1. Spherical Bessel functions are defined by

$$j_l(\rho) = (-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\sin\rho}{\rho}.$$

Show that these functions satisfy the differential equation

$$\left[\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} + \left(1 - \frac{l(l+1)}{\rho^2}\right)\right] R(\rho) = 0.$$

2. Consider a particle trapped in a spherical box of radius a. Find the eigenvalues and eigenfunctions for angular momentum quantum number $l = 0$.

3. Consider a potential (3D square well)

$$V(\mathbf{r}) = \begin{cases} -V_0, & |\mathbf{r}| < a, \\ 0, & |\mathbf{r}| > a. \end{cases}$$

Assume $E < 0$.

 (a) In both cases, interior ($r < a$) and exterior ($r > a$), obtain the radial differential equation and write down the solutions.
 (b) Write down the BC at $r = a$. Obtain the energy quantization condition.
 (c) For $l = 0$, simplify the energy quantization condition.

4. Starting with the generating function

$$U(\rho, s) = \frac{1}{1-s}\exp\left[-\frac{\rho s}{1-s}\right] = \sum_{q=0}^{\infty} \frac{L_q(\rho)}{q!} s^q$$

where $|s| < 1$, prove the recurrence relations,

$$L_{q+1}(\rho) + (\rho - 1 - 2q)\, L_q(\rho) + q^2 L_{q-1}(\rho) = 0$$

and

$$\frac{d}{d\rho}L_q(\rho) - q\frac{d}{d\rho}L_{q-1}(\rho) + L_{q-1}(\rho) = 0.$$

Using these, prove

$$\left[\rho\frac{d^2}{d\rho^2} + (1-\rho)\frac{d}{d\rho} + q\right] L_q(\rho) = 0.$$

5. Prove, using generating function, for hydrogen atom

$$\langle r\rangle_{nlm} = a_0\frac{n^2}{Z}\left\{1 + \frac{1}{2}\left[1 - \frac{l(l+1)}{n^2}\right]\right\}.$$

6. Statistically, the probabilty that the hydrogen atom is in a state nlm is $\exp[-E_n/kT]$ where E_n is the energy of the state, k is the Boltzman constant and T is temperature. What is the ratio of the probability of hydrogen being in the ground state to its being in the first excited state?

7. The state of the electron in the hydrogen atom is given by the wave function

$$\Psi(\mathbf{r}) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{3/2} e^{-\alpha^2 r^2/2}.$$

What is the probability that the electron will be found in the ground state? And in state $nlm = 110$?

8. Let $\hat{\mathbf{S}}$ be the spin angular momentum operator with $s = \frac{1}{2}$.

   (a) Anticommutator of two operators $\hat{A}$ and $\hat{B}$, denoted by $\left[\hat{A}, \hat{B}\right]_+$ is defined as $\hat{A}\hat{B}+\hat{B}\hat{A}$. Prove that $\left[\hat{S}_x, \hat{S}_y\right]_+ = \left[\hat{S}_y, \hat{S}_z\right]_+ = \left[\hat{S}_z, \hat{S}_x\right]_+ = 0$.

   (b) Prove that $\hat{S}_i\hat{S}_j = \frac{i\hbar}{2}\hat{S}_k$, where $i, j, k = x$ or y or z, but only in cyclic order.

   (c) Prove $\hat{S}_i^2 = I$.

9. Let $\hat{\mathbf{S}}$ be the spin angular momentum operator of a particle with $s = \frac{1}{2}$. Find the eigenfunctions and eigenvalues of operators $\hat{S}_x$ and $\hat{S}_y$. If the state of the particle is

$$\Psi = \begin{bmatrix} \cos a \\ \sin a\, e^{ib} \end{bmatrix}$$

where a, b are real constants, what is the probability that a measurement of $\hat{S}_y$ yields $-\hbar/2$.

10. Obtain the eigenvalues and corresponding normalized eigenvectors of $\hat{S}_n = \hat{\mathbf{n}} \cdot \hat{\mathbf{S}}$ for a particle of spin 1, where $\hat{\mathbf{n}}$ is a unit vector defined by polar coordinates (θ_0, ϕ_0).

11. Consider a spin-orbit interaction of a particle with spin 1. Write down the Hamiltonian in matrix form. Find all eigenvalues.

12. Two particles of the spin 1 have spin operators $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$. Let $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$. Find simultaneous eigenfunctions of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$.

13. An electron is at rest in an oscillating magnetic field

$$\mathbf{B} = B_0 \cos(\omega t)\, \hat{k}$$

where B_0 and ω are constants.

   (a) Construct the Hamiltonian of the system.

   (b) The state of the electron at $t = 0$ is given by 'up' state with respect to $x$axis, that is,

$$\Psi(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   Find $\Psi(t)$ for $t > 0$. [Since the Hamiltonian is dependent on time, you must solve time-dependent Schrodinger equation directly.]

   (c) If you measure S_x, what is the probability that the result would be $-\hbar/2$?

   (d) What is the minimum value of B_0 that is required surely yield $-\hbar/2$ as a result of S_x measurement?