1. Given $\sigma(\theta, \phi) = P\cos\theta$.

   (a) Let $\mathbf{x} = d\mathbf{k}$. Then the potential is given by

$$\begin{aligned}
\phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}dS' \\
&= \frac{P}{4\pi\epsilon_0}\int \frac{\cos\theta'}{\sqrt{R^2+d^2-2Rd\cos\theta'}}R^2\sin\theta'\,d\theta'\,d\phi' \\
&= \frac{PR^2}{2\epsilon_0}\int_{-1}^{1}\frac{t\,dt}{\sqrt{R^2+d^2-2Rdt}} \qquad \text{where } t=\cos\theta' \\
&= \frac{Pd}{3\epsilon_0} \qquad \text{if } d<R \\
&= \frac{PR^3}{3\epsilon_0 d^2} \qquad \text{if } d>R
\end{aligned}$$

   (b) Comparing this form with the general form of solution, we get

$$\phi(r,\theta,\phi) = \begin{cases} \frac{P}{3\epsilon_0}r\cos\theta & r<R \\ \frac{PR^3}{3\epsilon_0 r^2}\cos\theta & r>R \end{cases}$$

   (c) Electric Field is given by

$$\mathbf{E}(r,\theta,\phi) = \begin{cases} \frac{P}{3\epsilon_0}\left(\hat{\mathbf{r}}\cos\theta - \hat{\theta}\sin\theta\right) = \frac{P}{3\epsilon_0}\hat{\mathbf{k}} & r<R \\ -\frac{PR^3}{3\epsilon_0 r^3}\left(2\hat{\mathbf{r}}\cos\theta + \hat{\theta}\sin\theta\right) & r>R \end{cases}$$

   (d) Electrostatic energy is given by

$$\begin{aligned}
W &= \frac{1}{2}\int \sigma(\mathbf{x})\phi(\mathbf{x})dS \\
&= \frac{P^2R^3}{6\epsilon_0}\int \cos^2\theta\sin\theta\,d\theta\,d\phi \\
&= \frac{2\pi P^2R^3}{9\epsilon_0}
\end{aligned}$$

2. Given: $2ql = P$

   (a) Now $q_1 = -qR/(L+l)$ and $q_2 = qR/(L-l)$ and $d_1 = R^2/(L+l)$ and $d_2 = R^2/(L-l)$.

   (b) $Q' = q_1 + q_2 = 2qlR/(L^2-l^2)$ and $P' = 2qlR^3L/(L^2-l^2)^2$.

   (c) Limiting values are $Q' = PR/L^2$ and $P' \to PR^3/L^3$.

3. Pipe with square cross section

(a) The general solution of Laplace equation for the given geometry and boundary condtions at $y = 0$, $y = a$ and at $x = 0$, is

$$\phi(x,y) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\frac{m\pi x}{a}\right)$$

Applying the fourth boundary condition, we get

$$\begin{aligned}
V_0 &= \sum_{m=1}^{\infty} C_m \sin\left(\frac{m\pi y}{a}\right) \sinh(m\pi) \\
\Longrightarrow C_m \sinh(m\pi)\frac{a}{2} &= \int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right) dy \\
&= \frac{V_0 a}{m\pi} \left(\cos(m\pi) - 1\right) \\
\Longrightarrow C_m &= \frac{2V_0}{m\pi \sinh(m\pi)}(-2) \qquad \text{odd } m \\
&= 0 \qquad \text{even } m
\end{aligned}$$

Finally the potential is given by

$$\phi(x,y) = -\frac{4V_0}{\pi} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m \sinh(m\pi)} \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\frac{m\pi x}{a}\right)$$

(b) Charge denstity can be found by

$$\begin{aligned}
\sigma(y) &= -\epsilon_0 \frac{\partial}{\partial x}\phi(0,y) \\
&= +\frac{4\epsilon_0 V_0}{a} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{\sinh(m\pi)} \sin\left(\frac{m\pi y}{a}\right)
\end{aligned}$$

The net charge is

$$\begin{aligned}
Q &= \int_0^a \sigma(y) dy \\
&= \frac{4\epsilon_0 V_0}{a} \sum_{m=1}^{\infty} \frac{1}{\sinh(m\pi)} \int_0^a \sin\left(\frac{m\pi y}{a}\right) dy \\
&= -\frac{8\epsilon_0 V_0}{\pi} \sum_{m=1}^{\infty} \frac{1}{m \sinh(m\pi)} \\
&= -\frac{\epsilon_0 V_0}{\pi} \ln(2)
\end{aligned}$$