Here are some practice problems in vector calculus and curvilinear coordinates.

1. Find equations for the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(2, -1, 5)$.
2. Find the unit outward normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point $(3, 1, -4)$.
3. Find the divergence and curl of $\hat{\mathbf{r}}/r^2$.
4. Prove:
   (a) $\nabla \cdot (\phi \mathbf{A}) = (\nabla \phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A})$.
   (b) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.
5. Show that $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ is a conservative force field. Find the scalar potential. Find the work done in moving an abject in this field from $(1, -2, 1)$ to $(3, 1, 4)$.
6. Given $\phi = 2xyz^2$and a curve $C(t) = (t^2, 2t, t^3)$ from $t = 0$ to $t = 1$. Find $\int_C \phi d\mathbf{r}$.
7. Evaluate $\int_S \mathbf{A} \cdot d\mathbf{S}$ where $\mathbf{A} = 18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}$ and $S$ is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.
8. Prove Green's theorem in a plane. (See any textbook)
9. For a given $R > 0$, define $r_a = \left(x^2 + y^2 + (z - R/2)^2\right)^{1/2}$ and $r_b = \left(x^2 + y^2 + (z + R/2)^2\right)^{1/2}$. The prolate ellipsoidal coordinates are defined as

$$\begin{aligned} \xi &= \frac{1}{R}(r_a + r_b) \\ \eta &= \frac{1}{R}(r_a - r_b) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

   Find inverse transformations, basis vectors $e_\xi, e_\eta, e_\phi$ , scale factors $h_\xi, h_\eta, h_\phi$ , differential vector $d\mathbf{r}$, length element $ds^2$ and volume element $dv$.

10. Let

$$\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & \frac{1}{2n} < x \end{cases}$$

   Show that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \delta_n(x)\, dx = f(0)$$

   assuming that the function $f$ is continuous at $x = 0$.

**Some Answers:**

1. Equation of the tangent plane: $-4x + 2y + z = 5$
   Equation of the normal line:
$$\frac{x-2}{-4} = \frac{y+1}{2} = \frac{z-5}{1}$$

2. Unit *outward* normal vector: $(2/3, 1/3/-2/3)$.

3. $\nabla \times \left(\frac{\mathbf{r}}{r^3}\right) = 0$ and $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = 4\pi\delta(\mathbf{r})$. Proof: Show by explicit differentiation that if $r \neq 0$ then $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = 0$. Now choose $\epsilon > 0$. Let $S$ be the spherical surface of radius $\epsilon$ enclosing the volume $V$. Then,
$$\int_v \nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) dv = \oint_S \frac{\mathbf{r}}{r^3} \cdot ds = \int d\Omega = 4\pi$$

4. Explicitly differentiate to prove these results.

5. Since $\nabla \times \mathbf{F} = 0$, $\mathbf{F}$ is conservative. Scalar potential is given by
$$\begin{aligned}\phi(x,y,z) &= -\int \mathbf{F} \cdot \mathbf{dr} \\ &= -\int F_x dx + \int (\text{Terms in y and z only}) dy + \int (\text{Terms in z only}) dz \\ &= -(x^2y + xz^3)\end{aligned}$$
   Thus work done from $(1,-2,1)$ to $(3,1,4)$ is $\phi(1,-2,1) - \phi(1,-2,1) = -202$.

6. $\int \phi d\mathbf{r} = (8/11, 8/10, 1)$

7. Note this general procedure: Suppose the given surface, say $\phi(x,y,z) = c$, has a one-one projection on some domain $D$ in $z = 0$ plane then the surface can also be expressed by equation $z = f(x,y)$. In this case,
$$\int_S \mathbf{A} \cdot \mathbf{ds} = \int_D \mathbf{A} \cdot \hat{\mathbf{n}} \frac{dx\, dy}{|\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}|}.$$
   Where $\hat{\mathbf{n}}$ is the unit normal to the surface. For our example, (See diagram) $S$ is the trangular section of the of the plane in the first octant. $D$ is its projection on $XY$ plane and is bounded by $x$ axis, $y$ axis and the line $2x + 3y = 12$. $\hat{\mathbf{n}} = (2,3,6)/7$. Thus required integral is 24.

8. See Arfken.

9. For a given $R > 0$, define $r_a = \left(x^2 + y^2 + (z - R/2)^2\right)^{1/2}$ and $r_b = \left(x^2 + y^2 + (z + R/2)^2\right)^{1/2}$. The prolate ellipsoidal coordinates are defined as
$$\begin{aligned}\xi &= \frac{1}{R}(r_a + r_b) \\ \eta &= \frac{1}{R}(r_a - r_b) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right)\end{aligned}$$

The inverse transformations are given by

$$\begin{aligned} x &= \frac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi \\ y &= \frac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi \\ z &= -\frac{R}{2}\xi\eta \end{aligned}$$

The differentials are related by

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \left(\frac{R}{2}\right)\xi\alpha\cos\phi & -\left(\frac{R}{2}\right)\eta\alpha^{-1}\cos\phi & -\left(\frac{R}{2}\right)\beta\sin\phi \\ \left(\frac{R}{2}\right)\xi\alpha\sin\phi & -\left(\frac{R}{2}\right)\eta\alpha^{-1}\sin\phi & \left(\frac{R}{2}\right)\beta\cos\phi \\ -\left(\frac{R}{2}\right)\eta & -\left(\frac{R}{2}\right)\xi & 0 \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \\ d\phi \end{pmatrix}$$

where $\alpha = \sqrt{\frac{1-\eta^2}{\xi^2-1}}$ and $\beta = \sqrt{(\xi^2-1)(1-\eta^2)}$
The basis vectors are given by the three columns of the matrix given above. The system is orthogonal. The differential vector is given by

$$ds = \left(\frac{R}{2}\right)\sqrt{\frac{(\xi^2-\eta^2)}{(\xi^2-1)}}\mathbf{e}_\xi d\xi + \left(\frac{R}{2}\right)\sqrt{\frac{(\xi^2-\eta^2)}{(1-\eta^2)}}\mathbf{e}_\eta d\eta + \left(\frac{R}{2}\right)\sqrt{(\xi^2-1)(1-\eta^2)}\mathbf{e}_\phi d\phi$$

The volume element is given by

$$dv = \left(\frac{R}{2}\right)^3\left(\xi^2-\eta^2\right)d\xi d\eta d\phi$$

10. Let

$$S_n = \int_{-\infty}^{\infty} f(x)\delta_n(x)\,dx = n\int_{-1/2n}^{1/2n} f(x)\,dx$$

Let $\epsilon > 0$ be any infinitesimally small number. Since $f$ is continuous at $x = 0$, $\exists\,\delta > 0$ such that $\forall\,|x| < \delta$, $|f(x) - f(0)| < \epsilon$. Now choose an integer $N$ such that $\frac{1}{2N} < \delta$. Then, for every $n > N$, clearly, $|x| < \frac{1}{2n} \Rightarrow |x| < \frac{1}{2n} < \frac{1}{2N} < \delta$

$$\begin{aligned} |S_n - f(0)| &= \left| n\int_{-1/2n}^{1/2n} f(x)\,dx - f(0)\right| \\ &= \left| n\int_{-1/2n}^{1/2n} (f(x) - f(0))\,dx\right| \\ &\le n\int_{-1/2n}^{1/2n} |f(x) - f(0)|\,dx \\ &\le n\cdot\frac{1}{n}\cdot\epsilon = \epsilon \end{aligned}$$

Thus, $S_n \to f(0)$ as $n \to \infty$. Here it is also true that for a given $x \neq 0$, $\delta_n(x) \to \delta(x)$ as $n \to 0$. Remember, such pointwise convergence is not necessary in the definition of Dirac delta function. For example,

$$\delta_n(x) = \frac{\sin nx}{\pi x}$$

converges to delta function as $n \to \infty$ (that is $S_n \to f(0)$) , however for a given $x$, $\delta_n(x)$ is oscillatory.