Tutorial 5: Theory of Residues and Applications
PH 503 Mathematical physics

1. Determine all the isolated singularities of each of the following functions and compute the residue at each singularity
   (a) $e^{3z}/(z-2)$
   (b) $(z+1)/(z^2-3z+2)$
   (c) $(\cos z)/z^2$
   (d) $\left(\frac{z-1}{z+1}\right)^3$
   (e) $\sin(1/3z)$
   (f) $(z-1)/\sin z$

2. Evaluate each of the following integrals by means of the Cauchy residue theorem.
   (a) $\int_C \frac{\sin z}{z^2-4}dz$ where $C: |z| = 5$.
   (b) $\int_C \frac{e^z}{z(z-2)^3}dz$ where $C: |z| = 3$.
   (c) $\int_C \tan z dz$ where $C: |z| = 2\pi$.
   (d) $\int_C \frac{1}{z^2 \sin z}dz$ where $C: |z| = 1$.

3. Let f have an isolated singularity at z_0 (f analytic in punctured nbd of z_0). Show that the residue of the derivative f' is equal to zero.

4. Using method of residues, verify each of the following.
   (a) $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}$.
   (b) $\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)} = \frac{3\pi\sqrt{5}}{25}$.
   (c) $\int_0^{2\pi} \frac{d\theta}{a^2\sin^2\theta+b^2\cos^2} = \frac{2\pi}{ab}$
   (d) $\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}$

5. Verify the following integral formulae with the help of residues.
   (a) $\text{pv}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$.
   (b) $\text{pv}\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2} = \frac{\pi}{6}$.
   (c) $\int_0^{\infty} \frac{x^2+1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}$.

6. Show that

$$pv\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}dx = \sec 1$$

   by integrating $e^{2z}/\cosh(\pi z)$ around a rectangle with vertices at $z = \pm R, \pm R+i$ and then taking a limit $R \to \infty$.

7. Show that

$$\int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi\sqrt{3}}{9}$$

   by integrating $1/\left(z^3+1\right)$ around the boundary of the circular sector $S: \left\{z = re^{i\theta} : 0 \leq \theta \leq 2\pi/3,\ 0 \leq r \leq R\right\}$ and then letting $R \to \infty$.

8. Using the method of residues, verify:

   (a) $\text{pv}\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1} dx = \frac{\pi}{e^2}$.

   (b) $\text{pv}\int_{-\infty}^{\infty} \frac{x \sin x}{x^2-2x+10} dx = \frac{\pi}{3e^3} (3\cos 1 + \sin 1)$.

9. Given that $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2$, integrate e^{iz^2} around the boundary of the circular sector $S: \{z = re^{i\theta} : 0 \le \theta \le \pi/4,\, 0 \le r \le R\}$ and letting $R \to \infty$, prove that

$$\int_0^{\infty} e^{ix^2} dx = \frac{\sqrt{2\pi}}{4} (1+i).$$

10. Using the technique of residues, verify:

    (a) $\int_{-\infty}^{\infty} \frac{e^{2ix}}{x+1} dx = \pi i e^{-2i}$;

    (b) $\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x-2)} dx = \pi i \left(e^{2i} - e^{i}\right)$.

11. Compute pv $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x-1} dx$ for $0 < a < 1$. (Use rectangular contour).