof that does not use A as a premise:

1. Proof of $A \to B_i$ not using A as a premise	Induction assumption
2. Proof of $A \to (B_i \to B_n)$ not using A as a premise	Induction assumption
3. $(A \to (B_i \to B_n)) \to ((A \to B_i) \to (A \to B_n))$	Axiom 2
4. $(A \to B_i) \to (A \to B_n)$	2, 3, MP
5. $A \to B_n$	1, 4, MP QED.

Since $B_n = B$, we have a proof of $A \to B$ that does not use A as a premise. QED.