Tutorial 2: Elementary Functions and Mapping
PH 503 Mathematical physics

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1. Find for each function given below, the domain of definition:
   (a) $f(z) = \frac{1}{z^2+1}$;
   (b) $f(z) = Arg(\frac{1}{z})$;
   (c) $f(z) = \frac{z}{\bar{z}+z}$;
   (d) $f(z) = \frac{1}{1-|z|^2}$.

2. Write each of the following functions in the form $f(z) = u(x,y) + iv(x,y)$:
   (a) $f(z) = z^3 - 1$;
   (b) $f(z) = \sin z$;
   (c) $\mathrm{f}(z) = \log z$.

3. A line segment is given by $z_1(t) = (1, t)$ where $0 \le t \le 4\pi$.
   (a) Let $f(t) = \exp(z_1(t)) = u(t) + iv(t)$. Plot $u(t)$ and $v(t)$ as a function of t.
   (b) Do the same for $z_2(t) = (2, t)$ and $z_3 = (t, \pi/6)$.

4. Show that
   (a) $\sin^2 z + \cos^2 z = 1$;
   (b) $\sin^2(1+i) = 1.2828 + 1.6489i$ and $\cos^2(1+i) = -.2828 - 1.6489i$;
   (c) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$;
   (d) $\cosh^2 z - \sinh^2 z = 1$;
   (e) $\cosh^2(1+i) = -.2828 + 1.6489i$ and $\sinh^2(1+i) = -1.2828 + 1.6489i$.
   (f) $\log(z_1 z_2) = \log z_1 + \log z_2$.

5. Let $w = 1/z$ and $z = x + iy$.
   (a) Find u and v if $w = u + iv$.
   (b) Show that a curve in z-plane given by
   $$A(x^2 + y^2) + Bx + Cy + D = 0$$
   $(B^2 + C^2 > 4AD)$ transforms into a curve in w-plane given by
   $$D(u^2 + v^2) + Bu - Cv + A = 0$$
   (c) Show that a line, not passing through origin in z-plane, maps to a circle passing through origin in w-plane.
   (d) Find and sketch a level curve in z-plane for $u(x,y) = 5$.

6. Show that the lines $ay = x$ $(a \neq 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the function $w = \exp z$, where $w = \rho e^{i\phi}$.

7. Sketch the region onto which the sector $r \le 1$, $0 \le \theta \le \pi/4$ is mapped by transformation
   (a) $f(z) = z^2$;
   (b) $f(z) = z^3$;
   (c) $f(z) = z^4$.

8. A particle constrained to move in a two dimensional plane, where its coordinates can be given by a complex number z. It is acted upon by a central force $F(z) = f(|z|)z$. Derive the equations of motion

$$\begin{aligned} 2r'\theta' + r\theta'' &= 0 \\ r'' - r\left(\theta'\right)^2 &= \frac{r}{m} f\left(|z|\right) \end{aligned}$$

9. Find each of the following limits.

   (a) $\lim\limits_{z \to 2+3i} (z - 5i)^2$

   (b) $\lim\limits_{z \to 2} \frac{z^2+3}{iz}$

   (c) $\lim\limits_{z \to 3i} \frac{z^2+9}{z-3i}$

   (d) $\lim\limits_{z \to 1-i} \left[x + i\left(2x + y\right)\right]$

   (e) $\lim\limits_{z \to \pi i/2} \left(z + 1\right) e^z$

10. Show that the limit of the function $f\left(z\right) = \left(z/\bar{z}\right)^2$ as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin.

11. Find $f'(z)$ when

   (a) $f\left(z\right) = 3z^2 - 2z + 4$;

   (b) $f(z) = (1 - 4z^2)^3$;

   (c) $f\left(z\right) = \frac{z-1}{2z-1}$;