Time Dependent Perturbation Theory

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1 Time Evolution of a Quantum System

The time evolution of a quantum system is given by Schrödinger equation

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi(t) &= \mathbf{H}(t)\Psi(t) \\ \Psi(t_0) &= \Psi_0. \end{aligned} \tag{1}$$

Definition 1 *A time evolution operator $T_{\mathbf{H}}(t, t_0)$ for a system described by a Hamiltonian $\mathbf{H}$, is defined as*

$$T_{\mathbf{H}}(t, t_0)\Psi_0 = \Psi(t) \tag{2}$$

where $\Psi(t)$ is solution of Equation 1.

Clearly, time evolution operator must obey

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}T_{\mathbf{H}}(t, t_0) &= \mathbf{H}(t)T_{\mathbf{H}}(t, t_0) \\ T_{\mathbf{H}}(t_0, t_0) &= I. \end{aligned} \tag{3}$$

Some trivial properties are

$$T_{\mathbf{H}}(t, t) = 1 \quad \forall t \tag{4}$$

$$[T_{\mathbf{H}}(t, t_0)]^{-1} = T_{\mathbf{H}}(t_0, t) \tag{5}$$

$$T_{\mathbf{H}}(t + \epsilon, t) = I - \frac{i}{\hbar}\epsilon\mathbf{H} \tag{6}$$

We will drop the subscript $\mathbf{H}$ without loss of clarity.

Let $\mathbf{L}$ be an operator corresponding to an obseravable and $\mathbf{L}\,|L_i\rangle = L_i\,|L_i\rangle$. The probability amplitudes for an outcome L_i at time t is given by

$$\langle L_i\,|\Psi(t)\,\rangle = \langle L_i\,|T(t, t_0)\Psi_0\,\rangle \tag{7}$$

$$= \sum_j \langle L_i\,|T(t, t_0)L_j\,\rangle\,\langle L_j\,|\Psi_0\,\rangle \tag{8}$$

Typically, one needs to find the matrix elements of the time evolution operator. In general, this may be very difficult problem. Some general theory using Green's function is discussed later. However, for time independent Hamiltonian, the form of time evolution operator is easy to obtain.

Theorem 1 *If the Hamiltonian $\mathbf{H}$ of a system is time independent, then*

$$T_{\mathbf{H}}(t, t_0) = \exp\left(-\frac{i\mathbf{H}t}{\hbar}\right) \tag{9}$$

In this case, if $\mathbf{H}\,|n\rangle = \epsilon_n\,|n\rangle$, then

$$T_{\mathbf{H}}(t,t_0) = \sum_n |n\rangle \exp\left(-\frac{i\epsilon_n t}{\hbar}\right)\langle n| \tag{10}$$

and

$$\Psi(t) = T_{\mathbf{H}}(t,t_0)\Psi_0 = \sum_n |n\rangle \exp\left(-\frac{i\epsilon_n t}{\hbar}\right)\langle n\,|\Psi_0\,\rangle \tag{11}$$

Example 1

Consider a particle trapped in a 1D box of width π. Let $\hbar = m = 1$. Energy eigenfunctions are $|n\rangle = \sqrt{2/\pi}\,\sin(nx)$ with eigenvalues $\epsilon_n = n^2/2$. Let at $t = 0$

$$\Psi(0) = \Psi_0 = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \tag{12}$$

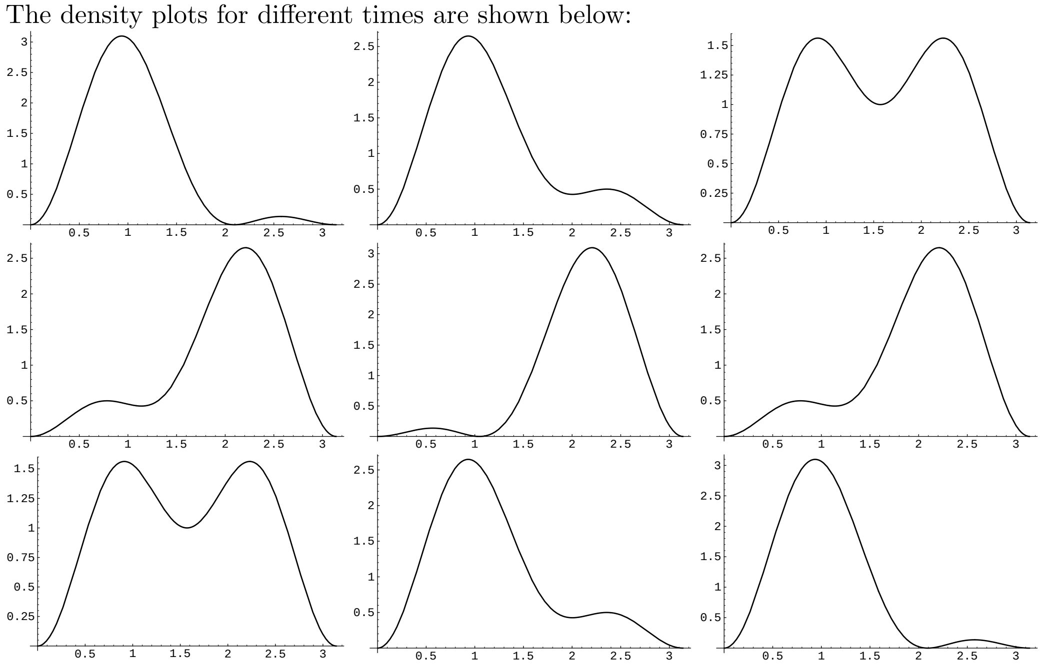
Then,

$$\Psi(t) = \frac{1}{\sqrt{2}}\left[\exp\left(-it/2\right)|1\rangle + \exp\left(-2it\right)|2\rangle\right] \tag{13}$$

Now the average values position and momentum are functions of time

$$\begin{aligned}\langle\mathbf{x}\rangle &= \frac{\pi}{2} - \frac{16}{9\pi}\cos\left(\frac{3}{2}t\right)\\ \langle\mathbf{p}\rangle &= \frac{8}{3\pi}\sin\left(\frac{3}{2}t\right)\end{aligned}$$

The density plots for different times are shown below:



2 Interaction Picture

If the form of the Hamiltonian is

$$\mathbf{H} = \mathbf{H}_0 + V(t) \tag{14}$$

where $\mathbf{H}_0$ is time independent, one can formulate a perturbation theory if $V(t)$ is weak. Let $\mathbf{H}_0 |n\rangle = \epsilon_n |n\rangle$. Let us define (put $t_0 = 0$)

$$\begin{aligned} \bar{\Psi}(t) &= \exp\left(\frac{i\mathbf{H}_0 t}{\hbar}\right)\Psi(t) & (15) \\ \bar{V}(t) &= \exp\left(\frac{i\mathbf{H}_0 t}{\hbar}\right)V(t)\exp\left(-\frac{i\mathbf{H}_0 t}{\hbar}\right) & (16) \end{aligned}$$

Now,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\bar{\Psi}(t) &= i\hbar\frac{\partial}{\partial t}\exp\left(\frac{i\mathbf{H}_0 t}{\hbar}\right)\Psi(t) \\ &= \exp\left(\frac{i\mathbf{H}_0 t}{\hbar}\right)\left(-\mathbf{H}_0 + i\hbar\frac{\partial}{\partial t}\right)\Psi(t) \\ &= \exp\left(\frac{i\mathbf{H}_0 t}{\hbar}\right)(V(t))\exp\left(\frac{i\mathbf{H}_0 t}{\hbar}\right)\exp\left(-\frac{i\mathbf{H}_0 t}{\hbar}\right)\Psi(t) \\ \Rightarrow i\hbar\frac{\partial}{\partial t}\bar{\Psi}(t) &= \bar{V}(t)\bar{\Psi}(t) \end{aligned}$$

Time evolution operator for the barred quantities can be written, by analogy with Equation 2,

$$\bar{T}(t, t_0)\Psi_0 = \bar{\Psi}(t) \tag{17}$$

and

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\bar{T}(t, t_0) &= \bar{V}(t)\bar{T}(t, t_0) \\ \bar{T}(t_0, t_0) &= I. & (18) \end{aligned}$$

A formal solution can be obtained by integrating this equation

$$\bar{T}(t, t_0) = I - \frac{i}{\hbar}\int_{t_0}^{t}\bar{V}(t')\bar{T}(t', t_0)dt' \tag{19}$$

Iterating, one can obtain formal power series in terms of V

$$\bar{T}(t, t_0) = I - \frac{i}{\hbar}\int_{t_0}^{t}\bar{V}(t')dt' + \left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t}\bar{V}(t')dt'\int_{t_0}^{t'}\bar{V}(t'')dt'' + \cdots \tag{20}$$

In case V is weak, in some sense, one may truncate the series to the first order in V. Matrix elements of $\bar{T}$ are given by

$$\begin{aligned} \left\langle n\left|\bar{T}(t)\right|s\right\rangle &= \delta_{n,s} - \frac{i}{\hbar}\int_{t_0}^{t}\left\langle n\left|\bar{V}(t')\right|s)\right\rangle dt' & (21) \\ &= \delta_{n,s} - \frac{i}{\hbar}\int_{t_0}^{t}e^{i\omega_{ns}t}\left\langle n\left|V(t')\right|s)\right\rangle dt' & (22) \end{aligned}$$

where $\omega_{ns} = (\epsilon_n - \epsilon_s)/\hbar$.