Given Hamiltonian

$$\mathbf{H} = A\mathbf{L}^2 + B\mathbf{L}_z + C\mathbf{L}_y \tag{1}$$

can be diagonalized by a rotation operator. Let $\theta = \tan^{-1}(C/B)$ and $\mathbf{U} = \exp(-i\mathbf{L}_x\theta/\hbar)$.

$$\begin{aligned} \mathbf{U}^\dagger\mathbf{L}_z\mathbf{U} &= \mathbf{L}_z\cosh(i\theta) + (-i\mathbf{L}_y)\sinh(i\theta) && (2)\\ &= \mathbf{L}_z\cos(\theta) + \mathbf{L}_y\sin(\theta) && (3)\\ &= \frac{B}{\sqrt{B^2+C^2}}\mathbf{L}_z + \frac{C}{\sqrt{B^2+C^2}}\mathbf{L}_y && (4) \end{aligned}$$

Let $\mathbf{H'} = A\mathbf{L}^2 + \sqrt{B^2+C^2}\mathbf{L}_z$. Then it is easy to see that $\mathbf{U}^\dagger\mathbf{H'}\mathbf{U} = \mathbf{H}$. Then eigenvalues of $\mathbf{H}$ and $\mathbf{H'}$ are same and are given by $E_{lm} = Al(l+1)\hbar^2 + \sqrt{B^2+C^2}m\hbar$. Eigenfunctions of $\mathbf{H}$ are given by

$$\begin{aligned} \mathbf{U}^\dagger|lm\rangle &= \exp(i\mathbf{L}_x\theta/\hbar)|lm\rangle && (5)\\ &= \sum_{m'}|lm'\rangle\left[\delta_{m,m'} - \frac{i\theta}{2}\sqrt{(l-m)(l+m+1)}\delta_{m',m+1} - \frac{i\theta}{2}\sqrt{(l+m)(l-m+1)}\delta_{m',m-1}\right] && (6)\\ &= |lm\rangle - \frac{i\theta}{2}\sqrt{(l-m)(l+m+1)}|l,m+1\rangle - \frac{i\theta}{2}\sqrt{(l+m)(l-m+1)}|l,m-1\rangle && (7) \end{aligned}$$

Some useful formulae:

$$\exp(\lambda A)B\exp(-\lambda A) = B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \cdots \tag{8}$$

If $[A,[A,B]] = \beta B$ then

$$\exp(\lambda A)B\exp(-\lambda A) = B\cosh(\lambda\sqrt{\beta}) + [A,B]\sinh(\lambda\sqrt{\beta})/\sqrt{\beta} \tag{9}$$