Tutorial 4: Contour Integrals, Taylor and Laurent Series
PH 503 Mathematical physics

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1. For each of the following smooth curves give an admissible parametrization that is consistent with the indicated direction.

   (a) The line segment from $z = 1 + i$ to $z = -1 - 3i$.

   (b) the circle $|z - 2i| = 4$ traversed once in the clockwise direction starting from the point $z = -2i$.

   (c) the segment of the parabola $y = x^2$ from point $(1, 1)$ to $(3, 9)$

2. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when} \quad m \neq n \\ 2\pi & \text{when} \quad m = n. \end{cases}$$

3. A semicircular contour is given by two separate parametrization

$$\begin{aligned} z_1(t) &= 2e^{it} \qquad \left(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right) \\ z_2(\tau) &= \sqrt{4 - \tau^2} + i\tau \qquad (-2 \leq \tau \leq 2). \end{aligned}$$

   (a) Find the length of the contour using each parametrization.

   (b) Find a function $t = \phi(\tau)$ such that $z_2(\tau) = z_1(\phi(\tau))$.

4. Let C be the perimeter of the square with vertices at $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$ traversed once in that order. Compute following integrals using primary deønition:

   (a) $\int_C e^z dz$;

   (b) $\int_C \bar{z}^2 dz$.

5. Evaluate $\int_C (x - 2xyi)\, dz$ over the contour $C : z = t + it^2, 0 \leq t \leq 1$.

6. Show that

   (a) $\left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{3\pi}{4}$ if $C : |z| = 3$.

   (b) $\left|\int_C \frac{e^{3z}}{1+e^z} dz\right| \leq \frac{2\pi e^{3R}}{e^R - 1}$ if C is a vertical line segment from $z = R\,(> 0)$ to $z = R + 2\pi i$.

7. Use antiderivatives to evaluate following integrals:

   (a) $\int_i^{i/2} e^{\pi z} dz$;

   (b) $\int_0^{\pi+2i} \cos(z/2)\, dz$

   (c) $\int_1^3 (z - 2)^3\, dz$.

8. Use antiderivatives to show that

$$\int \frac{dz}{z^2 - a^2} = \frac{1}{2a} \log\left(\frac{z-a}{z+a}\right) + c_1 = \frac{1}{a} \coth^{-1}\left(\frac{z}{a}\right) + c_2$$

9. Let C be a square with vertices on $z = \pm 2 \pm 2i$. Evaluate following integrals using Cauchy integral formula to evaluate following integrals:

   (a) $\int_C \frac{e^{-z} dz}{z-(\pi i/2)}$;

(b) $\int_C \frac{\cos z}{z(z^2+8)}dz$;

(c) $\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz \qquad (-2 < x_0 < 2)$.

10. Show that $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz = \sin t$ if $t(>0)$ is a real constant and $C : |z| = 3$.

11. Verify each of the following Taylor expansions by ønding a general formula for $f^{(j)}(z_0)$.

(a) $\sinh z = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}$ with $z_0 = 0$.

(b) $\cosh z = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}$ with $z_0 = 0$.

(c) $\frac{1}{1-z} = \sum_{j=0}^{\infty} \frac{(z-i)^j}{(1-i)^{j+1}}$ with $z_0 = i$.

(d) $z^3 = 1 + 3(z-1) + 3(z-1)^2 + (z-1)^3$ with $z_0 = 1$.

12. Find the Laurent Series for the function $1/(z+z^2)$ for each of the following domains:

(a) $0 < |z| < 1$;

(b) $1 < |z|$;

(c) $0 < |z+1| < 1$;

(d) $1 < |z+1|$.

13. Find Laurent series for

(a) $\sin(2z)/z^3$ in $|z| > 0$;

(b) $z^2\cos(1/3z)$ in $|z| > 0$.

14. Prove that the Laurent series expansion of the function

$$f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right]$$

in $|z| > 0$ is given by

$$\sum_{k=-\infty}^{\infty} J_k(\lambda)\, z^k,$$

where

$$J_k(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \cos(k\theta - \lambda\sin\theta)\, d\theta.$$

The functions $J_k(\lambda)$ are known as Bessel functions of the ørst kind.