1. **A particle in a two dimensional box of sides a. If a perturbation**

$$V' = \lambda xy$$

is applied, find the change in the energy of the ground state and the first excited state.

Unperturbed Wavefunctions and energies:

$$\psi_{m,n}(x,y) = \frac{2}{a}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \quad (1)$$

$$E^{(0)}_{m,n} = \frac{\hbar^2\pi^2}{2ma^2}\left(m^2+n^2\right) \quad (2)$$

where $m, n = 1, 2, \ldots$.

The ground state corresponds to $m = 1$ and $n = 1$ and is non-degenerate. First order correction:

$$E^{(1)}_{1,1} = \langle\psi_{1,1}|\lambda xy|\psi_{1,1}\rangle \quad (3)$$

$$= \lambda\left(\frac{2}{a}\right)^2\int_0^L x\sin^2\left(\frac{\pi x}{a}\right)dx\int_0^L y\sin^2\left(\frac{\pi y}{a}\right)dy \quad (4)$$

$$= \frac{\lambda a^2}{4} \quad (5)$$

The first excited level is two-fold degenerate. The two degenerate states are $\psi_{1,2}$ and $\psi_{2,1}$. The perturbation matrix in this subspace is

$$\lambda\begin{pmatrix} \frac{a^2}{4} & \left(\frac{8a}{9\pi^2}\right)^2 \\ \left(\frac{8a}{9\pi^2}\right)^2 & \frac{a^2}{4}\end{pmatrix} \quad (6)$$

After diagonalising the matrix, we get first order correction to energy

$$\lambda a^2\left(\frac{1}{4}\pm\left(\frac{8}{9\pi^2}\right)^2\right)$$

2. **A rotator whose orientation is specified by the angular coordinates θ and ϕ performs a *hindered rotation* described by the Hamiltonian**

$$H = A\mathbf{L}^2 + B\hbar^2\cos 2\phi$$

with $A >> B$. Calculate the S, P and D energy levels of this system in the first order perturbation theory, and work out unperturbed energy eigenfunctions.

Let $H_0 = A\mathbf{L}^2$ and $H_1 = B\hbar^2\cos 2\phi$. Eigenstates of H_0 are $Y_{lm}(\theta,\phi)$ with energies $E_l = A\hbar^2 l(l+1)$ and degeneracy $\left(2\bar{l}+1\right)$. $Y_{lm}(\theta,\phi) = P_{lm}(\cos\theta)\,e^{im\phi}$.

$$(Y_{lm}, \cos 2\phi Y_{lm'}) = \int P_{lm}(\cos\theta)\,P_{lm'}(\cos\theta)\,d\cos\theta\int e^{i(m'-m)\phi}\cos 2\phi d\phi$$

$$= C\delta_{m,m'\pm 2}$$

For S state, there is no first order correction. For P states, The hamiltonian matrix is

$$\begin{pmatrix} 0 & 0 & B\hbar^2 a \\ 0 & 0 & 0 \\ B\hbar^2 a & 0 & 0 \end{pmatrix} \quad (7)$$

where $a = \int Y^*_{1-1}Y_{11}d\Omega = -\frac{3}{8\pi}\int_0^\pi \sin^3\theta d\theta\int_0^{2\pi}\cos(2\phi)\,e^{2i\phi}d\phi = -\frac{1}{2}$. Hence the P level splits into three levels.

3. **For a particle of mass m moving in the potential**

$$V = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 + \lambda xy$$

Find the approximate expressions for energy of the ground state and the first excited state.

Unperturbed system is isotropic harmonic oscillator. Energies are given by $(n_x + n_y + 1)\hbar\omega$. Corresponding eigenstates are denoted by $|n_x, n_y\rangle$. Perturbation is

$$H' = \lambda xy = \frac{\hbar\lambda}{2m\omega}(a_x + a_x^\dagger)(a_y + a_y^\dagger)$$

Ground state is non-degenerate. First order correction is zero. Excited state is two-fold degenerate. The states are $|0, 1\rangle$ and $|1, 0\rangle$. The perturbation matrix is

$$\begin{pmatrix} 0 & \frac{\hbar\lambda}{2m\omega} \\ \frac{\hbar\lambda}{2m\omega} & 0 \end{pmatrix}$$

Perturbed energies are then $\pm\frac{\hbar\lambda}{2m\omega}$.

4. **A particle of mass m and a charge q is placed in a box of sides (a, a, b), where $b < a$. A weak electric field**

$$\vec{E} = \mathcal{E}(y/a, x/a, 0)$$

is applied. Find the energy of the ground state and the first excited state.

The potential energy due to applied electric field is $-q\mathcal{E}xy/a$. Unperturbed energies are

$$E_{l,m,n} = \frac{\hbar^2\pi^2}{2m}\left(\frac{l^2 + m^2}{a^2} + \frac{n^2}{b^2}\right) \tag{8}$$

Ground state is nondegenerate. First excited state is two-fold degerate. What is the degeneracy of the second excited state? Solution is simillar to that of problem 1.

5. **Find the shift in the ground state energy of a 3D harmonic oscillator due to relativistic correction to the kinetic energy.**

Relativistic Correction: $H' = -p^4/(8m^3c^2)$. Ground state wavefunction is

$$\psi_{000}(\vec{r}) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{3/2} e^{-\alpha^2 r^2/2}$$

where $\alpha = \sqrt{m\omega/\hbar}$. The correction term is given by

$$\begin{aligned}
E_n^{(2)} &= -\frac{1}{8m^3c^2}\langle p^4\rangle && (9)\\
&= -\frac{1}{2mc^2}\langle (p^2/(2m))^2\rangle && (10)\\
&= -\frac{1}{2mc^2}\langle (H_0 - (1/2)m\omega^2 r^2)^2\rangle && (11)\\
&= -\frac{1}{2mc^2}((E_0^{(0)})^2 - E_0^{(0)}m\omega^2\langle r^2\rangle + (1/4)m^2\omega^4\langle r^4\rangle) && (12)
\end{aligned}$$

Using

$$\begin{aligned}
\langle r^2\rangle &= \left(\frac{\alpha}{\sqrt{\pi}}\right)^3 \int_0^\infty r^2 e^{-\alpha^2 r^2}(4\pi r^2 dr) = \frac{3}{2\alpha^2} && (13)\\
\langle r^4\rangle &= \left(\frac{\alpha}{\sqrt{\pi}}\right)^3 \int_0^\infty r^4 e^{-\alpha^2 r^2}(4\pi r^2 dr) = \frac{15}{4\alpha^4} && (14)
\end{aligned}$$

We get, $E_n^{(2)} = (7/32)(\hbar\omega)^2/(mc^2)$

6. **If the general form of a spin-orbit coupling for a particle of mass m and spin S moving in a central potential $V(r)$ is**

$$H_{SO} = \frac{1}{2m^2c^2}\mathbf{S}\cdot\mathbf{L}\frac{1}{r}\frac{dV(r)}{dr},$$

what is the effect of the coupling on the spectrum of 3D harmonic oscillator?

For HO, the perturbation is

$$H_{SO} = \frac{\omega^2}{2mc^2}\mathbf{S}\cdot\mathbf{L}$$

We can solve the HO in spherical polar coordinates (See Mathews and Venkatesan pg:142). The energy eigenvalues are given by $E_{nlm} = (n+3/2)\hbar\omega$, where $n = 0, 1, \ldots$, $l = n, n-2, \ldots$. The corresponding wavefunctions are $\psi_{nlm} = R_{nl}Y_{lm}\chi$, where χ is a spin state. Here we can use the states $|nlsjm>_J$. Alternatively, use nondegenerate perturbation theory. The correction is given by

$$E^{(1)}_{nlsjm} = \frac{\hbar^2\omega^2}{2mc^2}\begin{cases} \frac{l}{2} & \text{if} \quad j = l+1/2 \\ -\frac{l+1}{2} & \text{if} \quad j = l-1/2 \end{cases}$$