1. A modified infinite well potential is given by

$$V(x) = \left\{ \begin{array}{lll} \epsilon x & \text{if} & 0 \le x \le b \\ \infty & \text{otherwise} & \end{array} \right| \tag{1}$$

Obtain approximate energy eigenvalues to the first order in ϵ. Also find the second order correction to the ground state energy.

Solution

$$E_n^{(1)} = \frac{\epsilon b}{2} \tag{2}$$

$$E_0^{(2)} = \left(\frac{\epsilon b}{\pi}\right)^2 \frac{2mb^2}{\hbar^2 \pi^2} \sum_{n=2}^{\infty} \frac{8n}{(4n^2-1)^3} \tag{3}$$

$$= \frac{8mb^2\epsilon^2}{\hbar^2\pi^2} \left(-\frac{3}{4} + \frac{7}{8}\zeta(3)\right) \tag{4}$$

$$= 0.3018 \frac{8mb^2\epsilon^2}{\hbar^2\pi^2} \tag{5}$$

2. The bottom of an infinite well is changed to have the shape

$$V(x) = \epsilon \sin\frac{\pi x}{b} \qquad 0 \le x \le b \tag{6}$$

Calculate the energy shifts for all the states to first order in ϵ.

Solution

$$E_n^{(1)} = \frac{\epsilon}{\pi} \frac{8n^2}{4n^2-1} \tag{7}$$

3. A simple harmonic oscillator is perturbed by a constant force. Find the energy eigenvalues. Calculate second order perturbation correction to the energy values and compare with the exact answer.

Solution

$$E_n^{(2)} = -\frac{\lambda^2}{2m\omega^2} \tag{8}$$

Same as exact answer.

4. A simple harmonic oscillator is perturbed by a potential $V(x) = \lambda x^4$. Show that the perturbation can be written as

$$V(x) = \lambda \left(\frac{\hbar}{2m\omega}\right)^2 \left(a^4 + a^2(4\hat{n} - 2) + (6\hat{n}^2 + 6\hat{n} + 3) + {a^\dagger}^2(4\hat{n} + 6) + {a^\dagger}^4\right) \tag{9}$$

where $\hat{n} = a^\dagger a$. Obtain the first order correction to the energies. Discuss validity of this approximation for states with large $\hat{n}$ eigenvalues.

Solution

$$E_n^{(1)} = \lambda \left(\frac{\hbar}{2m\omega}\right)^2 \left(6n^2 + 6n + 3)\right) \tag{10}$$

This is valid only if $E_n^{(1)} << E_n^{(0)}$.

5. The Hamiltonian of a rigid rotator in magnetic field is of the form $A\mathbf{L}^2 + BL_z + CL_y$, if quadratic terms are neglected. Obtain exact eigenvalues and eigenfunctions of this Hamiltonian. Assuming $B \gg C$, use second order perturbation theory to obtain approximate values of energy and compare with the exact answers.

   Solution

   Exact energies are

   $$E_{l,m} = Al(l+1)\hbar^2 + \sqrt{(B^2 + C^2)}m\hbar$$

   Second order correction by perturbation method

   $$E_{l,m}^{(2)} = \left(\frac{C^2}{2B^2}\right) Bm\hbar$$

6. A charged particle is constrained to move on a spherical shell in a weak uniform electric field. Obtain the energy spectrum to the second order in the field strength. (Is it ok to use non-degenerate perturbation theory?)

   Solution

   If perturbation is $B\cos\theta$ then

   $$E_{l=0,m=0}^{(2)} = -\frac{B^2}{6A\hbar^2} \tag{11}$$
   $$E_{1,0}^{(2)} = \frac{B^2}{10A\hbar^2} \tag{12}$$
   $$E_{1,\pm}^{(2)} = -\frac{B^2}{20A\hbar^2} \tag{13}$$
   $$\tag{14}$$

7. In hydrogenic atoms, assume that the nucleus is uniformly charged sphere of radius R. Calculate the energy shift for $n = 1$ and $n = 2$ states.

   Solution

   The electrostatic potential due to the nucleus is

   $$V(r) = \frac{Ze^2}{4\pi\epsilon_0 2R}\left(\frac{r^2}{R^2} - 3\right) \qquad r \le R \tag{15}$$
   $$= -\frac{Ze^2}{4\pi\epsilon_0 r} \qquad r > R \tag{16}$$

The first order correction due to the perturbation is

$$\Delta E = \frac{Ze^2}{4\pi\epsilon_0 2R}\int_0^R [R_{nl}(r)]^2\left(\frac{r^2}{R^2}+\frac{2R}{r}-3\right)r^2\,dr \tag{17}$$

$$= \frac{Ze^2}{(4\pi)^2\epsilon_0 2R}\left(\frac{a^2}{R^2}-3+\frac{2R}{a}+3\frac{(a+R)^2}{R^2}e^{-2R/a}\right) \tag{18}$$

8. Find out the energy shifts due to linear Stark effect in $n = 3$ state of hydrogen atom.

Solution

$n = 3$ level splits into 5 equispaced lines with separation given by $(9/2)e\mathcal{E}\dashv$, where $\mathcal{E}$ is external electric field and a is Bohr radius.

9. Consider a Hamiltonian of the form

$$\mathbf{H} = \begin{pmatrix} E_0 & 0 \\ 0 & -E_0 \end{pmatrix} + \lambda \begin{pmatrix} \alpha & U \\ U^* & \beta \end{pmatrix} \tag{19}$$

Find the energy shift to first and second order in λ. Compare your results with exact eigenvalues.

Solution

Exact solution:

$$\frac{\lambda}{2}(\alpha+\beta) \pm E_0\left(1+\frac{\lambda(\alpha-\beta)}{E_0}+\frac{\lambda^2((\alpha-\beta)^2/4-|U|^2)}{E_0^2}\right)^{1/2} \tag{20}$$

Approximate Solution:

$$E_0+\lambda\alpha+\frac{\lambda^2|U|^2}{2E_0} \tag{21}$$

$$-E_0+\lambda\beta+\frac{\lambda^2|U|^2}{2E_0} \tag{22}$$

10. The perturbing hamiltonian for Hydrogen atom in constant magnetic field is given by

$$V = -\frac{e}{2mc}\mathbf{B}\cdot(\mathbf{L}+\mathbf{S}) \tag{23}$$

Show that, in this case a level of given total angular momentum quantum number j splits in $(2j+1)$ levels according to the formula $E^{(1)}_{jm} = -g_j\mu Bm$ where g_j is Lande's factor.