1. [5 Marks] Calculate the coefficients of transmission and reflection at $x = 0$ for the step potential given by

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases}$$

assuming $0 < E < V_0$, where E is the energy of the incoming beam of particle (mass m) travelling in positive x direction.

Solution:

Let $\alpha^2 = 2mE/\hbar$ and $\beta^2 = 2m(V_0 - E)/\hbar$. For $x < 0$,

$$\psi(x) = \begin{cases} A\exp(i\alpha x) + B\exp(-i\alpha x) & x < 0 \\ C\exp(\beta x) + D\exp(-\beta x) & x > 0 \end{cases}$$

Now since ψ must not diverge as $x \to \infty$, $C = 0$. Since ψ and ψ' must be continuous at $x = 0$,

$$\begin{aligned} A + B &= D \\ i\alpha(A - B) &= -\beta D \end{aligned}$$

This gives us

$$\begin{aligned} D &= \frac{2\alpha A}{\alpha + i\beta} \\ B &= \frac{(\alpha - i\beta)}{(\alpha + i\beta)} A \end{aligned}$$

Thus the reflection coefficient

$$R = \left|\frac{B}{A}\right|^2 = 1$$

and hence $T = 0$. Alternatively, outgoing flux

$$\begin{aligned} J_{out}(x) &= \frac{\hbar}{2mi}\left(\psi\frac{d}{dx}\psi^* - \psi^*\frac{d}{dx}\psi\right) \\ &= 0 \qquad x > 0 \end{aligned}$$

Hence, $T = J_{out}/J_{in} = 0$.

2. [5 Marks] The **normalized** wave function of a particle (mass m), moving in the harmonic potential of natural angular frequency ω, at an instant is given by

$$\psi(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{\frac{1}{2}} \exp\left(i\frac{p_0 x}{\hbar} - \frac{\alpha^2 x^2}{2}\right)$$

where $\alpha = \sqrt{m\omega/\hbar}$ and p_0 is a positive constant. Find the average energy of the particle.

Solution:

First

$$\begin{aligned} \langle x^2 \rangle &= \left(\frac{\alpha}{\sqrt{\pi}}\right)\int_{-\infty}^{\infty} x^2 \exp\left(-\alpha^2 x^2\right) dx \\ &= \left(\frac{\alpha}{\sqrt{\pi}}\right)\frac{\sqrt{\pi}}{2\alpha^3} = \frac{1}{2\alpha^2} = \frac{\hbar}{2m\omega} \end{aligned}$$

Now, $\hat{P}\psi(x) = \left(p_0 + i\hbar\alpha^2 x\right)\psi(x)$. Then

$$\begin{aligned}
\left\langle \hat{P}^2 \right\rangle &= \left\langle \hat{P}\psi, \hat{P}\psi \right\rangle \\
&= \left\langle \psi, \left(p_0 - i\hbar\alpha^2 x\right)\left(p_0 + i\hbar\alpha^2 x\right)\psi \right\rangle \\
&= \left\langle \psi, \left(p_0^2 + \hbar^2\alpha^4 x^2\right)\psi \right\rangle \\
&= p_0^2 + \frac{1}{2}\hbar^2\alpha^2 = p_0^2 + \frac{1}{2}\hbar m\omega
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle E \rangle &= \frac{1}{2m}\left\langle \hat{P}^2 \right\rangle + \frac{1}{2}m\omega^2 \left\langle x^2 \right\rangle \\
&= \frac{p_0^2}{2m} + \frac{1}{2}\hbar\omega.
\end{aligned}$$

Useful Information:

$$\int_{-\infty}^{\infty} x^2 \exp\left(-a^2 x^2\right) dx = \frac{\sqrt{\pi}}{2a^3}$$