Tutorial 3: Analytic Functions
PH 503 Mathematical physics

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1. Prove that $\frac{d}{dz}z^n = nz^{n-1}$ where n is an integer.

2. Find the derivative of the given functions using the rules of differentiation:

   (a) $e^z = 1 + \sum_1^\infty z^n/n!$.
   (b) $\sin z = \left(e^{iz} - e^{-iz}\right)/2i$.
   (c) $\cos z = \left(e^{iz} + e^{-iz}\right)/2$.
   (d) $\tan z = \sin z/\cos z$.

3. Show that the derivative of $f(z)$ does not exist for any z for each of the following:

   (a) $f(z) = \bar{z}$.
   (b) $f(z) = \operatorname{Re} z$.
   (c) $f(z) = \operatorname{Im} z$.

4. Write the following functions in the form $f(z) = u(x, y) + iv(x, y)$ and find the derivative for each:

   (a) $\cosh z$.
   (b) $\sinh z$.
   (c) $\log z$.

5. Prove *L'Hospital rule*: If $f(z)$ and $g(z)$ are analytic at z_0 and $f(z_0) = g(z_0) = 0$, but $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z_0)}{g(z_0)} = \frac{f'(z_0)}{g'(z_0)}.$$

   Find $\lim_{z \to i} \left(1 + z^6\right)/\left(1 + z^{10}\right)$ using L'Hospital rule.

6. Let $f(z) = z^3 + 1$, and let $z_1 = \left(-1 + i\sqrt{3}\right)/2$, $z_2 = \left(-1 - i\sqrt{3}\right)/2$. Show that there is no point w on the line segment from z_1 to z_2 such that

$$f(z_2) - f(z_1) = f'(w)(z_2 - z_1).$$

   This shows that the mean-value theorem does not extend to complex functions.

7. If $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic at z, then prove Cauchy-Riemann conditions

$$\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ u_\theta &= -r v_r \end{aligned}$$

   and that $f'(z) = e^{-i\theta}(u_r + iv_r)$.

8. Show that following functions are harmonic and find their harmonic conjugates. Find functions $f(z)$ of which the following are real parts.

   (a) y
   (b) xy
   (c) $\log(x^2 + y^2)$

9. If $f(z) = u(x, y) + iv(x, y)$, the equations $u(x, y) = c_1, v(x, y) = c_2$ where c_1 and c_2 are constants generate a family of curves in xy plane, namely, *level curves*.

(a) Find the normal vector to these level curves.

(b) Show that the two sets of level curves, one for u function and other for v function are orthogonal to each other if f is analytic.

10. $f(z) = z + 1/z$. Show that the level curve for $\text{Im}\, f(z) = 0$ consists of a real axis (excluding $z = 0$) and the circle $|z| = 1$.

11. Consider a wedge bounded by the nonnegative real axis and a line $y = x$ $(x \geq 0)$. Find a harmonic function $\phi(x, y)$ which is zero on the sides of the wedge but is not identically zero.