In non-relativistic quantum mechanics, the wave function of an electron is a pair of two complex functions. Usually it is written as a column matrix

$$\Psi(r,t) = \begin{bmatrix} \psi_\uparrow(r,t) \\ \psi_\downarrow(r,t) \end{bmatrix} \tag{1}$$

The probability density function is given by $\Psi^\dagger(r,t)\Psi(r,t)$. The probability density of finding the particle at r with spin up is given by $|\psi_\uparrow(r,t)|^2$ etc.

The spin operator is given by $\vec{\mathbf{S}} = \frac{1}{2}\hbar\vec{\sigma}$, where $\vec{\sigma}$ are pauli matrices. $S^2 = \frac{3}{4}\hbar^2$ is a constant operator. The eigenfunctions of S_z operator are

$$\begin{bmatrix} \psi_\uparrow(r,t) \\ 0 \end{bmatrix} \quad \text{and} \begin{bmatrix} 0 \\ \psi_\downarrow(r,t) \end{bmatrix}$$

with eigenvalues $\hbar/2$ and $-\hbar/2$ respectively, where $\psi_\uparrow$ and $\psi_\downarrow$ are arbitrary functions.

Angular momentum operator must be promoted to the matrix form. It can be written as

$$\vec{\mathbf{L}} \otimes I_2 = \begin{bmatrix} \vec{\mathbf{L}} & 0 \\ 0 & \vec{\mathbf{L}} \end{bmatrix}$$

We will continue to denote this operator by simply $\vec{\mathbf{L}}$. It is possible to find a set of simultaneous eigenfunctions of operators L^2, L_z, S^2 and S_z. The set is given below:

Ψ	L^2	L_z	S^2	S_z
$\begin{bmatrix} Y_{lm}(\theta,\phi) \\ 0 \end{bmatrix}$	$l(l+1)\hbar^2$	$m\hbar$	$\frac{3}{4}\hbar^2$	$\frac{1}{2}\hbar$
$\begin{bmatrix} 0 \\ Y_{lm}(\theta,\phi) \end{bmatrix}$	$l(l+1)\hbar^2$	$m\hbar$	$\frac{3}{4}\hbar^2$	$-\frac{1}{2}\hbar$

The quantity of interest is the total angular momentum which is defined as $\vec{\mathbf{J}} = \vec{\mathbf{L}} + \vec{\mathbf{S}}$. Now the mutually commuting set of operators is L^2, S^2, J^2 and J_z. The operator J_z is given by

$$J_z = \begin{bmatrix} L_z + \frac{1}{2}\hbar & 0 \\ 0 & L_z - \frac{1}{2}\hbar \end{bmatrix} \tag{2}$$

and the J^2 operator is given by

$$J^2 = \begin{bmatrix} L^2 + \frac{3}{4}\hbar^2 + \frac{1}{2}\hbar L_z & \frac{1}{2}\hbar L_- \\ \frac{1}{2}\hbar L_+ & L^2 + \frac{3}{4}\hbar^2 + \frac{1}{2}\hbar L_z \end{bmatrix} \tag{3}$$

We can easily verify that the simultaneous eigenfunctions of L^2, S^2, J^2 and J_z

Ψ	L^2	S^2	J^2	J_z	
$\frac{1}{\sqrt{2l+1}} \begin{bmatrix} \sqrt{l+m+1/2}\,Y_{lm-1/2} \\ \sqrt{l-m+1/2}\,Y_{lm+1/2} \end{bmatrix}$	$l(l+1)\hbar^2$	$\frac{3}{4}\hbar^2$	$(l+\frac{1}{2})(l+\frac{3}{2})\hbar^2$	$m\hbar$	
	$(j-\frac{1}{2})(j+\frac{1}{2})\hbar^2$	$\frac{3}{4}\hbar^2$	$j(j+1)\hbar^2$	$m\hbar$	(4)
$\frac{1}{\sqrt{2l+1}} \begin{bmatrix} -\sqrt{l-m+1/2}\,Y_{lm-1/2} \\ \sqrt{l+m+1/2}\,Y_{lm+1/2} \end{bmatrix}$	$l(l+1)\hbar^2$	$\frac{3}{4}\hbar^2$	$(l-\frac{1}{2})(l+\frac{1}{2})\hbar^2$	$m\hbar$	
	$(j+\frac{1}{2})(j+\frac{3}{2})\hbar^2$	$\frac{3}{4}\hbar^2$	$j(j+1)\hbar^2$	$m\hbar$	

We will denote these functions by $\mathcal{Y}_l^{jm}$. It is also easy to see that $J^2 = L^2 + \frac{3}{4}\hbar^2 + \hbar L \cdot \sigma$. Hence $L \cdot \sigma$ has same eigenfunctions.