1. [10 Marks]Answer the following questions.

   (a) [2] Show that $[\hat{X}, \hat{P}] = i\hbar$, where $\hat{X}$ and $\hat{P}$ are position and momentum operators.

   (b) [2] For a particle in a box, the wave function is given by

$$\Psi(x) = A\sqrt{x\,(L-x)}.$$

   Find A and sketch the probability density function for position measurement.

   (c) [2] What is the probability that the particle in question 1(b) will be found in the interval $[0, \frac{L}{4}]$?

   (d) [2] Show that the eigenvalues of a unitary matrix are complex numbers with unit magnitude.

   (e) [2] Let $\hat{p}$ be the momentum operator. Show that $\exp[i\hat{p}a/\hbar]f(x) = f(x+a)$ where a is a real constant. [Assume that f is a smooth function of a real variable.]

Solutions

(a) $\left[\hat{X}\hat{P} - \hat{P}\hat{X}\right] f(x) = (-i\hbar)\left[x\frac{d}{dx}f(x) - \frac{d}{dx}(xf(x))\right] = (-i\hbar)\left(-f(x)\right) = i\hbar f(x)$ Thus, the result.

(b) $\int_0^L A^2 x(L-x)dx = A^2L^3\left(\frac{1}{2}-\frac{1}{3}\right) = A^2L^3/6$. Thus $A = \sqrt{6/L^3}$. Sketch is given below.

(c) $\int_0^{L/4} A^2x(L-x)dx = \frac{6}{L^3}L^3\left(\frac{1}{32}-\frac{1}{192}\right) = \frac{5}{32}$.

(d) Let O be a unitary operator with an eigenvalue λ and eigenvector u. Then,

$$\begin{aligned} Ou &= \lambda u \\ \text{Taking hermitian adjoint} \implies u^\dagger O^\dagger &= \lambda^* u^\dagger. \end{aligned}$$

Thus,

$$\begin{aligned} u^\dagger O^\dagger O u &= \left(\lambda^* u^\dagger\right)(\lambda u) \\ \implies u^\dagger u &= |\lambda|^2\, u^\dagger u \\ \implies |\lambda|^2 &= 1. \end{aligned}$$

(e) Now,

$$\begin{aligned} e^{i\hat{p}a/\hbar} f(x) &= e^{a\frac{d}{dx}} f(x) \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{dx^n} f(x) \\ &= f(x+a) \end{aligned}$$

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2. [10 Marks] The wave function of a free particle at some instant is given by

$$\Psi(x) = B\exp\left[i\frac{p_0 x}{\hbar}\right]\exp\left[-\frac{|x|}{2d}\right]$$

where, B, p_0, and d are positive real constants.

(a) [2] Find B by normalizing Ψ.

(b) [3] Find $\langle\hat{x}\rangle, \langle\hat{x}^2\rangle$.

(c) [4] Find $\langle\hat{p}\rangle, \langle\hat{p}^2\rangle$. [Hint: Calculate $\langle\hat{p}^2\rangle$ by evaluating $\langle\hat{p}\Psi, \hat{p}\Psi\rangle$]

(d) [1] Verify uncertainty principle.

Solution

(a) The square of the norm of Ψ is given by

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi^*(x)\Psi(x)dx &= B^2 \left[\left[\int_{-\infty}^{0} \Psi^*(x)\Psi(x)dx + \int_{0}^{\infty} \Psi^*(x)\Psi(x)dx\right]\right] \\
&= B^2 \left[\int_{-\infty}^{0} e^{x/d}dx + \int_{0}^{\infty} e^{-x/d}dx\right] \\
&= B^2 [d+d] = 2dB^2.
\end{aligned}$$

For the norm to be unity, $B = \sqrt{1/2d}$.

(b) Average of $\hat{x}$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi^*(x)x\Psi(x)dx &= B^2 \left[\int_{-\infty}^{0} \Psi^*(x)x\Psi(x)dx + \int_{0}^{\infty} \Psi^*(x)x\Psi(x)dx\right] \\
&= B^2 \left[\int_{-\infty}^{0} xe^{x/d}dx + \int_{0}^{\infty} xe^{-x/d}dx\right] \\
&= B^2 \left[-d^2 + d^2\right] = 0.
\end{aligned}$$

Average of $\hat{x}^2$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi^*(x)x^2\Psi(x)dx &= B^2 \left[\int_{-\infty}^{0} \Psi^*(x)x^2\Psi(x)dx + \int_{0}^{\infty} \Psi^*(x)x^2\Psi(x)dx\right] \\
&= B^2 \left[\int_{-\infty}^{0} x^2e^{x/d}dx + \int_{0}^{\infty} x^2e^{-x/d}dx\right] \\
&= B^2 \left[2d^3 + 2d^3\right] = 2d^2.
\end{aligned}$$

(c) Now,

$$\hat{p}\Psi(x) = \begin{cases} \left(p_0 - \frac{i\hbar}{2d}\right)\Psi(x) & x > 0 \\ \left(p_0 + \frac{i\hbar}{2d}\right)\Psi(x) & x < 0 \end{cases}$$

Average of $\hat{p}$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi^*(x)\left(\hat{p}\Psi(x)\right)dx &= B\left[\int_{-\infty}^{0} \Psi^*(x)\hat{p}\Psi(x)dx + \int_{0}^{\infty} \Psi^*(x)\hat{p}\Psi(x)dx\right] \\
&= B^2\left[\left(p_0 + \frac{i\hbar}{2d}\right)\int_{-\infty}^{0} \Psi^*(x)\Psi(x)dx + \left(p_0 - \frac{i\hbar}{2d}\right)\int_{0}^{\infty} \Psi^*(x)\Psi(x)dx\right] \\
&= B^2\left[\left(p_0 + \frac{i\hbar}{2d}\right)d + \left(p_0 - \frac{i\hbar}{2d}\right)d\right] = p_0.
\end{aligned}$$

and average of $\hat{p}^2$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\hat{p}\Psi(x)\right)^*\left(\hat{p}\Psi(x)\right)dx &= B\left[\int_{-\infty}^{0} \left(\hat{p}\Psi(x)\right)^*\hat{p}\Psi(x)dx + \int_{0}^{\infty} \left(\hat{p}\Psi(x)\right)^*\hat{p}\Psi(x)dx\right] \\
&= B^2\left[\left(p_0 + \frac{i\hbar}{2d}\right)\left(p_0 - \frac{i\hbar}{2d}\right)d + \left(p_0 - \frac{i\hbar}{2d}\right)\left(p_0 + \frac{i\hbar}{2d}\right)d\right] \\
&= \left(p_0^2 + \frac{\hbar^2}{4d^2}\right).
\end{aligned}$$

(d) Thus $\sigma_x = \sqrt{2}d$ and $\sigma_p = \hbar/2d$, and $\sigma_x\sigma_p = \hbar/\sqrt{2} > \hbar/2$.

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3. [6 Marks] A particle is in the ground state of an infinite potential well of width L. Now the well is suddenly expanded **symmetrically** to the width of $2L$, leaving the wavefunction undisturbed. Show that the probability of finding the particle in the ground state of the new well is $(8/3\pi)^2$.

Solution

The eigen energies and eigen functions of the new well are given by

$$\begin{aligned} u_n(x) &= \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right) \\ \epsilon_n &= \frac{\hbar^2\pi^2}{8mL^2} n^2 \end{aligned}$$

The wave function just after the well changes to the new well will be

$$\Psi(x) = \begin{cases} 0 & x \in [0, L/2] \\ \sqrt{\frac{1}{L}} \sin\left(\frac{\pi}{L}\left(x - \frac{L}{2}\right)\right) & x \in [L/2, 3L/2] \\ 0 & x \in [3L/2, L] \end{cases}$$

Then, if $\Psi = \sum_n c_n u_n$, then

$$\begin{aligned} c_1 &= \int_0^{2L} \Psi(x) u_n(x) dx \\ &= -\sqrt{\frac{2}{L}} \int_{L/2}^{3L/2} \cos\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x}{2L}\right) dx \\ &= \frac{8}{3\pi}. \end{aligned}$$

Thus the probability of finding the particle in the ground state of the new well is $(8/3\pi)^2$.

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4. [4 Marks] Consider a n dimensional complex inner product space V_n with elements given by a $n \times 1$ column matrix. The inner product of two elements is defined as

$$\langle u, v\rangle = u^\dagger v$$

where $u^\dagger$ is the complex conjugate of the transpose of u. Let $B = \{e_1, e_2, \dots e_n\}$ be an orthonormal basis. A family of projection operators is defined as

$$\mathbf{P}_j u = \langle e_j, u\rangle\, e_j \qquad j = 1, 2, \dots n.$$

(a) [1] Show that the projection operators are hermitian.

(b) [1] Find the matrix of $\mathbf{P}_j$ wrt the basis B.

(c) [1] Show that $\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_i$.

(d) [1] Show that $\sum_{i=1}^n \mathbf{P}_i = \mathbf{I}$, where $\mathbf{I}$ is an identity operator.

Solution

(a) For any $u, v \in V_n$, we must show that $\langle u, \mathbf{P}_j v\rangle = \langle \mathbf{P}_j u, v\rangle$.

$$\begin{aligned} \text{RHS} = \langle u, \mathbf{P}_j v\rangle &= \langle u, \langle e_j, v\rangle\, e_j\rangle = \langle u, e_j\rangle \langle e_j, v\rangle \\ \text{LHS} = \langle \mathbf{P}_j u, v\rangle &= \langle \langle e_j, u\rangle\, e_j, v\rangle = \langle u, e_j\rangle \langle e_j, v\rangle \end{aligned}$$

Thus $\mathbf{P}_j$ is hermitian.

(b) $[\mathbf{P}_j]_{ik} = \langle e_i, \mathbf{P}_j e_k \rangle = \langle e_i, \langle e_j, e_k \rangle e_j \rangle = \delta_{j,k}\delta_{j,i}$. Thus,

$$\mathbf{P}_j = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix} j^{\text{th}} \text{ row.}$$

(c) $\mathbf{P}_i\mathbf{P}_j u = \langle e_j, u \rangle \mathbf{P}_i e_j = \langle e_j, u \rangle \langle e_i, e_j \rangle e_i = \delta_{ij} \langle e_i, u \rangle e_i = \delta_{ij}\mathbf{P}_i u$.

(d) Let $u = \sum c_n e_n$. Then $\mathbf{P}_j u = \sum c_n \mathbf{P}_j e_n = \sum c_n \langle e_j, e_n \rangle e_j = c_j e_j$. Thus $\sum_j \mathbf{P}_j u = \sum_j c_j e_j = u$. Thus $\sum_j \mathbf{P}_j = I$
