Tutorial 1: Complex Numbers
PH 503 Mathematical physics

1. Verify that
   (a) $\left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}i\right)=-2i$;
   (b) $(2,-3)\,(-2,1)=(-1,8)$;
   (c) $\frac{5}{(1-i)(2-i)(3-i)}=\frac{i}{2}$;
   (d) $(1-i)^4=-4$.
2. Verify that each of the two numbers $z=1\pm i$ satisfies the equation $z^2-2z+2=0$.
3. Solve the equation $z^2+z+1=0$ for $z=(x,y)$ by writing
$$(x,y)\,(x,y)+(x,y)+(1,0)=(0,0)$$
   and then solving a pair of simultaneous equations in x and y.
4. If $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$, then obtain an expression for z_1/z_2.
5. Prove that $\sqrt{2}\,|z|\geq|\operatorname{Re}z|+|\operatorname{Im}z|$.
6. In each case, sketch the set of points determined by the given condition:
   (a) $|z-1+i|=1$;
   (b) $|z+i|\leq 3$;
   (c) $\operatorname{Re}(\overline{z}-i)=2$;
   (d) $|2z-i|=4$.
7. If z lies on a circle $|z|=2$, then show that
$$\left|\frac{1}{z^4-4z^2+3}\right|\leq\frac{1}{3}.$$
8. Show that the equation $|z-z_0|=R$ of a circle, centered at z_0 with radius R, can be written as
$$|z|^2-2\operatorname{Re}(z\overline{z}_0)+|z_0|^2=R^2.$$
9. Show that the hyperbola $x^2-y^2=1$ can be written as
$$z^2+\overline{z}^2=2.$$
10. Using the fact that $|z_1-z_2|$ is the distance between two points z_1 and z_2 give a geometric argument that
    (a) the equation $|z-4i|+|z+4i|=10$ represents an ellipse;
    (b) the equation $|z-1|=|z+i|$ represents a line through origin whose slope is -1.
11. Find the principal argument $\arg z$ when
    (a) $z=\frac{-2}{1+\sqrt{3}i}$;
    (b) $z=\frac{i}{-2-2i}$;
    (c) $z=(\sqrt{3}-i)^6$.

12. By writing the individual factors on the left in exponential form, performing the needed operations, and finally converting back to rectangular coordinates, show that

   (a) $i\left(1-\sqrt{3}i\right)\left(\sqrt{3}+i\right)=2\left(1+\sqrt{3}i\right)$;
   (b) $5i/\left(2+i\right)=1+2i$;
   (c) $\left(-1+i\right)^7=-8\left(1+i\right)$;
   (d) $\left(1+\sqrt{3}i\right)^{-10}=2^{-11}\left(-1+\sqrt{3}i\right)$

13. Use de Moivre's formula to derive following trigonometric identities:

   (a) $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$;
   (b) $\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$.

14. Establish the identity

$$1+z+z^2+\cdots+z^n=\frac{1-z^{n+1}}{1-z}$$

   $(z\neq 1)$ and then use it to derive *Lagrange's trigonometric formula*:

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\frac{1}{2}+\frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}$$

   where $0<\theta<2\pi$.

15. Prove de Moivre's formula by mathematical induction

$$\left(\cos\theta+i\sin\theta\right)^n=\cos n\theta+i\sin n\theta$$

16. In each case, find all of the roots in rectangular coordinates, exhibit them geometrically, and point out which is the principal root:

   (a) $(-1)^{1/3}$;
   (b) $(16)^{1/4}$;
   (c) $(8)^{1/6}$;
   (d) $\left(-8-8\sqrt{3}i\right)^{1/4}$.

17. Show that if c is any nth root of unity other than itself, then

$$1+c+c^2+\cdots+c^{n-1}=0$$